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A Classification of 3-Dimensional η-Einstein Paracontact Metric Manifolds

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Abstract. We show that a 3−dimensional η-Einstein paracontact metric manifold is either a manifold with $\mathrm{tr}h^2 = 0$, flat or of constant ξ−sectional curvature $k \neq -1$ and constant φ-sectional curvature $-k \neq 1$.

1. Introduction

The assumption that $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a paracontact metric manifold is very weak, since the set of metrics associated to the paracontact form η is huge. Even if the structure is η−Einstein we do not have a complete classification. It is known very little for the geometry of manifolds with $n = 1$ (see [5]). On the other hand if the structure is para-Sasakian, the Ricci operator Q commutes with φ (see [5]), but in general $Q\varphi \neq \varphi Q$ and the problem of the characterization of paracontact metric manifolds with $Q\varphi = \varphi Q$ is open. In [2] Tanno defined a special family of paracontact metric manifolds by the requirement that ξ belongs to the k−nullity distribution of g. These manifolds are not well studied also (see [5]). In this paper, we show that a 3−dimensional paracontact manifold on which $Q\varphi = \varphi Q$ is either a manifold with $\mathrm{tr}h^2 = 0$, flat or of constant ξ−sectional curvature $k \neq -1$ and constant φ-sectional curvature $-k \neq 1$.

2. Preliminaries

First we will give some known definitons and facts which we use in the next section. A C^∞ manifold $M^{(2n+1)}$ is said to be *paracontact manifold*, if it carries a global 1−form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. We assume throughout that all manifolds are connected. Given a paracontact form η, it is well known that there exists a unique vector field ξ, called *characteristic vector field* of η, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for all vector fields X. A pseudo–Riemannian metric g is said to be an *associated metric* if there exists a tensor field φ of type $(1, 1)$ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 = Id - \eta \otimes \xi. \tag{2.1}$$

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Using these conditions it is easily obtained that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2.2}$$

The structure (φ, ξ, η, g) is called a *paracontact metric structure*, and a manifold M^{2n+1} with paracontact metric structure (φ, ξ, η, g) is said to be a *paracontact metric manifold*.

Let £ and R be the Lie differentiation and the curvature tensor respectively. Using £ and R we define the operators l and h are defined in the following way

$$lX = R(X,\xi)\xi, \quad h = \frac{1}{2}£_{\xi}\varphi. \tag{2.3}$$

The tensors h and l of $(1,1)-$type are symmetric and satisfy all of the subsequent conditions

$$l\xi = 0, \quad h\xi = 0, \quad \mathrm{tr}h = 0, \quad \mathrm{tr}h\varphi = 0, \quad h\varphi = -\varphi h. \tag{2.4}$$

For a paracontact manifold the following statements are fulfilled:

$$\nabla_X\xi = -\varphi X + \varphi hX \; (\nabla_{\xi}\xi = 0) \tag{2.5}$$

$$\nabla_{\xi}\varphi = 0 \tag{2.6}$$

$$\mathrm{tr}l = g(Q\xi, \xi) = -2n + trh^2 \tag{2.7}$$

$$\varphi l\varphi + l = -2(\varphi^2 - h^2) \tag{2.8}$$

$$\nabla_{\xi}h = -\varphi - \varphi l + \varphi h^2, \tag{2.9}$$

where tr is the trace of the operator, Q is the Ricci operator and ∇ is the Levi-Civita connection of g. Detailed proof of these formulas can be found in [4].

A paracontact metric manifold for which ξ is Killing is called a *K-paracontact manifold*. A paracontact structure on $M^{(2n+1)}$ naturally gives rise to an almost paracomplex structure on the product $M^{(2n+1)} \times \mathbb{R}$. If this almost paracomplex structure is integrable, the given paracontact metric manifold is said to be a *para-Sasakian*. Equivalently, (see [4]) a paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.10}$$

is satisfied for all vector fields X and Y.

It can be easily shown that a $3-$dimentional paracontact manifold is para-Sasakian if and only if $h = 0$. For further details we refer to [3],[4].

A paracontact metric structure is said to be $\eta-$*Einstein* if

$$Q = a.id + b.\eta \otimes \xi, \tag{2.11}$$

where a, b are smooth functions on $M^{(2n+1)}$. We also recall that the $k-$nullity distribution $N(k)$ of a pseudo-Riemannian manifold (M, g), for a real number k, is the distribution

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}, \tag{2.12}$$

for any $X, Y \in T_pM$ (see [2]).

