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Covering Properties of $C_p(X)$ and $C_k(X)$

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Abstract. Let X be a Tychonoff space. We survey some classic and recent results that characterize the topology or cardinality of X when $C_p(X)$ or $C_k(X)$ is covered by certain families of sets (sequences, resolutions, closure-preserving coverings, compact coverings ordered by a second countable space) which swallow or not some classes of sets (compact sets, functionally bounded sets, pointwise bounded sets) in $C(X)$.

1. Preliminaries

Unless otherwise stated, X will stand for an infinite Tychonoff space. We denote by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions on X equipped with the pointwise topology τ_p. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology. We denote by $C_k(X)$ the space $C(X)$ equipped with the compact-open topology τ_k. A family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of a set X is a *resolution* for X if it covers X and verifies that $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$. A family of bounded sets in a locally convex space E that swallows the bounded sets is called a *fundamental family of bounded sets*. Definitions not included in this paper can be found in [6, 18, 49].

2. Countable coverings for $C_p(X)$

The following folklore result can be found in [49, Proposition 9.18]. Velichko's theorem can be found in [1, I.2.1 Theorem] or in [49, Theorem 9.12].

Theorem 2.1. *The space $C_p(X)$ admits a fundamental sequence of pointwise bounded sets if and only if X is finite.*

Theorem 2.2 (Velichko). *The space $C_p(X)$ is covered by a sequence of compact sets if and only if X is finite.*

Next theorem extends Velichko's result to relatively countably compact sets.

Theorem 2.3 (Tkachuk-Shakhmatov [75]). *$C_p(X)$ is covered by a sequence of relatively countably compact sets if and only if X is finite.*

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Theorem 2.5 below extends Tkachuk-Shakhmatov theorem to pointwise bounded relatively sequentially complete sets. Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions defined on X is *pointwise eventually constant* [34] if for each $x \in X$ there is a constant $f(x)$ such that $f_n(x) = f(x)$ for all but finitely many $n \in \mathbb{N}$.

Theorem 2.4 (Ferrando-Kąkol-Saxon [34, Theorem 3.1]). *$C_p(X)$ is covered by a sequence of relatively sequentially complete sets if and only if X is a P-space.*

Proof. Assume that $C_p(X) = \bigcup_{n=1}^{\infty} Q_n$ with Q_n relatively sequentially complete for every $n \in \mathbb{N}$ and let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded pointwise eventually constant sequence in $C_p(X)$ with limit f in $\mathbb{R}^X$. Let us denote by $C^b(X)$ the Banach space of all continuous and bounded functions on X equipped with the supremum norm $\|\cdot\|_\infty$. Fix $k > 0$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \leq k$.

Since $\left\{C^b(X) \cap Q_n : n \in \mathbb{N}\right\}$ is a countable covering of $C^b(X)$, according to the Baire category theorem there is $p \in \mathbb{N}$ such that the closure B_p of $C^b(X) \cap Q_p$ in $C^b(X)$ has an interior point in the norm topology. So, if D denotes the closed unit ball of $C^b(X)$, there are $\epsilon > 0$ and $h \in Q_p$ with $h + \epsilon D \subseteq B_p$. Since $f_n \in kD$ for each $n \in \mathbb{N}$, we have $\left\{h + \epsilon k^{-1} f_n : n \in \mathbb{N}\right\} \subseteq B_p$. As $C^b(X) \cap Q_p$ is norm dense in B_p, for each $n \in \mathbb{N}$ there is $g_n \in C^b(X) \cap Q_p$ with $\left|g_n(x) - \left(h + \epsilon k^{-1} f_n\right)(x)\right| < n^{-1}$ for all $x \in X$. Since $\left\{h + \epsilon k^{-1} f_n\right\}_{n=1}^{\infty}$ is a pointwise eventually constant sequence that converges to $h + \epsilon k^{-1} f$, clearly $g_n \to h + \epsilon k^{-1} f$ pointwise on X. Using the fact that Q_p is relatively sequentially complete, it turns out that $h + \epsilon k^{-1} f \in C(X)$. Hence $f \in C(X)$. But, as follows from [34, Theorem 1.1], a Tychonoff space X is a P-space if and only if each uniformly bounded pointwise eventually constant sequence in $C_p(X)$ converges in $C_p(X)$. So, X is a P-space. For the converse note that if X is a P-space, then $C_p(X)$ is sequentially complete [8]. □

Theorem 2.5 (Ferrando-Kąkol-Saxon [34, Corollary 3.2]). *$C_p(X)$ is covered by a sequence of pointwise bounded relatively sequentially complete sets if and only if X is finite.*

Proof. If $C_p(X) = \bigcup_{n=1}^{\infty} Q_n$ with each Q_n pointwise bounded and relatively sequentially complete, Theorem 2.4 ensures that X is a P-space. If $\{x_n\}_{n=1}^{\infty}$ is an infinite sequence in X, for each $n \in \mathbb{N}$ there is $\alpha_n > 0$ with $\sup_{g\in Q_n} |g(x_n)| < \alpha_n$. But [49, Lemma 9.5] provides $f \in C(X)$ with $f(x_n) = \alpha_n$, i. e., such that $f \notin Q_n$ for every $n \in \mathbb{N}$, a contradiction. Thus X must be finite. □

Theorem 2.6 (Tkachuk, [69, 3.11 Theorem]). *If $C_p(X)$ is covered by a sequence of functionally bounded sets, then X is pseudocompact and each countable subset of X is closed, discrete and C^*-embedded in X.*

Proof. (Sketch) Let us call *σ-bounded* a space which is covered by countably many functionally bounded sets and assume that $C_p(X)$ is σ-bounded. If X is not pseudocompact, it contains a closed homeomorphic copy Y of $\mathbb{N}$, hence C-embedded [39, Problem 3L]. Since the restriction map $T : C_p(X) \to C_p(Y)$ defined by $Tf = f|_Y$ is continuous and onto, this implies that $C_p(Y)$ is σ-bounded. Hence $C_p(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}$ is covered by a sequence of compact sets and Velichko's theorem ensures that $\mathbb{N}$ must be finite, a contradiction. On the other hand, since $C_p(X, I) = \{f \in C(X) : -1 \leq f \leq 1\}$ is a retract of $C_p(X)$, it turns out that $C_p(X, I)$ is σ-bounded. If Z is a non-closed countable subset of X and $y \in \overline{Z} \setminus Z$, it is not hard to show that $M = \left\{f \in C_p(X, I) : f(y) = 0\right\}$ is also covered by countably many functionally bounded sets $\{F_n : n \in \mathbb{N}\}$. But one can determine a function $f \in M$ such that $f \notin F_n$ for every $n \in \mathbb{N}$ (see [69, 3.7 Lemma] for details). So, such Z does not exist. Finally, it is well-known that a subspace S of X is C^*-embedded if and only if $\overline{S}^{\beta X} = \beta S$. If each countable set in X is closed, it can be seen that each countable set A is discrete and C^*-embedded if and only if $\overline{A}^{\beta X} = \beta A$, [69, 3.8 Proposition]. With the help of this result one can show that if $C_p(X, I)$ is σ-bounded, every countable subset of X is discrete and C^*-bounded [69, 3.9 Theorem]. □

3. Uncountable coverings for $C_p(X)$

Recall that X is a *Lindelöf* Σ-space if it is a continuous image of a space that can be perfectly mapped onto a second countable space [1, 57]. Also, X is a Lindelöf Σ-space if and only if is *countably K-determined* [63], i. e., if there is an upper semi-continuous (*usc*) map T from a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ into the family $\mathcal{K}(X)$ of compact subsets of X such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$. This is equivalent to saying that (*i*) $\{T(\alpha) : \alpha \in \Sigma\}$ covers X and (*ii*) if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$ the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point in $T(\alpha)$. A space X is *K-analytic* (resp. *quasi-Suslin*) if there is a map T from $\mathbb{N}^{\mathbb{N}}$ into $\mathcal{K}(X)$ (resp. into the family of countably compact sets in X) such that (*i*) $\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ covers X and (*ii*) if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$ the sequence $\{x_n\}$ has a cluster point contained in $T(\alpha)$ (see [76, I.4.2 and I.4.3]). Each σ-compact (σ-countably compact) space is K-analytic (resp. quasi-Suslin). A space X is *analytic* if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$. Each analytic space is K-analytic, each K-analytic space is quasi-Suslin and Lindelöf Σ, and each Lindelöf Σ-space is Lindelöf. A family $\mathcal{N}$ of subsets of X is a *network* for X if for any $x \in X$ and any open set U in X with $x \in U$ there is some $P \in \mathcal{N}$ such that $x \in P \subseteq U$. The *network weight* $nw(X)$ of X is the least cardinality of a network of X, and a space X is called *cosmic* if $nw(X) = \aleph_0$. Alternatively, X is a cosmic space if and only if it is a continuous image of a separable metric space [56]. So, each analytic space is cosmic. Conversely, every K-analytic cosmic space is analytic [49, Proposition 6.4]. Moreover, $C_p(X)$ is a cosmic space if and only if X is cosmic [56, Proposition 10.5]. A family $\mathcal{N}$ of subsets of a space X is a *network modulo a family* $\mathcal{A}$ of subsets of X if for each open set V of X and for every $A \in \mathcal{A}$ with $A \subseteq V$ there exists $N \in \mathcal{N}$ such that $A \subseteq N \subseteq V$. A space is Lindelöf Σ if and only if it admits a countable network modulo a covering by compact sets [49, Proposition 3.5]. Hence, every cosmic space is a Lindelöf Σ-space. A space X is *angelic* if relatively countably compact sets in X are relatively compact and for every relatively compact subset A of X each point of $\overline{A}$ is the limit of a sequence of A, [36]. A space X is *projectively σ-compact* if each separable metrizable space Y that is a continuous image of X is σ-compact. Clearly, every σ-bounded space (in the sense of Theorem 2.6) is projectively σ-compact [3, Proposition 1.1], and every projectively σ-compact cosmic space is σ-compact (see [49, Proposition 9.4] or [60]). A space $C_p(X)$ is said to be *Lindelöf Σ-framed* (or *K-analytic-framed*) *in* $\mathbb{R}^X$ if there is a Lindelöf Σ-space (resp. a K-analytic space) S in $\mathbb{R}^X$ such that $C(X) \subseteq S$. A family $\mathcal{N}$ of subsets of a topological space X is called a *cs^*-network at a point* $x \in X$ if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite [38]; $\mathcal{N}$ is a *cs^*-network* in X if $\mathcal{N}$ is a cs^*-network at each point $x \in X$.

Lemma 3.1. *If $C_p(X)$ is Lindelöf Σ-framed in $\mathbb{R}^X$, then υX is a Lindelöf Σ-space and $C_p(X)$ is angelic.*

Proof. First statement after the conditional comes from [59, Theorem 3.5] or [22, Theorem 3]. For the second use the first and [62, Theorem 3], since $C_p(X)$ is angelic whenever $C_p(\upsilon X)$ is angelic. □

Lemma 3.2 (Ferrando-Kąkol, [29, Lemma 1]). *Let X be nonempty and Z be a subspace of $\mathbb{R}^X$. If Z has a countable network modulo a cover $\mathcal{B}$ of Z by pointwise bounded subsets, then $Y = \bigcup\{\overline{B} : B \in \mathcal{B}\}$, closures in $\mathbb{R}^X$, is a Lindelöf Σ-space such that $Z \subseteq Y \subseteq \mathbb{R}^X$.*

Proof. Let $\mathcal{N} = \{T_n : n \in \mathbb{N}\}$ be a countable network modulo a cover $\mathcal{B}$ of Z consisting of pointwise bounded sets. Set $\mathcal{N}_1 = \{\overline{T_n} : n \in \mathbb{N}\}$, $\mathcal{B}_1 = \{\overline{B} : B \in \mathcal{B}\}$, closures in $\mathbb{R}^X$, and $Y = \cup\mathcal{B}_1$. Let us show that $\mathcal{N}_1$ is a network in Y modulo the compact cover $\mathcal{B}_1$ of Y. In fact, if U is a neighborhood in $\mathbb{R}^X$ of $\overline{B}$, use $\overline{B}$ compactness to get a closed neighborhood V of $\overline{B}$ in $\mathbb{R}^X$ contained in U. Since $\mathcal{N}$ is a network modulo $\mathcal{B}$ in Z there is $n \in \mathbb{N}$ with $B \subseteq T_n \subseteq V \cap Z$, which implies that $\overline{B} \subseteq \overline{T_n} \subseteq U$. According to Nagami's criterion [1, IV.9.1 Proposition], Y is a Lindelöf Σ-space such that $Z \subseteq Y \subseteq \mathbb{R}^X$. □

Theorem 3.3 (Ferrando-Kąkol, [29, Proposition 1]). *The following asserts are equivalent*

1. *$C_p(X)$ admits a resolution of pointwise bounded sets.*
2. *$C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$.*

Proof. Let $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a resolution for $C_p(X)$ of bounded sets, denote by B_α the closure of A_α in $\mathbb{R}^X$ and put $Z = \bigcup\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Clearly each B_α is a compact subset of $\mathbb{R}^X$ and Z is a quasi-Suslin space [11, Proposition 1] such that $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$. As each quasi-Suslin space Z has a countable network modulo a *resolution* $\mathcal{B}$ of Z consisting of countably compact sets (see [20, Proof Theorem 8]) and every countable compact subset of $\mathbb{R}^X$ is pointwise bounded, Lemma 3.2 assures that $Y = \bigcup\{\overline{B} : B \in \mathcal{B}\}$ is a Lindelöf Σ-space, hence Lindelöf, such that $Z \subseteq Y \subseteq \mathbb{R}^X$. As each set $\overline{B}$ with $B \in \mathcal{B}$ is compact, and $\{\overline{B} : B \in \mathcal{B}\}$ is a resolution for Y, again Y is a quasi-Suslin space. Since every Lindelöf quasi-Suslin space is K-analytic and $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, it turns out that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. For the converse, note that each K-analytic space has a resolution consisting of compact sets [67]. □

Theorem 3.4 (Arkhangel'skiĭ-Calbrix, [4, Theorem 2.3]). *If $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$, then X is projectively σ-compact.*