Finally, we call *ξ-sectional curvature* the sectional curvature

$$K(\xi, X) = \epsilon_X R(X, \xi, \xi, X),$$

where $|X| = \epsilon_X = \pm 1$, of a plane section spanned by ξ and the vector X orthogonal to ξ. Moreover the sectional curvature $K(X, \varphi X) = -R(X, \varphi X, \varphi X, X)$, where $|X| = -|\varphi X| = \pm 1$, of a plane section spanned by vectors X and φX orthogonal to ξ is called a *φ-sectional curvature*.

3. Main result

In this section we introduce our main results. Firstly we state the following lemma.

Lemma 1. *Let M^3 be a paracontact metric manifold with a paracontact metric structure (φ, ξ, η, g) such that $\varphi Q = Q\varphi$. Then the function* tr$l$ *is constant everywhere on M^3.*

Although it was previously proven in [5], we include this complete proof in this paper for completeness and since we use many of the formulas which appear in it.

Proof. Let us recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{scal}{2}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{3.13}$$

where *scal* is the scalar curvature of the manifold.
Using $\varphi Q = Q\varphi$, (2.7) and $\varphi\xi = 0$ we have that

$$Q\xi = (\mathrm{tr}l)\xi. \tag{3.14}$$

From (3.13) and using (2.3) and (3.14), we have that for any X,

$$lX = QX + \left(\mathrm{tr}l - \frac{scal}{2}\right)X + \eta(X)\left(\frac{scal}{2} - 2\mathrm{tr}l\right)\xi \tag{3.15}$$

and hence $\varphi Q = Q\varphi$ and $\varphi\xi = 0$ give

$$\varphi l = l\varphi. \tag{3.16}$$

As a result of (3.16), (2.8) and (2.9), we obtain

$$-l = \varphi^2 - h^2 \tag{3.17}$$

and hence $\nabla_\xi h = 0$.
By differentiating (3.17) along ξ and using formula (2.6) and $\nabla_\xi h = 0$, we find that $\nabla_\xi l = 0$ and therefore $\xi(\mathrm{tr}l) = 0$. If there exists $X \in T_pM$, $X \neq \xi$ at point $p \in M^3$ such that $lX = 0$, then $l = 0$ at the point p. In fact if Y is the projection of X on $\mathbb{D} = ker\eta$, we have $lY = 0$, since $l\xi = 0$. Using (3.16) we have $l\varphi Y = 0$. So $l = 0$ at the point p (and thus $\mathrm{tr}l = 0$ at the point p). Let us suppose that $l \neq 0$ on a neighborhood U of the point P. Using (3.16) and that φ is antisymmetric, we get $g(\varphi X, lX) = 0$. Hence lX is parallel to X for any X orthogonal to ξ. It is not hard to see that $lX = \frac{\mathrm{tr}l}{2}X$ for any X orthogonal to ξ. Thus for any X, we have

$$lX = \frac{\mathrm{tr}l}{2}\varphi^2 X \tag{3.18}$$

If we use (3.18) and substitute it in (3.15) we receive

$$QX = aX + b\eta(X)\xi, \tag{3.19}$$

where $a = \frac{scal - \mathrm{tr}l}{2}$ and $b = \frac{3\mathrm{tr}l - scal}{2}$. Differentiating (3.19) with respect to Y and using (3.19) and the fact that $\nabla_\xi\xi = 0$ we find

$$(\nabla_Y Q)X = (Ya)X + \left[(Yb)\eta(X) + bg(X, \nabla_Y\xi)\right]\xi + b\eta(X)\nabla_Y\xi. \tag{3.20}$$