Proof. Assume $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. Let Y be a separable metric space that is a continuous image of X, say $f : X \to Y$. Consider the pullback $f^* : \mathbb{R}^Y \to \mathbb{R}^X$ defined by $f^*(g) = g \circ f$, which is a linear homeomorphism onto $f^*\left(\mathbb{R}^Y\right)$ with closed range [1, 0.4.6 Proposition]. If S is a K-analytic space such that $C(X) \subseteq S \subseteq \mathbb{R}^X$, then $f^*(C(Y)) \subseteq S \cap f^*\left(\mathbb{R}^Y\right)$, which is a K-analytic subspace of $\mathbb{R}^X$, since $S \cap f^*\left(\mathbb{R}^Y\right)$ is closed in S. Hence $T := (f^*)^{-1}(S) \cap \mathbb{R}^Y$ is a K-analytic subspace of $\mathbb{R}^Y$ such that $C(Y) \subseteq T \subseteq \mathbb{R}^Y$, i. e., $C_p(Y)$ is K-analytic-framed in $\mathbb{R}^Y$. So, if $\mathbb{R}_+$ are the nonnegative real numbers, since there exists a (strictly increasing) homeomorphism from $\mathbb{R}$ onto $\mathbb{R}^+$, there exists a K-analytic subspace M of $\mathbb{R}_+^Y$ such that $C^+(Y) := C(Y) \cap \mathbb{R}_+^Y$ is contained in M. Let $\varphi : \mathbb{N}^\mathbb{N} \to \mathcal{K}(M)$, where $\mathcal{K}(M)$ designates the family of compact sets of M, an *usc* map such that $\bigcup\{\varphi(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} = M$. Define $\lambda : \mathbb{N}^\mathbb{N} \to \mathbb{R}_+^Y$ by $\lambda(\alpha) = \inf \varphi\left(\{\beta \in \mathbb{N}^\mathbb{N} : \beta \le \alpha\}\right)$. As $\{\beta \in \mathbb{N}^\mathbb{N} : \beta \le \alpha\}$ is a compact set in $\mathbb{N}^\mathbb{N}$, $\varphi\left(\{\beta \in \mathbb{N}^\mathbb{N} : \beta \le \alpha\}\right)$ is a compact set in M and the infimum is with respect to the pointwise ordering of $\mathbb{R}^Y$, hence $\lambda(\alpha)(y) = \inf\{\varphi(\beta)(y) : \beta \le \alpha\} > 0$ for each $y \in Y$. Clearly $\lambda(\alpha) \le \lambda(\beta)$ whenever $\beta \le \alpha$, and if $f \in C^+(Y) \subseteq M$ there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $f \in \varphi(\gamma)$, so that $\lambda(\gamma) \le f$. Let $\left(\overline{Y}, d\right)$ be a metric compactification of Y. For each $\alpha \in \mathbb{N}^\mathbb{N}$ set $K_\alpha = \bigcap\left\{\overline{Y} \setminus B(y, \lambda(\alpha)(y)) : y \in Y\right\}$, where $B(y, \lambda(\alpha)(y)) = \left\{z \in \overline{Y} : d(y, z) < \lambda(\alpha)(y)\right\}$ is the open ball in $\overline{Y}$ of center y and radius $\lambda(\alpha)(y) \ge 0$. Clearly K_α is a compact set in $\overline{Y} \setminus Y$, and we claim that $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution for $\overline{Y} \setminus Y$ that swallows the compact sets in $\overline{Y} \setminus Y$. The relation $K_\alpha \subseteq K_\beta$ comes from $\lambda(\beta) \le \lambda(\alpha)$ whenever $\alpha \le \beta$. In addition, if Q is a compact set in $\overline{Y} \setminus Y$, the function $h : \overline{Y} \to \mathbb{R}_+$ defined by $h(y) = d(y, Q)$ belongs to $C^+(Y)$ when restricted to Y. So, there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\lambda(\gamma) \le h|_Y$. Thus $d(y, z) \ge \lambda(\gamma)(y)$ for every $y \in Y$ and $z \in Q$. In other words, $Q \cap \bigcup\{B(y, \lambda(\alpha)(y)) : y \in Y\} = \emptyset$, which means that $Q \subseteq K_\gamma$. In this circumstances, Christensen's theorem [15, Theorem 3.3] shows that $\overline{Y} \setminus Y$ is a Polish space, so an absolute G_δ [51, Chapter 6, Problem K]. Consequently, Y is an F_σ of the compact space $\overline{Y}$, i. e., Y is a σ-compact space. □

Corollary 3.5. *If $C_p(X)$ admits a resolution consisting of pointwise bounded sets, then X is projectively σ-compact.*

Proof. This is a straightforward consequence of Theorems 3.3 and 3.4. □

Theorem 3.6 (Ferrando-Kąkol, [29, Corollary 1]). *Let X be a cosmic space. $C_p(X)$ has a resolution of pointwise bounded sets if and only if X is σ-compact.*

Proof. The 'only if' statement is consequence of Corollary 3.5 and the fact, mentioned earlier, that each projectively σ-compact cosmic space is σ-compact. For the 'if' part note that if $X = \bigcup_{n=1}^\infty K_n$ with each K_n compact, the family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with

$$A_\alpha = \left\{f \in C(X) : \sup_{x \in K_n} \left|f(x)\right| \le \alpha(n),\, n \in \mathbb{N}\right\}$$

is a resolution for $C(X)$ consisting of pointwise bounded sets. □

Theorem 3.7 (Calbrix [9, Theorem 2.3.1]). *If $C_p(X)$ is analytic, then X is σ-compact.*

Proof. If $C_p(X)$ is analytic, it is cosmic. Hence X is also a cosmic space [56, Proposition 10.5]. Since $C_p(X)$ is K-analytic, it has a resolution of pointwise bounded sets (actually, of compact sets [67]). So, Theorem 3.6 ensures that X is σ-compact. □

Corollary 3.8. *If X is metrizable, the following are equivalent.*

1. $C_p(X)$ *is analytic.*
2. X *is σ-compact.*
3. $C_p(X)$ *has a resolution of pointwise bounded sets.*

Proof. $1 \Rightarrow 2$ follows from Theorem 3.7 and, as mentioned above, $2 \Rightarrow 3$ always holds true. On the other hand, if $C_p(X)$ has a resolution of pointwise bounded sets, then $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ by Theorem 3.3 and angelic by Lemma 3.1. But if X is metrizable, $C_p(X)$ is angelic if and only if X is separable [49, Corollary 6.10]. Consequently, for metrizable X, the fact that $C_p(X)$ has a resolution of pointwise bounded sets entails that X is a cosmic space. So, Theorem 3.6 yields the implication $3 \Rightarrow 2$. Finally, if X is a metrizable σ-compact space then X is separable. Thus $C_p(X)$ is analytic by a clasic result of Christensen [15, Theorem 3.7] (*cf.* Theorem 4.4 below). Hence $2 \Rightarrow 1$. □

Corollary 3.9. *If $C_p\left(C_p(X)\right)$ has a resolution consisting of pointwise bounded sets, then X is pseudocompact.*

Proof. If X is not pseudocompact, then $C_p(X)$ contains a complemented (linearly homeomorphic) copy of $\mathbb{R}^\omega$. If P is a continuous linear projection from $C_p(X)$ onto the linear subspace $\mathbb{R}^\omega$ the (linear) restriction map $T : C_p\left(C_p(X)\right) \to C_p(\mathbb{R}^\omega)$ given by $T\varphi = \varphi\big|_{\mathbb{R}^\omega}$ is continuous and onto, for if $\psi \in C(\mathbb{R}^\omega)$ then $\psi \circ P \in C\left(C_p(X)\right)$ and $T(\psi \circ P) = \psi$ due to $Pg = g$ for every $g \in \mathbb{R}^\omega$. Hence T carries a resolution from $C_p\left(C_p(X)\right)$ onto $C_p(\mathbb{R}^\omega)$ made up of pointwise bounded sets. Since $\mathbb{R}^\omega$ is metrizable, Corollary 3.8 shows that $\mathbb{R}^\omega$ is a σ-space, which is not true. □

Theorem 3.10 (Tkachuk [71, 2.8 Theorem]). *$C_p(X)$ has a resolution consisting of compact sets if and only if it is K-analytic.*

Proof. If $C_p(X)$ has a resolution consisting of compact sets, then $C_p(X)$ is a quasi-Suslin space [11, Proposition 1]. But, according to Lemma 3.1, the space $C_p(X)$ is angelic, and every quasi-Suslin angelic space is K-analytic [11]. The converse can be found in [67] or in [49, Theorem 3.2]. □

The following result was stated and proved by Tkachuk, [71, 3.9 Theorem]. However, it can also be derived as a consequence of Valdivia's closed graph theorem for K-analytic spaces [76, Chapter I] (as mentioned in [71]), which is the approach we choose.

Theorem 3.11. *Assume $C_p(X)$ is a Baire space. $C_p(X)$ has a resolution of compact sets if and only if X is countable and discrete.*

Proof. According to Theorem 3.10, if $C_p(X)$ has a resolution of compact sets then $C_p(X)$ is K-analytic. Hence $C_p(X)$ is a locally convex space which is both Baire and K-analytic, so a separable Fréchet space by [76, I.4.3.(21)]. This forces to $C_p(X) = \mathbb{R}^X$ with X countable. Hence X is countable and discrete. □

Theorem 3.12 (Arkhangel'skiĭ). *If $C_p(X)$ is both Baire and a Lindelöf Σ-space, then X is countable.*

Proof. Let us prove this result with the additional assumption that X is realcompact. A proof of the general case can be found in [71, 3.8 Theorem]. If $C_p(X)$ is a Baire space, it is barrelled, i. e., each closed absorbing absolutely convex set is a neighborhood of the null function. Hence, by the Buchwalter-Schmets theorem, the functionally bounded sets in X are finite [8] (see also[1, I.3.4 Theorem]). If $C_p(X)$ is a Lindelöf Σ-space, then υX is a Lindelöf Σ-space by Lemma 3.1 (see also [59, Theorem 3.5]). Since by assumption $X = \upsilon X$, it turns out that X is a Lindelöf Σ-space with finite compact sets. Consequently X must be countable [1, IV.6.15 Proposition]. □

Theorem 3.13. *Let $C_p(X)$ be a Baire space. If $C_p(X)$ has a resolution of pointwise bounded sets, then X is countable.*

Proof. This follows from a general property of locally convex spaces which assures that each locally convex Baire space E with a resolution of bounded sets is metrizable (see [50, Corollary 1]). Let us try a direct approach. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution for $C_p(X)$ consisting of absolutely convex pointwise bounded sets. Define $\beta(1) = n_1$, $\beta(i+1) = \alpha(i)$ for each $i \in \mathbb{N}$, and set $B_\beta := n_1 \overline{\text{abx}(A_\alpha)}$ where $\text{abx}(A_\alpha)$ stands for the absolutely convex cover of A_α and the closure is in $\mathbb{R}^X$. Thus $Z := \bigcup\{B_\beta : \beta \in \mathbb{N}^{\mathbb{N}}\}$ is a linear subspace of $\mathbb{R}^X$, and each set B_β is compact with $B_\alpha \subseteq B_\beta$ if $\alpha \leq \beta$. So, Z is a locally convex Baire space with a resolution of compact sets. By [31, Theorem 1], Z is a separable Fréchet space. Hence $C_p(X)$ is metrizable, so X must be countable. □

Theorem 3.14. *Let X be a paracompact locally compact space. $C_p(X)$ has a resolution of pointwise bounded sets if and only if X is σ-compact.*

Proof. As follows from [7, 9.10 Theorem 5] the space X is the topological sum $\bigoplus_{\alpha\in A} X_\alpha$ of a family $\{X_\alpha : \alpha \in A\}$ of locally compact σ-compact (pairwise disjoint) subspaces of X. Consequently, $C_p(X) = \prod_{\alpha\in A} C_p(X_\alpha)$ isomorphically. By the previous equality, $C_p(X)$ contains a copy of $\mathbb{R}^A$. If $C_p(X)$ has a resolution of pointwise bounded sets, the subspace $\mathbb{R}^A$ of $C_p(X)$ also has a resolution of pointwise bounded sets. Since $\mathbb{R}^A$ is a Baire space, Theorem 3.13 shows that A must be countable. So, X is σ-compact. The converse also holds as shown in the 'if' part of Theorem 3.6. □

The preceding theorem was originally stated as a part of [10, Proposition 2.2] assuming $C_p(X)$ is K-analytic.

Theorem 3.15 (Tkachuk, [71, 3.7 Theorem]). *$C_p(X)$ has a resolution of compact sets that swallows the compact sets if and only if X is countable and discrete.*

Proof. Assume $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution for $C_p(X)$ of compact sets that swallows the compact sets of $C_p(X)$. We claim that compact subsets of X are finite. Otherwise there exists an infinite compact set K in X. Since, according to Theorem 3.10, $C_p(X)$ is K-analytic, it turns out that $C_p(C_p(X))$ is angelic [24, Theorem 78]. As K is embedded in $C_p(C_p(X))$, it must be a Fréchet-Urysohn compact, so there is a non trivial sequence $\{x_n\}_{n=1}^{\infty}$ that converges to some $x \in K$. Let $S = \{x_n : n \in \mathbb{N}\} \cup \{x\}$, so that S is a countable compact set, hence metrizable. Thus, there is a linear extender map $\varphi : C_p(S) \to C_p(X)$, i. e., such that $\varphi\left(f|_S\right) = f$ for every $f \in C(X)$, which embeds $C_p(S)$ into a closed linear subspace of $C_p(X)$, [5, Proposition 4.1]. Therefore the metrizable space $C_p(S)$ also has a resolution of compact sets that swallows the compact sets in $C_p(S)$. According to Christensen's theorem [24, Theorem 94] this means that $C_p(S)$ is a Polish space. Hence, [1, I.3.3 Corollary] ensures that the compact set S is discrete, hence finite. This contradiction ensures that the compact sets in X are finite.

Since $C_p(X)$ is K-analytic, Lemma 3.1 asserts that υX is a Lindelöf Σ-space. But a Lindelöf Σ-space with finite compact sets is countable [1, IV.6.15 Proposition], so X is countable. On the other hand, if Q is a compact set in $C_p(X)$ there is $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $Q \subseteq A_\gamma$. Hence, $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution of compact sets for the metrizable space $C_p(X)$ that swallows the compact sets of $C_p(X)$. So, again $C_p(X)$ is a Polish space by Christensen's theorem, and one more time [1, I.3.3 Corollary] asserts that X is discrete.

For the converse, note that $C_p(X)$ coincides with $\mathbb{R}^{\mathbb{N}}$ whenever X is countable and discrete. Then $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $A_\alpha = \{x \in \mathbb{R}^{\mathbb{N}} : |x_n| \leq \alpha_n\}$ is a resolution for $C_p(X) = \mathbb{R}^{\mathbb{N}}$ consisting of compact sets that swallows the compact sets in $\mathbb{R}^{\mathbb{N}}$. □

Theorem 3.16 (Ferrando-Gabriyelyan-Kąkol [28, Theorem 3.3]). *$C_p(X)$ has a resolution of pointwise bounded sets that swallows the pointwise bounded sets if and only if X is countable. In other words, $C_p(X)$ has a fundamental resolution of pointwise bounded sets if and only if X is countable.*

Proof. (Sketch) If $C_p(X)$ admits a fundamental resolution of pointwise bounded sets one can fix [28, Theorem 3.3] a countable family of closed sets (some of them may be empty) $\mathcal{K} = \{K_n(\alpha) : n \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in X enjoying the properties:

1. $K_n(\alpha) \subseteq K_{n+1}(\alpha)$ for every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$.
2. $K_n(\alpha) \supseteq K_n(\beta)$ for every $n \in \mathbb{N}$ whenever $\alpha \leq \beta$.
3. $\bigcup_{n\in\mathbb{N}} K_n(\alpha) = X$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$.
4. For every increasing closed covering $\{V_n : n \in \mathbb{N}\}$ of X there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $K_n(\gamma) \subseteq V_n$ for all $n \in \mathbb{N}$.

Then it turns out that the family $\mathcal{N} := \{N_{mn}(\alpha) : m, n \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where

$$N_{mn}(\alpha) := \left\{ f \in C(X) : |f(x)| \leq \frac{1}{m} \ \ \forall x \in K_n(\alpha) \right\}$$

and $N_{mn}(\alpha) := \{0\}$ if $K_n(\alpha)$ is empty, is a countable cs^*-network at the origin in $C_p(X)$ (see [28, Proposition 3.2] or [24, Claim 108] for details). So, according to [65, Theorem 2.3], X must be countable. □

Recall that a locally convex space E is a *quasi-(LB)-space* if E has a resolution consisting of *Banach disks*, i. e., of absolutely convex bounded sets D whose linear span E_D is a Banach space when equipped with the Minkowski functional of D as a norm.

Theorem 3.17 (Valdivia, [77]). *If E is a quasi-(LB)-space, there exists a resolution for E consisting of Banach disks that swallows the Banach disks of E.*

Proof. (Sketch) Let $\{D_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution for E consisting of Banach disks. For $(n_1, \ldots, n_k) \in \mathbb{N}^{\mathbb{N}}$ define the absolutely convex set

$$C_{n_1,\ldots,n_k} = \bigcup \{D_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}, \ \alpha(i) = n_i, \ 1 \leq i \leq k\}.$$

If $\alpha \in \mathbb{N}^{\mathbb{N}}$ and U a neighborhood of the origin in E it can be easily seen that there exists $k(\alpha, U) \in \mathbb{N}$ such that $C_{\alpha(1),\ldots,\alpha(k)} \subseteq kU$. So, if we set $F_{\alpha(1),\ldots,\alpha(k)} := \operatorname{span}\left(C_{\alpha(1),\ldots,\alpha(k)}\right)$ for every $k \in \mathbb{N}$ and $F_\alpha := \bigcap \left\{F_{\alpha(1),\ldots,\alpha(k)} : k \in \mathbb{N}\right\}$, the sequence

$$\left\{F_\alpha \cap k^{-1} C_{\alpha(1),\ldots,\alpha(k)} : k \in \mathbb{N}\right\}$$

is a base of absolutely convex neighborhoods of the origin in the linear subspace F_α of a locally convex topology τ_α stronger than the relative topology of E. In fact, it turns out that $\{(F_\alpha, \tau_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a family Fréchet spaces [77, Proposition 21] which covers E. Now, for $\alpha \in \mathbb{N}^{\mathbb{N}}$ set $\overline{\alpha}(i) = \alpha(2i - 1)$ for each $i \in \mathbb{N}$ and define

$$Q_\alpha = \bigcap_{k=1}^{\infty} \alpha(2k) \cdot \left(F_{\overline{\alpha}} \cap C_{\overline{\alpha}(1),\ldots,\overline{\alpha}(k)}\right)$$

The family $\{Q_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is clearly a resolution for E, and consists of Banach disks. It remains to prove that this family swallows the Banach disks of E. In order to establish this statement, choose a Banach disk D in E and consider the Banach space E_D. Then consider the canonical inclusion $J : E_D \to E$ and put $U_{n_1,\ldots,n_k} := J^{-1}(C_{n_1,\ldots,n_k})$. As $E_D = \bigcup \{U_{n_1} : n_1 \in \mathbb{N}\}$ and $U_{n_1,\ldots,n_k} = \bigcup \{U_{n_1,\ldots,n_k,n_{k+1}} : n_{k+1} \in \mathbb{N}\}$ for each $k \in \mathbb{N}$, there is $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\overline{U_{\beta(1),\ldots,\beta(k)}}$, closure in E_D, is a neighborhood of the origin in E_D for each $k \in \mathbb{N}$. So, using the fact that $\operatorname{Int}\left(\overline{U_{\beta(1),\ldots,\beta(k)}}\right) \subseteq U_{\beta(1),\ldots,\beta(k)}$ for each $k \in \mathbb{N}$ (see [77] or [49, Proposition 3.21] for details), if $x \in E_D$ and $k \in \mathbb{N}$ there is $\lambda > 0$ such that $\lambda x \in U_{\beta(1),\ldots,\beta(k)}$, which implies that $J(x) \in F_{\beta(1),\ldots,\beta(k)}$. This shows that $J(E_D) \subseteq F_\beta$. So, by the closed graph theorem J is a continuous linear map from E_D into F_β. Hence, if we choose a sequence $\{m_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that

$$D \subseteq m_k \cdot \left(F_\beta \cap C_{\beta(1),\ldots,\beta(k)}\right)$$

for every $k \in \mathbb{N}$, setting $\gamma(2k) = m_k$ and $\gamma(2k-1) = \beta(k)$ for each $k \in \mathbb{N}$, it follows that $D \subseteq Q_\gamma$. □

Theorem 3.18 (Ferrando-Gabriyelyan-Kąkol [28, Proposition 3.6]). *Let X be a P-space. $C_p(X)$ has a resolution of pointwise bounded sets if and only if X is countable and discrete.*

Proof. If X is a P-space then $C_p(X)$ is locally complete [34, Theorem 1.1], i. e., each pointwise bounded set is contained in a Banach disk. So, according to Theorem 3.17 there exists a resolution for $C_p(X)$ consisting of Banach disks that swallows the pointwise bounded sets in $C_p(X)$. Hence, X is countable by Theorem 3.16. But every countable P-space is discrete. □

Alternatively, one may use the fact that $C_p(X)$ is a Baire space (note that $C_p(X)$ is pseudocomplete [72, Section 1.5, p. 46] whenever X is a P-space and use [72, Problem 464]). Then apply Theorem 3.13 to conclude that X must be countable, hence discrete.

Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ in a locally convex space E is called *local null* or *Mackey convergent* to zero [52, 28.3] if there is a closed disk B in E such that $x_n \to \mathbf{0}$ in the normed space E_B. Each local null sequence in E is a null sequence.

Theorem 3.19 (Ferrando, [25, Theorem 12]). *$C_p(X)$ admits a resolution of convex compact sets that swallows the local null sequences in $C_p(X)$ if and only if X is countable and discrete.*

Proof. We may assume that $C_p(X)$ admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of absolutely convex compact sets swallowing the local null sequences in $C_p(X)$. If $T : C_p(\upsilon X) \to C_p(X)$ denotes the restriction map $Tg = g\big|_X$ we proceed as in [49, Proposition 9.14] to show that the family $\mathcal{A} = \{T^{-1}(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution for $C_p(\upsilon X)$ consisting of (absolutely convex) compact sets, with the additional benefit that $\mathcal{A}$ swallows the local null sequences in $C_p(\upsilon X)$. So, we may assume without loss of generality that X is realcompact or, equivalently, that $C_p(X)$ is bornological [8]. Hence, we denote as above by $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ a resolution for $C_p(X)$, with X realcompact, consisting of absolutely convex compact sets that swallows the local null sequences in $C_p(X)$.

Let $\mathcal{M}$ denote the family of all local null sequences in $C_p(X)$. Since $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows the members of $\mathcal{M}$, the Mackey* topology $\mu(L(X), C(X))$ of $L(X)$ is stronger than the topology τ_{c_0} on $L(X)$ of the uniform convergence on the local null sequences of $C_p(X)$. As in addition $\sigma(L(X), C(X)) \leq \tau_{c_0}$, we conclude that $(L(X), \tau_{c_0})' = C(X)$. Moreover, since we are assuming that $C_p(X)$ is bornological, its τ_{c_0}-dual $(L(X), \tau_{c_0})$ is complete by [52, 28.5.(1)].

We claim that every compact set in X is finite. Indeed, if K is a compact set in X, the homeomorphic copy $\delta(K)$ of K in $L_p(X)$ is compact, i. e., $\delta(K)$ is a $\sigma(L(X), C(X))$-compact set in $L(X)$. So, the completeness of $(L(X), \tau_{c_0})$, together with Krein's theorem and the fact that τ_{c_0} is a locally convex topology of the dual pair $\langle L(X), C(X)\rangle$, ensures that the weak* closure $Q = \overline{\operatorname{abx}(\delta(K))}$ in $L(X)$, where $\operatorname{abx}(\delta(K))$ stands for the absolutely convex hull of $\delta(K)$, is a compact set in $L_p(X)$, hence a strongly bounded set. Since $C_p(X)$ is quasi-barrelled [47, 11.7.3 Corollary], the strongly bounded sets in $L(X)$ are finite-dimensional. Therefore the set $\delta(K)$, as a linearly independent system of vectors in $L(X)$, must be finite. Thus K is finite as well.

Since $\upsilon X = X$ is a Lindelöf Σ-space by Lemma 3.1 and as we know each Lindelöf Σ-space with finite compact sets is countable [1, IV.6.15 Proposition], X is countable. So $C_p(X)$ is a metrizable space. But in a metrizable locally convex space, the local null sequences and the null sequences are the same [52, 28.3.(1) c)]. Furthermore, if M is a compact set in the metrizable space $C_p(X)$, then M lies in the closed absolutely convex cover of a null sequence $\{f_n\}_{n=1}^{\infty}$, [52, 21.10.(3)]. So, if $\{f_n\}_{n=1}^{\infty} \subseteq A_\gamma$, thanks to the fact that A_γ is a closed absolutely convex set, it turns out that $M \subseteq A_\gamma$. Therefore $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution for $C_p(X)$ that swallows the compact sets of $C_p(X)$. So, $C_p(X)$ is a Polish space by Christensen's theorem [24, Theorem 94]. But then [1, I.3.3 Corollary] asserts that X is discrete. The converse is obvious. □

Theorem 3.20 (Ferrando, [25, Theorem 16]). *$C_p(X)$ has a resolution of absolutely convex pointwise bounded sequentially complete sets that swallows the null sequences if and only if X is countable and discrete.*

Proof. It can be readily seen that there is no loss of generality if we assume X to be realcompact. If $C_p(X)$ has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of the stated characteristics and $\{f_n\}_{n=1}^{\infty}$ is a null sequence in $C_p(X)$, there is

$\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $f_n \in A_\gamma$ for every $n \in \mathbb{N}$. Since $\sum_{i=1}^{n} \xi_i f_i \in A_\gamma$ for every $\xi \in \ell_1$ with $\|\xi\|_1 \leq 1$ and A_γ is sequentially complete, it follows that $\sum_{i=1}^{\infty} \xi_i f_i \in A_\gamma$ for every $\xi \in \ell_1$ with $\|\xi\|_1 \leq 1$. So, the Banach disk

$$Q := \left\{ \sum_{i=1}^{\infty} \xi_i f_i : \xi \in \ell_1, \|\xi\|_1 \leq 1 \right\}$$

is contained in A_γ. Now, it can be proved as in [52, 20.10.(6)] that $Q = \{f_n : n \in \mathbb{N}\}^{00}$, the absolute bipolar of the null sequence $\{f_n : n \in \mathbb{N}\}$. Since each local null sequence is a null sequence, the dual of $(L(X), \tau_{c_0})$ is $C(X)$, so $\sigma(L(X), C(X)) \leq \tau_{c_0} \leq \mu(L(X), C(X))$. As $C_p(X)$ is bornological, the space $L(X)$ is $\mu(L(X), C(X))$-complete. So, proceeding as in the proof of Theorem 3.19, with the help of Krein's theorem we establish that each compact set in X is finite. Now, using the fact that the resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consists of pointwise bounded sets, Lemma 3.1 asserts that X is a Lindelöf Σ-space. Thus X must be countable, [1, IV.6.15 Proposition], so $C_p(X)$ is metrizable.

If M is a compact set in the metrizable space $C_p(X)$, as mentioned above M lies in the closed absolutely convex cover of a null sequence $\{f_n\}_{n=1}^{\infty}$. So, if $\{f_n\} \subseteq A_\gamma$ then $M \subseteq A_\gamma$. Thus $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution for $C_p(X)$ that swallows the compact sets of $C_p(X)$. Since each set A_α is precompact in $C_p(X)$ and sequentially complete, the metrizability of $C_p(X)$ ensures that A_α is compact in $C_p(X)$. Hence $C_p(X)$ is a Polish space by Christensen's theorem. Thus X is discrete. The converse is clear, since each (absolutely convex) compact set in $\mathbb{R}^{\mathbb{N}}$ is pointwise bounded and sequentially complete. □

Another result of this type, which we state without proof is the following.