So using $\xi(\mathrm{tr}l) = 0$ and $\nabla_\xi\xi = 0$, from (3.20) with $X = Y = \xi$, we have $(\nabla_\xi Q)\xi = 0$. Also using $h\varphi = -\varphi h$, and (2.5), from (3.20) with $X = Y$ orthogonal to ξ, we get

$$g((\nabla_X Q)X - (\nabla_{\varphi X}Q)\varphi X, \xi) = 0. \tag{3.21}$$

But it is well known that

$$(\nabla_X Q)X - (\nabla_{\varphi X} Q)\varphi X + (\nabla_\xi Q)\xi = \frac{1}{2}\mathrm{grad}(scal),$$

for any unit vector X orthogonal to ξ. Hence, we easily get from the last two equations that $\xi(scal) = 0$, and thus $\nabla_\xi Q = 0$. Therefore, differentiating (3.13) with respect to ξ and using $\nabla_\xi Q = 0$, we have $\nabla_\xi R = 0$. So from the second identity of Bianchi, we get

$$(\nabla_X R)(Y, \xi, Z) = (\nabla_Y R)(X, \xi, Z). \tag{3.22}$$

Now, substituting (3.19) in (3.13), we obtain

$$R(X, Y)Z = [\gamma g(Y, Z) + b\eta(Y)\eta(Z)]\, X - [\gamma g(X, Z) + b\eta(X)\eta(Z)]\, Y + \tag{3.23}$$
$$+ b\,[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\,\xi,$$

where $\gamma = \frac{scal}{2} - \mathrm{tr}l$. For $Z = \xi$, (3.23) gives

$$R(X, Y)\xi = \frac{\mathrm{tr}l}{2}\,[\eta(Y)X - \eta(X)Y]\,. \tag{3.24}$$

Using (3.24), we obtain $(\nabla_X R)(Y, \xi, \xi) = \frac{X(\mathrm{tr}l)}{2}Y$, for X, Y orthogonal to ξ. From this and (3.22) for $Z = \xi$, we get $X(\mathrm{tr}l)Y = Y(\mathrm{tr}l)X$. Therefore $X(\mathrm{tr}l) = 0$ for X orthogonal to ξ, but $\xi(\mathrm{tr}l) = 0$, so the function $\mathrm{tr}l$ is constant and this completes the proof of the Lemma. □

Remark 1. *When $l = 0$ everywhere, then using* (3.13), (3.14) *and* (3.15) *we get $R(X, Y)\xi = 0$. This together with Theorem 3.3 in [6] gives that M^3 is flat.*

Now we can state the following proposition

Proposition 1. *[5] Let M^3 be a paracontact metric manifold with paracontact metric structure (φ, ξ, η, g). Then the following conditions are equivalent:*
i) M^3 is a $\eta-$Einstein;
ii) $Q\varphi = \varphi Q$;
iii) ξ belongs to the $k-$nullity distribution.

Next we present our main theorem.

Theorem 1. *Let M^3 be a paracontact metric manifold with paracontact metric structure (φ, ξ, η, g) on which $Q\varphi = \varphi Q$. Then M^3 is either a manifold with $trh^2 = 0$, flat or of constant $\xi-$sectional curvature $k \neq -1$ and constant φ-sectional curvature $-k \neq 1$.*

Proof. From the proof of *Lemma* 1 and *Remark* 1 it follows straightforward that if $\mathrm{tr}l = 0$, $l = 0$ it turns out that M^3 is flat. By the proof of *Lemma* 1 it is easy to show that if $k = -1$, then $\mathrm{tr}l = -2$ and by using (2.7), we have that M^3 is a manifold with $\mathrm{tr}h^2 = 0$.

Let us firstly consider the case where $trl \neq 0$. From *Proposition* 1 and (2.12) it follows that

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{3.25}$$

where $k = \frac{\mathrm{tr}l}{2} \neq -1$ and $k \neq 0$. This implies that

$$(\nabla_X \varphi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX) \tag{3.26}$$

as was pointed out by S. Zamkovoy in ([4]). In fact this is true for any 3-dimensional paracontact manifold ([3]). Computing $R(X, Y)\xi$ from (2.5) we receive

$$R(X, Y)\xi = -(\nabla_X \varphi)Y + (\nabla_Y \varphi)X + (\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X =$$

$$= -(\nabla_X\varphi)Y + (\nabla_Y\varphi)X + (\nabla_X\varphi)hY - (\nabla_Y\varphi)hX + \varphi(\nabla_X h)Y - \varphi(\nabla_Y h)X.$$

Then using (3.25) and (3.26) we obtain either

$$k(\eta(Y)X - \eta(X)Y) = \eta(X)(Y - hY) - \eta(Y)(X - hX) + \varphi((\nabla_X h)Y - (\nabla_Y h)X).$$

or

$$(k+1)(\eta(Y)X - \eta(X)Y) = -\eta(X)hY + \eta(Y)hX + \varphi((\nabla_X h)Y - (\nabla_Y h)X). \tag{3.27}$$