Theorem 3.21 (Ferrando, [25, Theorem 33]). *Let X be first countable. $C_p(X)$ has a resolution of pointwise bounded sets that swallows the Cauchy sequences if and only if X is countable.*

4. Uncountable coverings for $C_k(X)$

Theorem 4.1 (Ferrando-Moll, [35, Corollary 5]). *The space $C_k(X)$ has a resolution consisting of compact sets if and only if it is K-analytic.*

Proof. If $C_k(X)$ has a resolution consisting of compact sets, so does $C_p(X)$. So, Lemma 3.1 and Theorem 3.3 ensure that υX is a Lindelöf Σ-space and $C_p(X)$ is angelic. Therefore $C_k(X)$ is angelic as well [36, 3.3 Theorem]. Since $C_k(X)$ is a quasi-Suslin space, necessarily $C_k(X)$ must be K-analytic [11]. □

Theorem 4.2 (Gabriyelyan-Kąkol [37, Corollary 2.10]). *Let X be metrizable. $C_k(X)$ has a resolution of compact sets that swallows the compact sets if and only if X is σ-compact.*

Proof. If $C_k(X)$ has a resolution of compact sets, $C_p(X)$ has a resolution of pointwise bounded sets. So, Corollary 3.8 assures that X is σ-compact. Conversely, if $\{K_m : m \in \mathbb{N}\}$ is an increasing sequence of compact sets in X covering X then $\Delta_m = \{(x, x) : x \in K_m\}$ is compact in the metric space $(X \times X, d)$. Hence, the sequence $\{U_{m,n} : n \in \mathbb{N}\}$ where

$$U_{m,n} = \left\{(x, y) \in X \times X : d((x, y), \Delta_m) < n^{-1}\right\}$$

is a basis of the system of neighborhoods of Δ_m. Let us encode in each $\alpha \in \mathbb{N}^{\mathbb{N}}$ a whole sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements of $\mathbb{N}^{\mathbb{N}}$ by considering a bidimensional array whose i^{th} file is formed by coordinates $(\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(n), \ldots)$ of α_i and defining α by setting $\alpha(1) = \alpha_1(1)$, $\alpha(2) = \alpha_1(2)$, $\alpha(3) = \alpha_2(1)$, $\alpha(4) = \alpha_1(3)$, $\alpha(5) = \alpha_2(2)$, $\alpha(6) = \alpha_3(1)$, $\alpha(7) = \alpha_1(4)$, … and so on. Conversely, given $\alpha \in \mathbb{N}^{\mathbb{N}}$ we may extract a sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{N}^{\mathbb{N}}$ from α as indicated above. Then let A_α be the absolutely convex set

$$\left\{ f \in C(X) : \sup_{(x,y) \in U_{m,\alpha_m(n)}} \left| f(x) - f(y) \right| \leq \frac{1}{n}, \sup_{x \in K_m} \left| f(x) \right| \leq \alpha_m(1) \ \forall m, n \in \mathbb{N} \right\}.$$

Let $x \in X$ and $\epsilon > 0$ be given. Take $m \in \mathbb{N}$ such that $x \in K_m$ and $1/n < \epsilon$. Setting $U_{m,n}(x) := \{y \in X : (x, y) \in U_{m,n}\}$, each $f \in A_\alpha$ satisfies

$$\sup_{y \in U_{m,\alpha_m(n)}(x)} \left| f(x) - f(y) \right| \leq \sup_{(z,y) \in U_{m,\alpha_m(n)}} \left| f(z) - f(y) \right| \leq n^{-1} < \epsilon.$$

As $U_{m,\alpha_m(n)}(x)$ is a neighborhood of x, this means that A_α is equicontinuous at x. So all sets A_α are equicontinuous. In addition, since $\sup_{f \in A_\alpha} \left| f(z) \right| \leq \alpha_m(1)$ if $z \in K_m$, we see that A_α is pointwise bounded and closed. Hence A_α is a compact set in $C_k(X)$.

On the other hand, if $\mathcal{K}$ is a compact set in $C_k(X)$, the fact that X is a $k_{\mathbb{R}}$-space guarantees that $\mathcal{K}$ is equicontinuous (Ascoli's theorem). Since $\mathcal{K}$ is equicontinuous at each $x \in K_m$, for each $n \in \mathbb{N}$ there is $\epsilon(m, n, x) > 0$ such that

$$\sup_{y \in B(x, \epsilon(m,n,x))} \left| f(x) - f(y) \right| \leq \frac{1}{2n} \tag{1}$$

for all $f \in \mathcal{K}$, where $B(x, \epsilon)$ stands for the open ball of center at x and radius $\epsilon > 0$.

Setting $U = \bigcup_{z \in K_m} B(z, \epsilon(m, n, z)) \times B(z, \epsilon(m, n, z))$, if $(x, y) \in U$ there is $z \in K_m$ such that $x, y \in B(z, \epsilon(m, n, z))$, so $\left| f(x) - f(y) \right| \leq \left| f(x) - f(z) \right| + \left| f(z) - f(y) \right| < n^{-1}$ for all $f \in \mathcal{K}$. As $\Delta_m \subseteq U$ there is $r(m, n) \in \mathbb{N}$ with $\Delta_m \subseteq U_{m,r(m,n)} \subseteq U$. Thus

$$\sup_{(x,y) \in U_{m,r(m,n)}} \left| f(x) - f(y) \right| \leq \frac{1}{n}.$$

On the other hand, the fact that $\mathcal{K}$ is a compact set for the compact-open topology ensures that for each $m \in \mathbb{N}$ there is $k_m \in \mathbb{N}$ such that $\sup_{f \in \mathcal{K}} \sup_{x \in K_m} \left| f(x) \right| \leq k_m$. Hence, setting α such that $\alpha_m(n) = r(m, n)$, we may assume that $\alpha_m(1) \geq k_m$. All this says that $\mathcal{K} \subseteq A_\alpha$. As $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, the family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is as stated. □

Corollary 4.3 (Ferrando [23, Proposition 3]). *Let X be a metrizable space. $C_k(X)$ has a fundamental bounded resolution if and only if X is σ-compact.*

Proof. If X is σ-compact, Theorem 4.2 ensures that $C_k(X)$ has a resolution consisting of compact sets that swallows the compact sets. So, $C_k(X)$ has a bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of closed absolutely convex bounded sets. As X is a $k_{\mathbb{R}}$-space, $C_k(X)$ is complete and consequently each A_α is a Banach disk. So, Theorem 3.17 provides a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for $C_k(X)$ consisting of Banach disks that swallows the Banach disks, hence the bounded sets in $C_k(X)$. Thus, $C_k(X)$ has a fundamental bounded resolution. The converse comes from Corollary 3.8. □

Theorem 4.4 (Christensen [15, Theorem 3.7]). *Let X be a separable metric space. $C_k(X)$ is analytic if and only if X is σ-compact.*

Proof. If $C_k(X)$ is analytic then $C_p(X)$ is analytic as well, so Calbrix's theorem ensures that X is σ-compact. If X is σ-compact then $C_k(X)$ has a resolution of compact sets by Theorem 4.2. Hence $C_k(X)$ is K-analytic by Theorem 4.1. As X is a separable metric space, it is a cosmic space, and so is $C_p(X)$. So, $C_p(X)$ being K-analytic and cosmic is analytic. Hence $C_p(X)$ must be submetrizable by the second statement of [24, Theorem 85] (see [66, Proposition 6.3]). Consequently, $C_k(X)$ is K-analytic and submetrizable, hence analytic by the first statement of [24, Theorem 85]. □

If $\mathcal{N}$ is a uniformity for a (nonempty) set X, we denote by $\tau_{\mathcal{N}}$ the uniform topology defined by $\mathcal{N}$. A base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of $\mathcal{N}$ is called a $\mathfrak{G}$*-base* if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. There is no loss of generality by assuming that each U_α is a symmetric vicinity. On the other hand, if $\{\mathcal{U}_\lambda : \lambda \in \Lambda\}$ is the family of all admissible uniformities for a completely regular space (X, τ), the smallest uniformity $\mathcal{U}_{\lambda_0}$ that makes all τ-continuous functions $f : X \to \mathbb{R}$ uniformly continuous, is called the *Nachbin uniform structure* of X, [61].

Theorem 4.5 (Ferrando, [21, Theorem 1]). *$C_k(X)$ has a resolution consisting of equicontinuous sets if and only if there exists an admissible uniformity for X, larger than or equal to the Nachbin uniformity, with a $\mathfrak{G}$-base.*

Proof. Assume $\mathcal{N}$ is a uniformity for X which contains the Nachbin uniform structure and let $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base of $\mathcal{N}$. If $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $\mathbb{N}^\mathbb{N}$, encode $\{\alpha_n\}_{n=1}^\infty$ in α as indicated in the proof of Theorem 4.2 and define

$$P_\alpha = \left\{ f \in C(X) : \sup_{(x,y)\in U_{\alpha_n}} \left| f(x) - f(y) \right| \leq \frac{1}{n} \ \forall n \in \mathbb{N} \right\}.$$

We claim that $\{P_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a resolution for $C_k(X)$ consisting of equicontinuous sets. In fact, since if $\alpha \leq \beta$ then $\alpha_n \leq \beta_n$ for every $n \in \mathbb{N}$, clearly $P_\alpha \subseteq P_\beta$. On the other hand, if $f \in C(X)$, since $\mathcal{N}$ is larger than the Nachbin uniformity, f is $\mathcal{N}$-uniformly continuous on X. Bearing in mind that $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base of $\mathcal{N}$, for each $n \in \mathbb{N}$ there exists $\alpha_n \in \mathbb{N}^\mathbb{N}$ such that $\left| f(x) - f(y) \right| \leq 1/n$ whenever $(x, y) \in U_{\alpha_n}$, which shows that $f \in P_\alpha$ for α defined as above. Finally, let us see that each set P_α is equicontinuous. Indeed, given $\epsilon > 0$ take $n \in \mathbb{N}$ such that $1/n < \epsilon$. According to the definition of P_α there is $\alpha_n \in \mathbb{N}^\mathbb{N}$, which we extract from α as explained earlier, such that $\left| f(x) - f(y) \right| < \epsilon$ whenever $(x, y) \in U_{\alpha_n}$ and this happens for every $f \in P_\alpha$, which shows that P_α is uniformly equicontinuous, hence equicontinuous.

For the converse, suppose that $\{P_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a resolution of $C_k(X)$ consisting of equicontinuous sets. For each $\alpha \in \mathbb{N}^\mathbb{N}$ define

$$V_\alpha = \{(x, y) \in X \times X : \sup_{f\in P_\alpha} \left| f(x) - f(y) \right| < \alpha(1)^{-1}\}.$$

If $\alpha \leq \beta$ then $P_\alpha \subseteq P_\beta$, which implies that $V_\beta \subseteq V_\alpha$. Let us see that $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a base of some uniformity $\mathcal{N}$ for X. First observe that the diagonal $\Delta(X) = \{(x, x) : x \in X\}$ is contained in each V_α, so no V_α is empty. On the other hand, clearly $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a filter-base with $V_\alpha^{-1} = V_\alpha$. In addition, if $\beta \in \mathbb{N}^\mathbb{N}$ satisfies that $\beta \geq \alpha$ with $\beta(1) \geq 2\alpha(1)$ we claim that $V_\beta \circ V_\beta \subseteq V_\alpha$. Indeed, if $(x, y) \in V_\beta \circ V_\beta$ there is $z \in X$ with $(x, z), (z, y) \in V_\beta$. Hence $\left| f(x) - f(z) \right| < \beta(1)^{-1}$ and $\left| f(z) - f(y) \right| < \beta(1)^{-1}$ for every $f \in P_\beta$. So, $\left| f(x) - f(y) \right| < 2\beta(1)^{-1} \leq \alpha(1)^{-1}$ for all $f \in P_\alpha \subseteq P_\beta$, which shows that $(x, y) \in V_\alpha$.

Let us check that $\mathcal{N}$ is an admissible uniformity for X, i. e., that $\tau_\mathcal{N}$ coincides with the original topology of X. Since X is completely regular, it suffices to show that X and $(X, \tau_\mathcal{N})$ have the same continuous functions. Take $f \in C(X)$, pick an arbitrary point $x_0 \in X$ and choose $\epsilon > 0$. Then select $\alpha \in \mathbb{N}^\mathbb{N}$ such that $f \in P_\alpha$ and $\alpha(1)^{-1} < \epsilon$. Clearly

$$V_\alpha(x_0) = \{y \in X : (x_0, y) \in V_\alpha\}$$

is a $\tau_\mathcal{N}$-neighborhood of x_0, and since $\left| f(x) - f(y) \right| < \alpha(1)^{-1} < \epsilon$ for every $(x, y) \in V_\alpha$, we have in particular that $\left| f(x_0) - f(y) \right| < \epsilon$ for all $y \in V_\alpha(x_0)$. This shows that f is continuous at x_0 under $\tau_\mathcal{N}$. Assume conversely that $f \in C(X, \tau_\mathcal{N})$ and fix $x_0 \in X$ and $\epsilon > 0$. Then there is $\alpha \in \mathbb{N}^\mathbb{N}$ with $\left| f(x_0) - f(y) \right| < \epsilon$ for every $y \in V_\alpha(x_0)$. But, since P_α is equicontinuous at x_0, there exists a neighborhood V of x_0 of the original topology of X such that $\sup_{h\in P_\alpha} \left| h(y) - h(x_0) \right| < \alpha(1)^{-1}$ for every $y \in V$. Hence if $x \in V$ then $\sup_{h\in P_\alpha} |h(x) - h(x_0)| < \alpha(1)^{-1}$, which according to the definition of V_α means that $x \in V_\alpha(x_0)$. This shows that $V \subseteq V_\alpha(x_0)$ and thus $\left| f(x_0) - f(y) \right| < \epsilon$ for all $y \in V$. So f is continuous at x_0 under the original topology of X and $f \in C(X)$.

Let us finally check that the uniformity $\mathcal{N}$ generated by the base $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is larger than the Nachbin uniformity. We have to prove that every real-valued continuous function on X is $\mathcal{N}$-uniformly continuous. Now, given $f \in C(X)$ and $\epsilon > 0$, taking advantage of the fact that $\{P_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a resolution of $C(X)$, we can choose $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\gamma(1)^{-1} < \epsilon$ and $f \in P_\gamma$. Consequently, for each $(x, y) \in V_\gamma$ it happens that $\left| f(x) - f(y) \right| < \gamma(1)^{-1} < \epsilon$, which shows that f is $\mathcal{N}$-uniformly continuous, as stated. □

Corollary 4.6. *Let X be a $k_\mathbb{R}$-space. If $C_k(X)$ is K-analytic then there exists an admissible uniformity for X, larger than or equal to the Nachbin uniformity, with a $\mathfrak{G}$-base.*

Theorem 4.7 (Ferrando-Gabriyelyan-Kąkol, [27, Theorem 1.8]). *$C_k(X)$ has a resolution consisting of weakly compact sets that swallows the weakly compact sets if and only if X is countable and discrete.*

Proof. First we claim that if $C_k(X)$ has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ consisting of weakly compact sets that swallows the weakly compact sets in $C_k(X)$, each compact set in X is finite. As $C_p(X)$ admits a resolution of compact sets, it is K-analytic by Theorem 3.10, so $C_p\left(C_p(X)\right)$ is angelic by Lemma 3.1. Hence, each compact set of $X \hookrightarrow C_p\left(C_p(X)\right)$ is Fréchet-Urysohn. If there exists an infinite compact set K in X, then K contains an infinite convergent sequence that, together with its limit, is homeomorphic to a metrizable compact subset Q of βX. Thus, there is a continuous linear extender map $\varphi : C_p(Q) \to C_p(\beta X)$, [5]. If $S : C_p(\beta X) \to C_p(X)$ is the restriction map $Sg = g\big|_X$, the mapping $\psi = S \circ \varphi$ is a continuous linear extender, i. e., $\psi(f)\big|_Q = f$ for every $f \in C(Q)$. This ensures that the linear map $\psi : C(Q) \to C_k(X)$ (weak), where $C_k(X)$ (weak) stands for the space $C_k(X)$ equipped with its weak topology, has closed graph. Since $C(Q)$ is a Banach space and $C_k(X)$ (weak) has a resolution of compact sets, the closed graph theorem [31, Theorem 1] ensures that $\psi : C(Q) \to C_k(X)$ (weak) is continuous, so weakly continuous.

A routine procedure shows that the family $\{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a resolution for the Banach space $C(Q)$ consisting of weakly compact sets. If P is a compact set under the weak topology of $C_k(Q)$, then $\psi(P)$ is a compact set in $C_k(X)$ (weak). Hence, there is a $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\psi(P) \subseteq A_\gamma$, so that $P \subseteq \psi^{-1}\left(A_\gamma\right)$. This means that $\{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows the compact sets of $C(Q)$ (weak). But it is shown in [55] that for compact Q, if the Banach space $C(Q)$ has a resolution of weakly compact sets that swallows the weakly compact sets, then Q is finite. Thus Q must be a finite set, a contradiction

Finally, since each compact set in X is finite, one has $C_k(X) = C_k(X)$ (weak) $= C_p(X)$. So X must be countable and discrete by Theorem 3.15. □

A Fréchet space E is called a *Strongly Weakly Countably Generated* (briefly a SWCG) space if every bounded set in $(E', \mu(E', E))$ is metrizable. Equivalently, E is a SWCG space if given a base of closed absolutely convex neighborhoods of zero $\{U_n : n \in \mathbb{N}\}$ with $2U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$ there exists an absolutely convex weakly compact set $K \subseteq E$ such that for every weakly compact (absolutely convex) set $L \subseteq E$ and every $n \in \mathbb{N}$ there is $\alpha(n) \in \mathbb{N}$ with $L \subseteq \alpha(n) K + U_n$ [30, Theorem 9]. A Fréchet space E is called *Strongly Weakly K-Analytic* (briefly SWKA) space if $(E, \sigma(E, E'))$ admits a compact resolution that swallows the $\sigma(E, E')$-compact sets.

If E is a Fréchet space with a base of closed absolutely convex neighborhoods of zero $\{U_n : n \in \mathbb{N}\}$ such that $2U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$, a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ for E is called *weakly compactly generated* if there exists an absolutely convex weakly compact set K such that

$$A_\alpha = \bigcap_{n=1}^{\infty} (\alpha(n) K + U_n)$$

for every $\alpha \in \mathbb{N}^\mathbb{N}$. Clearly $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, and the condition imposed to the base implies that each A_α is closed. Hence $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a weakly compact resolution for E, as follows from [30, Claim 6].

Theorem 4.8. *A Fréchet space E is SWCG if and only if E has a weakly compactly generated resolution that swallows the weakly compact sets.*

Proof. Assume that E is a SWCG space, and let $\{U_n : n \in \mathbb{N}\}$ be a base of closed absolutely convex neighborhoods of the origin such that $2U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$ set $A_\alpha := \bigcap_{n=1}^{\infty}(\alpha(n) K + U_n)$, where K is the absolute convex weakly compact set mentioned after the definition of SWCG space. Clearly $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a weakly compactly generated resolution for E. If $L \subseteq E$ is a weakly compact set in E, for each $n \in \mathbb{N}$ there exists $\gamma(n) \in \mathbb{N}$ such that $L \subseteq \gamma(n) K + U_n$, so that $L \subseteq A_\gamma$. Hence $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows the weakly compact sets of E.

Assume conversely that E contains a weakly compactly generated resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ that swallows the weakly compact sets. Then there exists a weakly compact absolutely convex set Q such that $A_\alpha = \bigcap_{n=1}^{\infty} (\alpha(n) Q + U_n)$ for every $\alpha \in \mathbb{N}^\mathbb{N}$. If L is any weakly compact set in E there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $L \subseteq A_\gamma$, hence for each $n \in \mathbb{N}$ one gets $L \subseteq \gamma(n) Q + U_n$. So E is a SWCG space. □

Theorem 4.9 (Ferrando-Kąkol, [30, Theorem 22]). *If $C_k(X)$ is a Fréchet space, the following statements are equivalent*

1. *$C_k(X)$ is a SWCG space.*
2. *$C_k(X)$ is a SWKA space.*
3. *X is countable and discrete.*

Proof. Clearly 1 ⇒ 2. Equivalence 2 ⇔ 3 is consequence of Theorem 4.7. If X is countable and discrete then $C_k(X) = \mathbb{R}^X$ is reflexive, so 3 ⇒ 1. □

5. Closure-preserving coverings for $C_p(X)$

A closure-preserving covering of $C_p(X)$ is a generalization of a locally finite covering. A covering $\mathcal{F}$ of a space X is called *closure-preserving* if

$$\overline{\bigcup \{F : F \in \mathcal{G}\}} = \bigcup \left\{\overline{F} : F \in \mathcal{G}\right\}$$

for any $\mathcal{G} \subseteq \mathcal{F}$.

Theorem 5.1 (Guerrero, [40, Corollary 2.7]). *$C_p(X)$ admits a closure-preserving covering by closed σ-countably compact sets if and only if X is finite.*

Proof. First let us suppose that $\mathcal{F}$ is a closure-preserving covering of $C_p(X)$ by closed σ-compact subspaces. Note that X must be pseudocompact. Otherwise $C_p(X)$ has a closed homeomorphic copy of $\mathbb{R}^{\mathbb{N}}$ and hence $C_p(\mathbb{N})$ has a closure-preserving covering $\mathcal{G}$ by closed σ-compact subspaces. As the space $C_p(\mathbb{N})$ is separable, there exists a countable subfamily $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \overline{\bigcup \mathcal{G}_0} = C_p(\mathbb{N})$, which means that $C_p(\mathbb{N})$ is covered by a countable family of compact sets. Thus $\mathbb{N}$ should be finite by Velichko's theorem, a contradiction.

If $f \in C(X)$ we claim that $f(X)$ is finite. Indeed, if $Y := f(X)$ since Y is a separable metric space the space $C_p(Y)$ is separable. On the other hand, as X is pseudocompact and Y is second countable f is an $\mathbb{R}$-quotient map [72, S.154, Fact 3], so the pullback $f^* : C_p(Y) \to C_p(X)$ defined by $f^*(g) = g \circ f$ embeds $C_p(Y)$ in $C_p(X)$ as a closed subspace [1, 0.4.10 Proposition]. Therefore, $C_p(Y)$ is covered by a closure-preserving family $\mathcal{M}$ of closed σ-compact subspaces and there exists a countable subfamily $\mathcal{N}$ of $\mathcal{M}$ such that $\bigcup \mathcal{N} = \overline{\bigcup \mathcal{N}} = C_p(Y)$. Again Velichko's theorem implies that Y must be finite.

Since $f(X)$ is finite for every $f \in C(X)$, the space X must be finite. If not there is a countable discrete subspace $D = \{x_n : n \in \mathbb{N}\}$ in X and a countable family of open sets $\{U_n : n \in \mathbb{N}\}$ such that $U_n \cap D = \{x_n\}$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $i \neq j$. So, for each $n \in \mathbb{N}$ there is $f_n \in C(X)$ with $0 \leq f_n \leq 1$ such that $f_n(x_n) = 1$ and $f_n(x) = 0$ if $x \in X \setminus U_n$. Then clearly $f = \sum_{n=1}^{\infty} f_n \in C(X)$ but $f(X) \supseteq D$, which is infinite, a contradiction.

If the closure-preserving covering consists of closed σ-countably compact sets instead of closed σ-compact sets, we get the same conclusion by using the Tkachuk-Shakhmatov theorem instead of Velichko's theorem.

Conversely, $\mathbb{R}^n$ can always be covered by a countable family of compact balls. □

Theorem 5.2 (Guerrero, [40, Corollary 2.8]). *If $C_p(X)$ admits a closure-preserving covering by countably compact sets then X is finite.*

Proof. Let $\mathcal{F}$ be a closure-preserving cover of $C_p(X)$ by countably compact sets. If X is not pseudocompact, there is a sequence $\{F_n : n \in \mathbb{N}\}$ in $\mathcal{F}$ with $\bigcup_{n=1}^{\infty} \overline{F_n} \cap \mathbb{R}^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}}$. So, $\mathbb{R}^{\mathbb{N}}$ is covered by a countable family of pseudocompact sets. In this case Theorem 2.6 forces $\mathbb{N}$ to be pseudocompact, a contradiction. So, X is pseudocompact.

Then [1, 3.4.23 Theorem] shows that each member of $\mathcal{F}$ is a compact set. Hence, $\mathcal{F}$ is a closure-preserving cover of $C_p(X)$ by compact sets, and the conclusion follows from the preceding theorem. □

Lemma 5.3 (Guerrero, [40, Lemma 2.10]). *Let X be an infinite compact space. If $C_p(X)$ admits a closure-preserving covering by subspaces of density less than or equal to an infinite cardinal κ then $w(X) \leq \kappa$.*

Proof. We shall restrict ourselves to the case $\kappa = \aleph_0$, what will be used later. So, assume $C_p(X)$ admits a closure-preserving covering $\mathcal{F}$ by closed separable subspaces. Proceed by contradiction by supposing $w(X) > \aleph_0$. It suffices to consider the case $w(X) = \aleph_1$.

Since $d\left(C_p(X)\right) = iw(X) = w(X) = \aleph_1$ [58], there is $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $|\mathcal{F}_0| \leq \aleph_1$ and $C_p(X) = \bigcup \mathcal{F}_0$. This covering can be rewritten as $\{F_\alpha : 0 \leq \alpha < \omega_1\}$, and if we define $G_\alpha = \bigcup \left\{F_\beta : 0 \leq \beta < \alpha\right\}$ for every $0 \leq \alpha < \omega_1$, clearly $\mathcal{G} = \{G_\alpha : 0 \leq \alpha < \omega_1\}$ is an increasing closure-preserving covering of $C_p(X)$ by separable subspaces which swallows the separable sets in $C_p(X)$.

As X is embeddable in $[0,1]^{\omega_1}$, let us consider the natural projections $\pi_\alpha : X \to [0,1]^\alpha$ for $0 \leq \alpha < \omega_1$. For each $\alpha < \omega_1$ define $Z_\alpha = \pi_\alpha(X)$ and set $M_\alpha := \pi_\alpha^*(C(Z_\alpha))$, where $\pi_\alpha^* : C_p(Z_\alpha) \to C_p(X)$ is the pullback of π_α, defined as usual by $\pi_\alpha^*(h) = h \circ \pi_\alpha$. It can be easily seen that the family $\mathcal{M} = \{M_\alpha : 0 \leq \alpha < \omega_1\}$ is another increasing covering of $C_p(X)$ such that $d(M_\alpha) = w(Z_\alpha) \leq \aleph_0$. So, for each $\alpha < \omega_1$ there exists $\alpha \leq \beta < \omega_1$ with $M_\alpha \subseteq G_\beta$. Conversely, for each $\beta < \omega_1$ there exists $\beta \leq \gamma < \omega_1$ with $\overline{G_\beta} \subseteq M_\gamma$.

Note that X cannot be embedded in $[0,1]^\alpha$ for any $\alpha < \omega_1$, otherwise if $X \hookrightarrow [0,1]^\gamma$ then $w(X) = \left|\gamma\right| = \aleph_0$, a contradiction. This entails that for each $\alpha < \omega_1$ there exists $\beta < \omega_1$ with $\beta \geq \alpha$ such that both $\overline{G_\alpha} \subseteq M_\beta$ and the natural projection $\pi_{\alpha,\beta} : Z_\beta \to Z_\alpha$ is not injective. So we may get an increasing sequence of countable ordinals $\{\alpha_n : n \in \mathbb{N}\}$ such that $\overline{G_{\alpha_{2n-1}}} \subseteq M_{2n} \subseteq G_{2n+1}$ and the projection $\pi_{\alpha_{2n-1},\alpha_{2n}} : Z_{\alpha_{2n}} \to Z_{\alpha_{2n-1}}$ is not injective. Let $\gamma := \sup\{\alpha_n : n \in \mathbb{N}\}$ and for each n choose two different points $x_n, y_n \in Z_{\alpha_{2n}} \subseteq Z_\gamma$ with $\pi_{\alpha_{2n-1},\gamma}(x_n) = \pi_{\alpha_{2n-1},\alpha_{2n}}(x_n) = \pi_{\alpha_{2n-1},\alpha_{2n}}(y_n) = \pi_{\alpha_{2n-1},\gamma}(y_n)$.

According to [40, Lemma 2.9] there is $g \in C\left(Z_\gamma\right)$ whose restriction to $\{x_n, y_n : n \in \mathbb{N}\}$ is injective, so that $g(x_n) \neq g(y_n)$ for every $n \in \mathbb{N}$. This means that $\operatorname{supp} g \not\subseteq Z_{\alpha_{2n}}$ for all $n \in \mathbb{N}$, in other words, g does not belong to $C(Z_{\alpha_{2n}})$ for any $n \in \mathbb{N}$. Hence, the function $f = \pi_\gamma^*(g) \in \pi_\alpha^*\left(C\left(Z_\gamma\right)\right) = M_\gamma$ does not belong to $M_{\alpha_{2n}}$ for any $n \in \mathbb{N}$. Thus $f \notin \bigcup_{n=1}^\infty M_{2n} = G_\gamma$, the latter equality because both $\mathcal{G}$ and $\mathcal{M}$ are increasing, $G_{\alpha_{2n-1}} \subseteq M_{\alpha_{2n}} \subseteq G_{\alpha_{2n+1}}$ and $\bigcup \mathcal{G} = C(X)$.

On the other hand, let a finite subset A of X and $\epsilon > 0$ be given. Let

$$U_f = \left\{h \in M_\gamma : \left|h(x) - f(x)\right| < \epsilon,\ x \in A\right\}$$

be a neighborhood of f in the relative topology of M_γ. If $\pi_\gamma(x) = \pi_\gamma(y)$ for $x, y \in A$ then $f(x) = f(y)$, so we may assume $\pi_\gamma(x) \neq \pi_\gamma(y)$ for each pair $x, y \in A$. In this case there is $l \in \mathbb{N}$ such that $\pi_{\alpha_{2l},\gamma}$ is one-to-one on $\pi_\gamma(A)$. Hence $\pi_{\alpha_{2l}} = \pi_{\alpha_{2l},\gamma} \circ \pi_\gamma$ is one-to-one on A. So, we can choose $\varphi \in C(Z_{\alpha_{2l}})$ such that $\varphi(\pi_{\alpha_{2l}}(x)) = f(x)$ for each $x \in A$.