Next we consider the case in which $k > -1$. In this case the operator h is diagonalizable (see [1]). Now let X be a unit eigenvector of h (i.e. $|X| = \epsilon_X = \pm 1$), say $hX = \lambda X$, $X \perp \xi$. Since $\mathrm{tr}h^2 = 2(k+1)$, $\lambda = \pm\sqrt{k+1}$ and hence is a constant. Setting $Y = \varphi X$, (3.27) yields

$$\varphi((\nabla_X h)\varphi X - (\nabla_{\varphi X} h)X) = 0.$$

From which we receive the following equation

$$\varphi(-\lambda\nabla_X\varphi X - h\nabla_X\varphi X - \lambda\nabla_{\varphi X}X + h\nabla_{\varphi X}X) = 0. \tag{3.28}$$

Let us recall that $\varphi h + h\varphi = 0$. Now we take the inner product of (3.28) with X and obtain

$$\lambda g(\nabla_{\varphi X}X, \varphi X) = 0.$$

Since $\lambda \neq 0$ ($k > -1$) and X is unit, $\nabla_{\varphi X}X$ is orthogonal to both X and φX and hence collinear with ξ. Now

$$\eta(\nabla_{\varphi X}X) = g(\nabla_{\varphi X}X, \xi) = -g(X, \nabla_{\varphi X}\xi) = -g(-X - hX, X) = \epsilon_X(\lambda + 1).$$

Therefore we receive

$$\nabla_{\varphi X}X = \epsilon_X(\lambda + 1)\xi.$$

Similarly if we take the inner product of (3.28) with φX it follows that

$$\nabla_X\varphi X = \epsilon_X(\lambda - 1)\xi$$

and in turn $\nabla_X X = 0$ and

$$[X, \varphi X] = -2\epsilon_X\xi.$$

Now from the form of the curvature tensor (3.23), we have

$$R(X, \varphi X)\varphi X = -\epsilon_X\left(\frac{scal}{2} - \mathrm{tr}l\right)\varphi X$$

and by direct computation using $\nabla_X\xi = (\lambda - 1)\varphi X$,

$$R(X, \varphi X)X = \nabla_X\nabla_{\varphi X}X - \nabla_{\varphi X}\nabla_X X - \nabla_{[X,\varphi X]}X =$$

$$= \epsilon_X(\lambda + 1)\nabla_X\xi + 2\epsilon_X\nabla_\xi X = \epsilon_X(\lambda^2 - 1)\nabla_X\xi + 2\epsilon_X\nabla_\xi X.$$

Thus

$$\nabla_\xi X = \left(\frac{\lambda^2 - 1}{2} - \frac{scal}{4}\right)\varphi X$$

and hence

$$[\xi, X] = \left(\frac{(\lambda - 1)^2}{2} - \frac{scal}{4}\right)\varphi X.$$

Now computing $R(\xi, X)\xi$, by ((3.25)) and by direct computation, we have

$$-(\lambda^2 - 1)X = \nabla_\xi(-\varphi X + \varphi hX) - \nabla_{\left(\frac{(\lambda-1)^2}{2} - \frac{scal}{4}\right)\varphi X}\xi =$$

$$= (\lambda - 1)\varphi\nabla_\xi X + \left(\frac{(\lambda-1)^2}{2} - \frac{scal}{4}\right)(X + hX) =$$

$$= \left((\lambda-1)^2(\lambda+1) - \lambda\frac{scal}{2}\right)X$$

from which

$$scal = 2(\lambda^2 - 1) = 2k.$$

From (3.25) and (3.23) it follows that for the ξ−sectional curvature $K(X,\xi)$ and φ−sectional curvature $K(X,\varphi X)$ are equal to

$$K(X,\xi) = k \quad \textit{and} \quad K(X,\varphi X) = -k$$

respectively as was desired.