Since $h := \varphi \circ \pi_{\alpha_{2l}} = \pi_{\alpha_{2l}}^*(\varphi) \in M_{2l} \subseteq M_\gamma$, clearly $h \in U_f$. So, $f \in \overline{M_{2l}}$ and consequently $f \in \overline{M_{2l}} \subseteq \overline{G_{\alpha_{2n+1}}} \subseteq G_{\alpha_{2n+3}} \subseteq G_\gamma$, a contradiction. □

Theorem 5.4 (Guerrero, [40, Corollary 2.13]). *Let X be an infinite compact space. $C_p(X)$ admits a closure-preserving covering by separable subspaces if and only if X is metrizable.*

Proof. If X is a compact metrizable space, then $C_p(X)$ is separable. Let D be a countable dense subspace of $C_p(X)$. For every $f \in C(X)$ put $D_f := D \cup \{f\}$. Then clearly $\mathcal{F} = \left\{D_f : f \in C(X)\right\}$ is a closure-preserving covering of $C_p(X)$ by separable subspaces. Conversely, if the space $C_p(X)$ admits a closure-preserving covering by separable subspaces, Lemma 5.3 with $\kappa = \aleph_0$ yields $w(X) \leq \aleph_0$. Since X is compact, this implies that X must be metrizable. □

Theorem 5.5 (Guerrero, [40, Corollary 2.14]). *Let X be an infinite compact space. $C_p(X)$ admits a closure-preserving covering by second countable subspaces if and only if X is countable.*

Proof. If $C_p(X)$ admits a closure-preserving covering by second countable subspaces, then $C_p(X)$ admits a closure-preserving cover by separable subspaces. Hence X is metrizable by the previous theorem and, consequently, $C_p(X)$ is separable. This clearly implies that $C_p(X)$ has indeed a countable covering by second countable subspaces, so we may apply [70, Corolary 1.7] to guarantee that X is countable. □

For the following lemma, given a function $f \in C^b(X)$ and a number $\epsilon > 0$ let

$$I(f,\epsilon) = \left\{g \in C^b(X) : \left\|f - g\right\|_\infty \leq \epsilon\right\}.$$

Lemma 5.6 (Guerrero-Tkachuk [43, Proposition 2.1 (a)]). *If $\mathcal{F}$ is a closure-preserving covering of $C_p(X)$ by closed subspaces, there exist $F \in \mathcal{F}$ and $f \in C^b(X)$ such that $I(f,\epsilon) \subseteq F$ for some $\epsilon > 0$.*

Proof. We claim that the family $\left\{F \cap C^b(X) : F \in \mathcal{F}\right\}$ is also a closure-preserving covering by closed subspaces of the Banach space $C^b(X)$ equipped with the supremum-norm $\|\cdot\|_\infty$. Indeed, since the Banach topology τ_u is stronger than the pointwise topology, denoting $C^b(X)$ by G, if $\mathcal{F}' \subseteq \mathcal{F}$ one has

$$\bigcup_{F\in\mathcal{F}'} F \cap G = G \cap \overline{\bigcup_{F\in\mathcal{F}'} F}^{\tau_p} \supseteq G \cap \overline{\bigcup_{F\in\mathcal{F}'} F \cap G}^{\tau_p} \supseteq \overline{\bigcup_{F\in\mathcal{F}'} F \cap G}^{\tau_u} \supseteq \bigcup_{F\in\mathcal{F}'} F \cap G.$$

By [68, Theorem 2.5] there exist $F \in \mathcal{F}$ and $f \in C^b(X)$ for which there is an open ball

$$B(f,\delta) = \left\{g \in C^b(X) : \left\|f - g\right\|_\infty < \delta\right\}$$

centered at f in the Čech-complete space $\left(C^b(X), \|\cdot\|_\infty\right)$, such that $B(f,\delta) \subseteq F \cap G$. Hence, if $\epsilon = \delta/2$ we get $I(f,\epsilon) \subseteq F$. □

Theorem 5.7 (Guerrero-Tkachuk [43, Corollary 2.5]). *If $\mathcal{P}$ is a hereditary topological property and $C_p(X)$ has a closure-preserving cover $\mathcal{F}$ by closed subspaces such that each $F \in \mathcal{F}$ has property $\mathcal{P}$, both $C_p(X,[0,1])$ and $C_p(X)$ have property $\mathcal{P}$.*

Proof. Under these hypotheses we claim that some $F \in \mathcal{F}$ contains a homeomorphic copy of $C_p(X)$. By Lemma 5.6 there exist $F \in \mathcal{F}$ and $f \in C^b(X)$ such that $I(f,\epsilon) \subseteq F$ for some $\epsilon > 0$. Then the map $\varphi : C_p(X,[0,1]) \to C_p^b(X)$ defined by

$$\varphi(g) = 2\epsilon\left(g - \tfrac{1}{2}\right) + f$$

is a homeomorphism such that $\varphi(C(X,[0,1])) = I(f,\epsilon)$. Since F has the hereditary property $\mathcal{P}$, the set $I(f,\epsilon)$ also has property $\mathcal{P}$ and consequently $C_p(X,[0,1])$ has property $\mathcal{P}$. But $C_p(X,[0,1])$ contains $C_p(X,(0,1))$, which is homeomorphic to $C_p(X)$. So $C_p(X)$ also has property $\mathcal{P}$. □

Theorem 5.8 (Guerrero-Tkachuk [43, Theorem 2.7]). *Let $\mathcal{P}$ be a closed hereditary topological property. If $C_p(X)$ has a closure-preserving cover $\mathcal{F}$ by closed subspaces such that each $F \in \mathcal{F}$ has property $\mathcal{P}$, then $C_p(X,[0,1])$ has property $\mathcal{P}$.*

Proof. Again Lemma 5.6 provides $F \in \mathcal{F}$ and $f \in C^b(X)$ such that $I(f,\epsilon) \subseteq F$ for some $\epsilon > 0$. By the proof of Theorem 5.7 the subspace $I(f,\epsilon)$ of $C_p^b(X)$ is homeomorphic to $C_p(X,[0,1])$ and closed in F, so $C_p(X,[0,1])$ has property $\mathcal{P}$. □

Remark 5.9. *Applications of the preceding results.* Theorem 5.7 applies for instance to the Fréchet-Urysohn property and metrizability. Theorem 5.8 applies to K-analyticity, Lindelöf Σ-property and normality. Concerning realcompactness, if $C_p(X)$ has a closure-preserving cover $\mathcal{F}$ by closed subspaces such that each $F \in \mathcal{F}$ is realcompact, Theorem 5.8 ensures that the space $C_p(X,[0,1])$ is realcompact. Since $C(X,(0,1))$ can be obtained from $C_p(X,[0,1])$ by throwing out a union of G_δ-subsets of $C_p(X,[0,1])$, it turns out that $C(X,(0,1))$ is realcompact (see [72, Problem 408]). Hence $C_p(X)$ is realcompact.

Corollary 5.10 (Guerrero-Tkachuk [43, Proposition 2.20]). *If X is a Lindelöf Σ-space and $C_p(X)$ has a closure-preserving cover by closed Lindelöf Σ-subspaces then $C_p(X)$ is a Lindelöf Σ-space.*

Proof. According to Theorem 5.8, $C_p(X,[0,1])$ must be a Lindelöf Σ-space. So [1, IV.9.17 Theorem] ensures that $C_p(X)$ is a Lindelöf Σ-space. □

6. Domination by a second countable space

Given a Tychonoff space M, a family of sets $\mathcal{A}$ of another Tychonoff space X is said to be *M-ordered* (or ordered by M) if $\mathcal{A} = \{A_K : K \in \mathcal{K}(M)\}$, where $\mathcal{K}(M)$ denotes the family of all compact sets in M, and $P \subseteq Q$ in M implies $A_P \subseteq A_Q$. The space X is said to be *M-dominated* (or dominated by the space M) if X has an M-ordered covering $\mathcal{A}$ consisting of compact sets (an M-ordered compact covering).

Theorem 6.1 (Cascales-Orihuela-Tkachuk, [14, 2.1(a) Theorem]). *Every Lindelöf Σ-space is dominated by a second countable space.*

Proof. An equivalent definition of Lindelöf Σ-space says that X is a Lindelöf Σ-space if and only if there exists a second countable space M and a compact-valued *usc* map $T : M \to \mathcal{K}(X)$ such that $\bigcup\{T(x) : x \in M\} = X$. If K is a compact set in M, define $A_K = \bigcup\{T(x) : x \in K\}$. Clearly $\mathcal{A} = \{A_K : K \in \mathcal{K}(M)\}$ is an M-ordered cover consisting of compact sets. □

The class of spaces dominated by second countable spaces has good stability properties [14, 2.1 Theorem].

Theorem 6.2 (Cascales-Orihuela-Tkachuk, [14, 2.2 Proposition]). *The following relations are equivalent for a Tychonoff space X.*

1. *X has a resolution consisting of compact sets.*
2. *X is $\mathbb{N}^{\mathbb{N}}$-dominated.*
3. *X is dominated by a Polish space.*

Proof. 1 ⇒ 2. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution for X of compact sets. If $P \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$, define $\alpha_P \in \mathbb{N}^{\mathbb{N}}$ by $\alpha_P(i) = \max \pi_i(P)$, where $\pi_i : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is the canonical i^{th}-projection. Clearly $\alpha_P \le \alpha_Q$ if $P \subseteq Q$ and if we set $A_P := A_{\alpha_P}$ for every $P \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$, then $\mathcal{A} = \{A_P : P \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})\}$ is an $\mathbb{N}^{\mathbb{N}}$-ordered family of compact sets which covers X. The latter because if $x \in X$ there is $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $x \in A_\gamma$, and the set $Q_\gamma := \{\alpha \in \mathbb{N}^{\mathbb{N}} : \alpha(i) \le \gamma(i)\ \forall i \in \mathbb{N}\}$ is compact in $\mathbb{N}^{\mathbb{N}}$ and verifies that $\alpha_{Q_\gamma} = \gamma$. So $x \in A_{Q_\gamma}$.

2 ⇒ 1. Let $\{A_P : P \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})\}$ be an $\mathbb{N}^{\mathbb{N}}$-ordered compact cover of X. If $\gamma \in \mathbb{N}^{\mathbb{N}}$ let $Q_\gamma \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$ be the previously defined set that verifies the equality $\alpha_{Q_\gamma} = \gamma$. Then the family $\mathcal{A} = \{A_\gamma : \gamma \in \mathbb{N}^{\mathbb{N}}\}$ with $A_\gamma := A_{Q_\gamma}$ verifies that $A_\gamma \subseteq A_\delta$ if $\gamma \le \delta$. Moreover, $\mathcal{A}$ covers X. For if $x \in X$ there is $P \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$ with $x \in A_P$. So, if $\sigma(i) = \max \pi_i(P)$ for every $i \in \mathbb{N}$ then $A_P \subseteq A_\sigma$ and hence $x \in A_\sigma$. Therefore $\mathcal{A}$ is a resolution for X by compact sets.

2 ⇒ 3 is clear. Finally, if X is dominated by a Polish space M, there is an M-ordered compact cover $\mathcal{A} = \{A_K : K \in \mathcal{K}(M)\}$. Since M is a Polish space, there is an open continuous map $\varphi : \mathbb{N}^{\mathbb{N}} \to M$ from $\mathbb{N}^{\mathbb{N}}$ onto M. Consider the family $\mathcal{F} = \{A_{\varphi(Q_\alpha)} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. If $x \in X$ there is a compact set K in M such that $x \in A_K$ and there exists $P \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$ such that $\varphi(P) = K$ [18, 5.5.8]. If $\sigma(i) = \max \pi_i(P)$ for every $i \in \mathbb{N}$ then $P \subseteq Q_\sigma$ and hence $K = \varphi(P) \subseteq \varphi(Q_\sigma)$ so that $x \in A_{\varphi(Q_\sigma)}$. Hence $\mathcal{F}$ covers X and clearly $\mathcal{F}$ is a $\mathbb{N}^{\mathbb{N}}$-ordered compact covering for X. So X is $\mathbb{N}^{\mathbb{N}}$-dominated. This shows that 3 ⇒ 2. □

Theorem 6.3 (Cascales-Orihuela-Tkachuk, [14, 2.4 Corollary]). *$C_p(X)$ is dominated by a Polish space if and only if it is K-analytic.*

Proof. If $C_p(X)$ is dominated by a Polish space, by the previous theorem $C_p(X)$ has a resolution consisting of compact sets. So $C_p(X)$ is K-analytic by Theorem 3.10. Conversely, if $C_p(X)$ is K-analytic, it has a resolution of compact sets [67]. Thus, according to Theorem 6.2, $C_p(X)$ is dominated by a Polish space. □

Lemma 6.4. *If X is dominated by a second countable space, X has a countable network modulo a covering by countably compact sets and $C_p(X)$ is Lindelöf Σ-framed in $\mathbb{R}^X$.*

Proof. If X is dominated by a second countable space M, the first statement of the consequent follows from [14, 2.6 Proposition], where one should notice that the fact that M is second countable is critical. The second follows from [71, 2.7 Proposition]. □

Theorem 6.5 (Cascales-Orihuela-Tkachuk, [14, 2.15 Theorem]). *$C_p(X)$ is dominated by a second countable space if and only if it is a Lindelöf Σ-space.*

Proof. Sufficiency is Theorem 6.1. For the necessity assume that $\{F_K : K \in \mathcal{K}(M)\}$ is an M-ordered compact covering of $C_p(X)$. Apply Lemma 6.4 to show that $C_p\left(C_p(X)\right)$ is Lindelöf Σ-framed in $\mathbb{R}^X$ and then Lemma 3.1 to get that $\upsilon C_p(X)$ is a Lindelöf Σ-space. Then apply [1, IV.9.5 Theorem] to conclude that υX is also a Lindelöf Σ-space, which guarantees that the space $C_p(\upsilon X)$ is angelic [25, Theorem 78].

If $T : C_p(\upsilon X) \to C_p(X)$ denotes the restriction map $Tf = f\big|_X$, it can be easily seen that $\{G_K : K \in \mathcal{K}(M)\}$, where $G_K = T^{-1}(F_K)$ is an M-ordered compact covering of $C_p(\upsilon X)$. Since $C_p(\upsilon X)$ is dominated by a second countable space, Lemma 6.4 asserts that $C_p(\upsilon X)$ has a countable network modulo a covering by countably compact subsets. But $C_p(\upsilon X)$ angelicity ensures that $C_p(\upsilon X)$ has a countable network modulo a covering by compact sets. So, according to [1, IV.9.1 Proposition], $C_p(\upsilon X)$ is a Lindelöf Σ-space. Consequently $C_p(X)$, as a continuous image of a Lindelöf Σ-space, is also a Lindelöf Σ-space. □

Domination of each subspace of $C_p(X)$ by a second countable space also leads to some interesting properties. We state the following theorem without proof (see [14] for details).

Theorem 6.6 (Cascales-Orihuela-Tkachuk, [14, 2.18 Proposition]). *If every subspace of $C_p(X)$ is dominated by a second countable space, then $C_p(X)$ is cosmic.*

A Tychonoff space X is *strongly dominated* by M if there exists an M-ordered compact covering $\mathcal{F}$ of X that swallows the compact sets in X. Strong domination by second countable spaces has been extensively studied in [14, 41, 45, 74]. Under CH it is shown in [14, 3.10 Theorem] that, for compact X, if $C_p(X)$ is strongly dominated by a second countable space, X must be countable. The CH is removed in [41], where it is proved that, assuming $C_p(X)$ is a strongly dominated by a second countable space, if X is separable, scattered, second countable, compact or pseudocompact, then X is countable. Theorem 6.8 below extends this result to all Tychonoff spaces.

Lemma 6.7 (Guerrero-Tkachuk, [45, Lemma 3.4.5]). *Let X be an uncountable Lindelöf Σ-space. Assume $C_p(X)$ is strongly dominated by a second countable space M, and let $\{F_K : K \in \mathcal{K}(M)\}$ be an M-ordered compact covering of $C_p(X)$ that swallows the compact sets in $C_p(X)$. Then there exists a family $\mathcal{Q} = \{Q_K : K \in \mathcal{K}(M)\}$ of compact sets of $\mathbb{R}^X$ such that $Q_K \subseteq Q_L$ if $K \subseteq L$ and $\bigcup \mathcal{Q}$ contains the linear subspace $\Sigma(X)$ of all countable supported functions of $\mathbb{R}^X$.*

Proof. If $K \in \mathcal{K}(M)$, let $a_K(x) = \inf\{g(x) : g \in F_K\}$ and $b_K(x) = \sup\{g(x) : g \in F_K\}$. Letting

$$Q_K = \prod_{x \in X} \left[a_K(x), b_K(x)\right],$$

the family $\mathcal{Q} = \{Q_K : K \in \mathcal{K}(M)\}$ consists of compact sets in $\mathbb{R}^X$ and clearly verifies that $Q_K \subseteq Q_L$ if $Q \subseteq L$. We claim that $\Sigma(X) \subseteq \bigcup \mathcal{Q}$.

Choose $f \in \Sigma(X)$ and denote by $A = \{x_i : i \in \mathbb{N}\}$ the countable support of f. By Theorem 6.5 we know that $C_p(X)$ is a Lindelöf Σ-space, hence [64, Theorem 5.4] provides a retraction $r : X \to F$ such that $A \subseteq F$ and $|F| \leq \aleph_0$. If $F = \{y_n : n \in \mathbb{N}\}$, put $U_1 = F$ and $U_{n+1} = F \setminus \{y_1, \ldots, y_n\}$ for each $n \in \mathbb{N}$. Clearly, the family $\{U_n : n \in \mathbb{N}\}$ consists of F-open sets and is point-finite in F, i. e., each $x \in F$ belongs at most to finitely-many sets U_n. Moreover $y_n \in U_n$ for every $n \in \mathbb{N}$. Since F is a retract of X, it follows that the family $\{V_n : n \in \mathbb{N}\}$, where $V_n := r^{-1}(U_n)$ for each $n \in \mathbb{N}$, consists of open sets in X, is point-finite in X and verifies that $y_n \in V_n$

for every $n \in \mathbb{N}$. If $x_i = y_{n_i}$ and we set $W_i = V_{n_i}$ for each $i \in \mathbb{N}$, the family $\{W_i : i \in \mathbb{N}\}$ consists of open sets in X, is point-finite in X and verifies that $x_i \in W_i$ for every $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$ choose $f_i \in C(X)$ with $0 \le f_i \le 1$ such that $f_i(x_i) = 1$ and $f_i(X \setminus W_i) = \{0\}$ and define $g_i = |f(x_i)| \cdot f_i$ and $h_i = -|f(x_i)| \cdot f_i$. As the family $\{W_i : i \in \mathbb{N}\}$ is point-finite, the set $P = \{g_i, h_i : i \in \mathbb{N}\} \cup \{\}$, where here stands for the identically null function on X, is compact in $C_p(X)$. Consequently, there exists some $K \in \mathcal{K}(M)$ such that $P \subseteq F_K$, which means that $g(x) \in [a_K(x), b_K(x)]$ for each $g \in P$. Since $f(x_i)$ coincides with $g_i(x_i)$ or with $h_i(x_i)$, clearly $f(x_i) \in [a_K(x_i), b_K(x_i)]$ for each $i \in \mathbb{N}$, whereas if $x \notin A$ then $f(x) = 0 \in [a_K(x), b_K(x)]$ since $\in P \subseteq F_K$. Therefore $f \in Q_K$ and the proof is over. □

Theorem 6.8 (Guerrero-Tkachuk, [45, Theorem 3.4]). *$C_p(X)$ is strongly dominated by a second countable space if and only if X is countable.*

Proof. Suppose that $C_p(X)$ is strongly dominated by a second countable space and let $\{F_K : K \in \mathcal{K}(M)\}$ be an M-ordered compact covering of $C_p(X)$ that swallows the compact sets in $C_p(X)$. Proceeding by contradiction, assume that X is uncountable. By [41, Theorem 3.10] there is no loss of generality if we assume that X is a Lindelöf Σ-space. So, according to Lemma 6.7, there exists a family $\mathcal{Q} = \{Q_K : K \in \mathcal{K}(M)\}$ of compact sets in $\mathbb{R}^X$ such that $Q_K \subseteq Q_L$ if $Q \subseteq L$ and $Y = \bigcup \mathcal{Q}$ contains the linear subspace $\Sigma(X)$ of countable supported functions of $\mathbb{R}^X$.

It is not hard to see that this implies that there exists a Lindelöf Σ-space Z such that $Y \subseteq Z \subseteq \mathbb{R}^X$, so that $\Sigma(X) \subseteq Z$. But $\Sigma(X)$ is not Lindelöf Σ-framed in $\mathbb{R}^X$ if X is uncountable [45, Proposition 3.1]. □

In [45, Theorem 3.9] it is showed that, for compact X, if $C_p(X, [0,1])$ is strongly dominated by a second countable space, then X is countable. In [74] the requirement of compactness of X is relaxed by the following result, which we state without proof.

Theorem 6.9 (Tkachuk, [74, 3.7 Theorem]). *Let X be a Lindelöf Σ-space. If $C_p(X, [0,1])$ is strongly dominated by a second countable space, then X is countable.*

7. Some examples

Example 7.1. *If Ω is a nonempty open subset of $\mathbb{R}^n$, both the space $\mathcal{D}(\Omega)$ of test functions equipped with its usual inductive limit topology and the space of distributions $\mathcal{D}'(\Omega)$ endowed with the Mackey* topology $\mu(\mathcal{D}'(\Omega), \mathcal{D}(\Omega))$, which coincides with the strong topology $\beta(\mathcal{D}'(\Omega), \mathcal{D}(\Omega))$* (see [46, Chapter 4])*, are analytic.* The first statement is consequence of the fact that the inductive limit of a sequence of Fréchet-Montel spaces is analytic, the second follows from the fact that the strong dual of an inductive limit of a sequence of Fréchet-Montel locally convex spaces is also analytic (see [76, I.4.4.(21) and I.4.4.(23)]).

Example 7.2. *The space $C_p(Z)$ with Z being the set of all weak P-points of $\mathbb{N}^\star$.* If X is a Tychonoff space, a point $x \in X$ is called a weak P-point of X if $x \notin \overline{A}$ for any countable set $A \subseteq X \setminus \{x\}$. Every P-point of X is a weak P-point of X. The subspace Z of all weak P-points of the remainder $\mathbb{N}^\star = \beta\mathbb{N} \setminus \mathbb{N}$ of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is dense in $\mathbb{N}^*$ [53], so it is infinite. But the space $C_p(Z)$ is covered by a sequence of pseudocompact sets (i. e., $C_p(Z)$ is σ-pseudocompact) [2, 6.4 Example]. By Theorem 2.6, the space Z is pseudocompact (see [2, 6.3 Proposition] for a direct proof of this property) and each countable subset of Z is closed, discrete and C^*-embedded in Z. Note that $C_p(Z)$ is not σ-compact, otherwise Z would be finite by Velichko's theorem.

Example 7.3. The Sorgenfrey line $\mathbb{S}$ is a (hereditarily) Lindelöf space which is not a Lindelöf Σ-space, since $\mathbb{S} \times \mathbb{S}$ is not Lindelöf. Hence Lemma 3.1 prevents the space $C_p(\mathbb{S})$ to have a resolution consisting of pointwise bounded sets.

Example 7.4. *The space $C_p(\mathbb{N}^\mathbb{N})$ is not K-analytic-framed in $\mathbb{R}^{\mathbb{N}^\mathbb{N}}$.* By Corollary 3.8 the space $C_p(\mathbb{N}^\mathbb{N})$ is not analytic and does not admit a resolution of pointwise bounded sets. Hence, $C_p(\mathbb{N}^\mathbb{N})$ is not K-analytic-framed in $\mathbb{R}^{\mathbb{N}^\mathbb{N}}$ because of Theorem 3.3.

Example 7.5. *The spaces $C_k(\mathbb{R})$ and $C_k(\mathbb{Q})$.* Both spaces have a resolution of compact sets that swallows the compact sets by virtue of Theorem 4.2. By Theorem 4.3 they also have a fundamental resolution of bounded sets, and according to Theorem 4.4 both spaces are analytic.

Example 7.6. *The spaces $C_p(\mathbb{R})$ and $C_p(\mathbb{Q})$.* Although both spaces have a resolution of compact sets, according to Theorem 3.15 they do not have a resolution of compact sets that swallows the compact sets. The space $C_p(\mathbb{Q})$ has a fundamental resolution of pointwise bounded sets but, as follows from Theorem 3.16, such a resolution lacks in $C_p(\mathbb{R})$.

Example 7.7. Let $\mathbb{N}$ be equipped with the discrete topology and choose $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then $X := \mathbb{N} \cup \{p\}$ with the relative topology of $\beta\mathbb{N}$ is a non discrete space with finite compact sets, hence hemicompact. So, $C_p(X)$ is analytic by Theorem 4.4, and $C_k(X)$ is a Fréchet space with a resolution of compact sets that swallows the compact sets by Theorem 4.2.

Example 7.8. *The space ω_1 of countable ordinals with the order topology.* It is essentially well-known that if $\aleph_1 = \mathfrak{d}$ (the dominating cardinal) the space ω_1 has a resolution of compact sets that swallows the compact sets in ω_1. However ω_1 is not even a μ-space, since ω_1 is pseudocompact but not compact.

Example 7.9. *The space $C_p(\omega_1)$.* Clearly $C_p(\omega_1)$ is not analytic because of Theorem 3.7. Actually, $C_p(\omega_1)$ does not admit a resolution of compact sets, since every topological space with a resolution of compact sets has countable extent (closed discrete sets are countable) [49, Corollary 3.5] whereas the extent of $C_p(\omega_1)$ is uncountable. Consequently, $C_p(\omega_1)$ is not K-analytic although, as is well-known, it is a Lindelöf space. Note that ω_1 is pseudocompact, hence projectively σ-compact.

Example 7.10. *If $C_k(X)$ admits a resolution of convex compact sets that swallows the local null sequences, X need not be countable or discrete.* If X is an infinite σ-compact metric space, then $C_k(X)$ has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets that swallows the compact sets (hence the local null sequences) of $C_k(X)$ by virtue of Theorem 4.2. But, if one looks at the proof of this theorem, the sets A_α are absolutely convex. So, $C_k(\mathbb{R})$ has a resolution of absolutely convex compact sets that swallows the local null sequences.

Example 7.11. *A K-analytic not analytic C_p-space.* Let X be the Reznichenko compact space mentioned in [2, 7.14 Example]. This is a Talagrand compact space with a point p such that $X = \beta Y$ with $Y = X \setminus \{p\}$. Hence Y is a pseudocompact not realcompact space, so that $X = \upsilon Y$. Since $C_p(Y)$ is a continuous image of $C_p(X)$, the space $C_p(Y)$ is K-analytic. $C_p(Y)$ is not analytic by Theorem 3.7.

Example 7.12. *The space $C_p(L(\aleph_1))$, where $L(\aleph_1)$ is the Lindelöfication of the discrete space of cardinality $\aleph_1$.* Since $L(\aleph_1)$ is a P-space, $C_p(L(\aleph_1))$ is Baire. So, by Theorem 3.13, $C_p(L(\aleph_1))$ lacks a resolution of pointwise bounded sets. As $L(\aleph_1)$ is a Lindelöf P-space, it is projectively σ-compact [3, Proposition 2.2]. Hence, the converse of Corollary 3.5 fails.

Example 7.13. *The space $C_p(L(\aleph_1), [0,1])$.* Under CH the space $C_p(L(\aleph_1), [0,1])$ has a compact resolution [71, 2.10 Example]. Since $C_p(L(\aleph_1), [0,1])$ is countably compact but not compact, $C(L(\aleph_1), [0,1])$ is not a μ-space, hence it is not K-analytic.

Example 7.14. *$C_p(X)$ need not be Lindelöf if $C_p(X, [0,1])$ is a Lindelöf Σ-space.* If X is a discrete space of cardinality $\aleph_1$ then $C_p(X, [0,1]) = [0,1]^{\omega_1}$ is compact, but $C_p(X) = \mathbb{R}^{\omega_1}$ is not Lindelöf. If both υX and $C_p(X, [0,1])$ are Lindelöf Σ-spaces, then $C_p(X)$ is a Lindelöf Σ-space (see [1, IV.9.17 Proposition] or [73, Problem 217]).

Example 7.15. *Neither $\mathbb{R}^{\mathbb{N}}$ nor $C_p([0,1])$ admits a closure-preserving covering by functionally bounded subspaces.* Otherwise, since both spaces are separable and the closure of a functionally bounded set is also functionally bounded, both would admit a countable covering by (closed) functionally bounded subspaces. So, Theorem 2.6 would require $\mathbb{N}$ to be pseudocompact, which is not, and every countable set in $[0,1]$ should be closed, which is neither the case since $[0,1]$ is uncountable and separable.