Let us now consider the case in which $k < -1$. As it was shown in (see [1]) the operator φh is diagonalizable. Now let X be a unit eigenvector of φh (i.e. $|X| = \epsilon_X = \pm 1$), say $\varphi hX = \lambda X$, $X \perp \xi$. Since $\mathrm{tr}h^2 = 2(k+1)$, $\lambda = \pm\sqrt{-(k+1)}$ and hence is a constant. We denote $Y = \varphi X$ (3.27). Hence

$$(\nabla_X \varphi h)\varphi X - (\nabla_{\varphi X}\varphi h)X = 0$$

from which we receive

$$-\lambda\nabla_X\varphi X - \varphi h\nabla_X\varphi X - \lambda\nabla_{\varphi X}X + \varphi h\nabla_{\varphi X}X = 0. \tag{3.29}$$

Taking the inner product of (3.29) with φX and recalling that $\varphi h + h\varphi = 0$, we have

$$\lambda g(\nabla_{\varphi X}X, \varphi X) = 0.$$

Since $\lambda \neq 0$ ($k < -1$) and X is unit, $\nabla_{\varphi X}X$ is orthogonal to both X and φX and hence collinear with ξ. Now

$$\eta(\nabla_{\varphi X}X) = g(\nabla_{\varphi X}X, \xi) = -g(X, \nabla_{\varphi X}\xi) = -g(-\varphi^2 X + \varphi h\varphi X, X) = \epsilon_X.$$

Therefore

$$\nabla_{\varphi X}X = \epsilon_X\xi.$$

Similarly taking the inner product of (3.29) with X yields

$$\nabla_X\varphi X = -\epsilon_X\xi$$

and in turn $\nabla_X X = -\epsilon_X\lambda\xi$ and

$$[X, \varphi X] = -2\epsilon_X\xi.$$

Now from the form of the curvature tensor (3.23), we have

$$R(X,\varphi X)\varphi X = -\epsilon_X\left(\frac{scal}{2} - \mathrm{tr}l\right)\varphi X$$

and by direct computation using $\nabla_X\xi = -\varphi X + \lambda X$,

$$R(X,\varphi X)X = \nabla_X\nabla_{\varphi X}X - \nabla_{\varphi X}\nabla_X X - \nabla_{[X,\varphi X]}X.$$

Thus

$$\nabla_\xi X = -\left(\frac{\lambda^2+1}{2} + \frac{scal}{4}\right)\varphi X$$

and hence

$$[\xi, X] = -\lambda X - \left(\frac{\lambda^2-1}{2} + \frac{scal}{4}\right)\varphi X.$$

We compute $R(\xi, X)\xi$, using ((3.25)) and by direct computation, we have

$$(\lambda^2+1)X = \nabla_\xi(-\varphi X + \lambda X) + \lambda\nabla_X\xi + \left(\frac{\lambda^2-1}{2} + \frac{scal}{4}\right)\nabla_{\varphi X}\xi =$$

$$= -\varphi\nabla_\xi X + \lambda\nabla_\xi X + \lambda\nabla_X\xi + \left(\frac{\lambda^2-1}{2} + \frac{scal}{4}\right)(-X - \lambda\varphi X) =$$

$$= (\lambda^2+1)X + \left(-\lambda - 2\lambda\left(\frac{scal}{4} + \frac{\lambda^2}{2}\right)\right)\varphi X$$

from which

$$scal = -2(\lambda^2+1) = 2k.$$

As a conclusion from (3.25) and (3.23) we see that

$$K(X,\xi) = k \quad and \quad K(X,\varphi X) = -k$$

as desired. □

Now we can state the following definition

Definition 1. *A paracontact metric structure* (φ, ξ, η, g) *is said to be* locally $\varphi-$ symmetric *if* $\varphi^2(\nabla_W R)(X, Y, Z) = 0$*, for all vector fields* W, X, Y, Z *orthogonal to* ξ*.*

The next theorem was proved in detail in [5]

Theorem 2. *Let* M^3 *be a paracontact metric manifold with* $Q\varphi = \varphi Q$*. Then* M^3 *is locally* $\varphi-$*symmetric if and only if the scalar curvature scal of* M^3 *is constant.*

Remark 2. *Using* (3.20) *with* $\mathrm{tr}l = const.$, *we obtain the following formula*

$$2|\nabla Q|^2 = |\mathrm{grad} scal|^2 - (3\mathrm{tr}l - scal)^2(4 + \mathrm{tr}l) \tag{3.30}$$

which is valid on any paracontact metric manifold M^3 *with* $Q\varphi = \varphi Q$*.*

From *Theorem* 2, we get the following

Corollary 1. *A locally* $\varphi-$*symmetric paracontact metric manifold* M^3 *is a manifold with either* $scal = 3\mathrm{tr}l$, $scal = -12$ *or* $\mathrm{tr}l = -4$.

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