Example 7.16 (Okunev, [59, Example 2.7]). *There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf.* If Y is the subspace of $[0,1]^{\omega_1}$ consisting of all function of finite support and $g \in [0,1]^{\omega_1}$ is the constant function $g(t) = 1$ for every $t \in \omega_1$, define $X = Y \cup \{g\}$. Then $C_p(X)$ is such space. Note that $g \in \overline{Y}$ but no countable subset of Y contains g in its closure, so that X has uncountable tightness $t(X)$. Hence $C_p(X)$ is not a Lindelöf space because of Asanov's theorem [1, I.4.1 Theorem]. Clearly $C_p(X)$ has a bounded resolution. So, according to Theorem 3.3, $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. But $C_p(X)$ is not K-analytic because it is not Lindelöf.

Example 7.17 (Guerrero-Tkachuk, [43, Example 3.8]). *There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but it contains a dense σ-compact subspace M.* Let Z be the subspace of $\{0,1\}^{\omega_1}$ consisting of those functions of finite support and define the function g as in the previous example. The space $X = Z \cup \{g\}$ is as promised. For each $f \in C(X)$ put $M_f := M \cup \{f\}$. Then $\mathcal{F} = \left\{M_f : f \in C(X)\right\}$ is a closure-preserving cover of $C_p(X)$ by σ-compact subspaces. This shows that the closedness condition of the sets of the closure-preserving covering in the statement of Theorem 5.1 cannot be dropped.

8. Further research

If $\Delta = \{(x,x) : x \in X\}$ is the diagonal of $X \times X$, much research has been developed on the (strong) domination of the space $(X \times X) \setminus \Delta$ by a second countable space. We provide a brief account of this investigation, but first let us point out a couple of facts.

In first place, according to [18, Excercise 4.2.B] a compact space X is metrizable if and only if Δ is a G_δ-set in $X \times X$. On the other hand, for compact X, *if $C_p(X,[0,1])$ is a Lindelöf Σ-space*, clearly $C_p(X)$ is also a Lindelöf Σ-space. Hence Baturov's theorem [1, III.6.1 Theorem] shows that for every subspace Y of $C_p\left(C_p(X)\right)$ the extent $ext(Y)$ of Y equals the Lindelöf number $l(Y)$ of Y. As X^n is embedded in $L_p(X)$, for each $n \in \mathbb{N}$, as a closed subspace [72, Problem 337], so in $C_p\left(C_p(X)\right)$, clearly $X^2 \setminus \Delta$ is embedded in $C_p\left(C_p(X)\right)$. Consequently, one gets $l\left(X^2 \setminus \Delta\right) = ext\left(X^2 \setminus \Delta\right)$. On the other hand, since each space which is dominated by a second countable space has countable extent [14, 2.1 (h) Theorem], if $(X \times X) \setminus \Delta$ is dominated by a second countable space, it follows that $l\left(X^2 \setminus \Delta\right) = \aleph_0$, i. e., $(X \times X) \setminus \Delta$ is a Lindelöf space. This implies that Δ is a G_δ-set in $X \times X$, so Δ must be metrizable.

The first result on this subject is [13, Theorem 1], whence it follows that, for compact X, if the space $(X \times X) \setminus \Delta$ is strongly $\mathbb{N}^{\mathbb{N}}$-dominated (equivalently, strongly dominated by a Polish space) then X is metrizable. This extends to the following.

Theorem 8.1 (Cascales-Muñoz-Orihuela [12, Corollary 22]). *For a compact space X the following statements are equivalent.*

1. *X is metrizable.*
2. *$(X \times X) \setminus \Delta$ is strongly dominated by a Polish space.*
3. *$(X \times X) \setminus \Delta$ is strongly dominated by a separable metric space.*

For strong domination of a Tychonoff space by a second countable space, one has

Theorem 8.2 (Guerrero-Tkachuk, [45, Corollary 3.6]). *If $(X \times X)\setminus\Delta$ is strongly dominated by a second countable space, then X is cosmic.*

Since each compact cosmic space is metrizable, one gets again

Corollary 8.3 (Cascales-Orihuela-Tkachuk, [14, 3.11 Theorem]). *A compact space X is metrizable if and only if $(X \times X) \setminus \Delta$ is strongly dominated by a second countable space.*

In [74, Example 4.6] it is shown that under MA the space $(X \times X) \setminus \Delta$ with X non-metrizable, first countable, compact space, is strongly dominated by a countable space (with a unique non-isolated point). So, under MA, for compact X strong domination of $(X \times X) \setminus \Delta$ by a countable space does not imply the metrizability of X.

In [16] it is shown that under CH a compact space X is metrizable whenever $(X \times X) \setminus \Delta$ is dominated by a Polish space. This result was extended to ZFC in [17] as follows.

Theorem 8.4 (Dow-Hart, [17, Theorem 8]). *A compact space X is metrizable if and only if $(X \times X) \setminus \Delta$ is dominated by a Polish space.*

Under CH one may change the Polish space domination of the previous theorem into second countable domination.

Theorem 8.5 (Guerrero-Tkachuk, [44, 3.3 Corollary]). *Under CH a compact space X is metrizable if and only if $(X \times X) \setminus \Delta$ is dominated by a second countable space.*

Recently, the following ZFC result has been published.

Theorem 8.6 (Feng, [19, Theorem 5.3]). *Let X be a compact space. If $(X \times X) \setminus \Delta$ is dominated by the space $\mathbb{Q}$, then X is metrizable.*

Since [44], when $(X \times X) \setminus \Delta$ is dominated by a space M, it is usual to say that X *has an M-diagonal*. With this new terminology and since the space $\mathbb{P}$ of irrationals is homeomorphic to the Polish space $\mathbb{N}^{\mathbb{N}}$, Theorems 8.4 and 8.6 can be stated as follows.

Theorem 8.7. *Let X be a compact space X. If X has either a $\mathbb{P}$-diagonal or a $\mathbb{Q}$-diagonal, then X is metrizable.*

The following result is a proper extension of Theorem 8.1.

Theorem 8.8 (Guerrero, [42, Theorem 2.3]). *If M is a separable metric space, every compact space with an M-diagonal is metrizable.*

In [44, Theorem 3.4 (a)] it is shown that under CH if a Tychonoff space X has a second countable diagonal, then X is cosmic. The following result show that the preceding statement also holds in ZFC.

Theorem 8.9 (Guerrero, [42, Corollary 2.4]). *For a Tychonoff space X, if $(X \times X) \setminus \Delta$ is dominated by a second countable space, then X is cosmic.*

Since, as mentioned earlier, each compact cosmic space is metrizable, this solves in the positive the following question originally posed by Cascales, Orihuela and Tkachuk in [14].

Problem 8.10 (Guerrero-Tkachuk, [44, Question 4.1]). *Let X be a compact space. If $(X \times X) \setminus \Delta$ is dominated by a second countable space, is it true in ZFC that X metrizable?*

It is proved in [70] that if $C_p(X)$ is covered by a countable family of countably tight sets, then $C_p(X)$ has countable tightness. In [78] is shown that a compact space with a closure-preserving covering by finite sets must be Eberlein compact. Related research about domination and strong domination of a space X by a locally compact second countable space M, by an ω-hyperbounded space M (i. e., an space in which the closure of each σ-compact subspace is compact) or by a κ-hemicompact space M (for a given infinite cardinal κ) can be found in [48].

On the other hand, the bidual $M(X)$ of $C_p(X)$ equipped with the relative topology of $\mathbb{R}^X$ has recently deserved some attention in relation to the distinguished property of $C_p(X)$ (see [33] and references therein), after the discovering that not always $M(X)$ coincides with $\mathbb{R}^X$ (in fact, it can be shown that $M(X) = \mathbb{R}^X$ exactly when $C_p(X)$ is distinguished, which is not always the case). Let us mention the following result (from which Theorem 3.16 is a straightforward consequence).

Theorem 8.11 (Ferrando, [25, Theorem 28]). *The bidual of $C_p(X)$ has a resolution consisting of pointwise bounded sets if and only if X is countable.*

9. Some open problems

Theorem 2.4 asserts that if $C_p(X)$ is covered by a sequence of relatively sequentially complete sets, then X is a P-space.

Problem 9.1. *If $C_k(X)$ is covered by a sequence of weakly relatively sequentially complete sets, is X is a P-space?*

Theorem 2.5 states that if $C_p(X)$ is covered by a sequence of pointwise bounded relatively sequentially complete sets, then X is finite.

Problem 9.2. *If $C_k(X)$ is covered by a sequence of bounded weakly relatively sequentially complete sets, must X be finite?*

By Corollary 3.5 if $C_p(X)$ has a resolution consisting of pointwise bounded sets, then X is projectively σ-compact. On the other hand, according to [26, Theorem 3.1] the space X is σ-compact if and only if there exists a metrizable locally convex topology τ on $C(X)$ such that $\tau_p \leq \tau \leq \tau_k$. If τ is a metrizable locally convex topology on $C(X)$ stronger than τ_p certainly $C_p(X)$ has a resolution consisting of pointwise bounded sets, but if X is a μ-space the τ_k-closures of a fundamental system of τ-neighborhoods of the origin in $C(X)$ define a metrizable locally convex topology η on $C(X)$ coarser than τ_k because of the Nachbin-Shirota theorem. In other words, if X is a μ-space and there is a metrizable locally convex topology τ on $C(X)$ such that $\tau_p \leq \tau$, there exists a metrizable locally convex topology η on $C(X)$ such that $\tau_p \leq \eta \leq \tau_k$. So, the following makes sense.

Problem 9.3 (Kąkol). *Assume that X is a μ-space. If $C_p(X)$ has a resolution consisting of pointwise bounded sets, is there always a stronger metrizable locally convex topology τ on $C(X)$ such that $\tau_p \leq \tau \leq \tau_k$?*

Observe that a positive answer to this question, also gives a positive answer to the following question.

Problem 9.4. *Assume that X is a μ-space. Is it true that $C_p(X)$ has a resolution consisting of pointwise bounded sets if and only if X is σ-compact? Equivalently, is it true that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if X is σ-compact?*

According to Theorem 3.16, the space $C_p(X)$ has a resolution of pointwise bounded sets that swallows the pointwise bounded sets if and only if X is countable.

Problem 9.5. *If $C_p(X)$ has a resolution of pointwise bounded sets that swallows the compact sets, is X countable?*

If X is first countable, Theorem 3.21 asserts that $C_p(X)$ has a resolution of pointwise bounded sets that swallows the Cauchy sequences if and only if X is countable.

Problem 9.6. *If $C_p(X)$ has a resolution of pointwise bounded sets that swallows the Cauchy sequences, is X countable?*

In Theorem 2.5 it is shown that $C_p(X)$ is covered by a sequence of pointwise bounded relatively sequentially complete sets if and only if X is finite.

Problem 9.7. *If $C_p(X)$ has a resolution of pointwise bounded relatively sequentially complete sets that swallows the pointwise bounded relatively sequentially complete sets, is X countable?*

Theorem 2.6 shows that if $C_p(X)$ is covered by a sequence of functionally bounded sets, X is pseudocompact and each countable set in X is closed, discrete and C^*-embedded.

Problem 9.8. *If $C_p(X)$ has a resolution of functionally bounded sets that swallows the functionally bounded sets, is X countable and discrete?*

If X is metrizable, according to Theorem 4.2 the space $C_k(X)$ has a resolution of compact sets that swallows the compact sets if and only if X is σ-compact.

Problem 9.9. *Characterize in terms of the topology of X those spaces $C_k(X)$ which admit a resolution of compact sets that swallows the compact sets.*

If X is metrizable, Theorem 4.3 shows that $C_k(X)$ has a resolution of bounded sets that swallows the bounded sets if and only if X is σ-compact, and in [23, Theorem 8] is proved that $C_k(X)$ has a resolution of bounded sets that swallows the bounded sets if and only if X is a so-called *cn-space* [23, p. 3].

Problem 9.10. *Is there a nicer characterization in terms of the topology of X of those spaces $C_k(X)$ which admit a resolution of bounded sets that swallows the bounded sets.*

By Theorem 6.9, if X is a Lindelöf Σ-space and $C_p(X,[0,1])$ is strongly dominated by a second countable space, then X is countable. So $C_p(X)$ is metrizable and separable, i. e., $C_p(X)$ is cosmic. Consequently $C_p(X)$ is a Lindelöf Σ-space.

Problem 9.11 (Guerrero-Tkachuk, [45, Question 4.1]). *Suppose that $C_p(X,[0,1])$ is dominated by a second countable space. Must $C_p(X,[0,1])$ be a Lindelöf Σ-space?*

Problem 9.12 (Guerrero-Tkachuk, [45, Question 4.2]). *If X is metrizable and $C_p(X,[0,1])$ is dominated by a second countable space, must $C_p(X,[0,1])$ be a Lindelöf Σ-space?*

Problem 9.13 (Guerrero-Tkachuk, [45, Question 4.3]). *If X is Lindelöf and $C_p(X,[0,1])$ is dominated by a second countable space, must $C_p(X,[0,1])$ be a Lindelöf Σ-space?*

Recalling again Theorem 3.16, the following natural question makes sense.

Problem 9.14. *Let M be a second countable space. If $C_p(X)$ is covered by an M-ordered family $\mathcal{A} = \{A_K : K \in \mathcal{K}(M)\}$ consisting of pointwise bounded sets that swallows the pointwise bounded sets in $C_p(X)$, must X be countable?*

Problem 9.15 (Guerrero-Tkachuk, [44, 4.4 Question]). *If X is a compact space with a σ-compact diagonal, is X metrizable?*

Problem 9.16 (Guerrero-Tkachuk, [44, 4.5 Question]). *If X is a compact space with a Lindelöf Σ diagonal, is X metrizable?*

Problem 9.17 (Tkachuk, [74, 5.6 Question]). *Is it true in ZFC that for any compact first countable space X there exists a countable space M that strongly dominates $(X \times X) \setminus \Delta$?*

Problem 9.18 (Guerrero, [42, Problem 4.1]). *Let X be a compact space. If $(X \times X) \setminus \Delta$ is dominated by a metric space, is X metrizable?*

In [25, Corollary 22] it is shown that for X realcompact, the weak* bidual $M(X)$ of $C_p(X)$ is a Lindelöf Σ-space if and only if X is countable.

Problem 9.19. *May we drop the condition that X is realcompact in the previous statement?*

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