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Scalar Extensions for Polar Topologies in Locally Convex Cones

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Abstract. We extend the scalar multiplications for dual pairs of cones and define the corresponding modular neighborhoods and linear polar topologies in locally convex cones. Endowed with the polar topology, every cone may be embedded in a larger cone carrying a linear polar topology over the extended scalars and the embedding is an isomorphism.

1. Introduction

In duality theory of locally convex cones, the bilinear mappings of dual pairs required only to be homogenous for positive scalars; however, the polar topologies in this theory may be considered as generalization of their notions in locally convex spaces [1, Ch II, 3]. Many results of functional analysis have been developed for locally convex cones so far; for example, the varies topics of the duality theory can be found in [3–10]. The extension of scalar multiplications over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ have been investigated in the general case of neighborhoods and topologies for locally convex cones in [12]. In this paper, we introduce the notions of extended linear dual pairs and the corresponding modular neighborhoods in locally convex cones which leads to the definition of linear polar topologies over the real or complex scalars. In particular, we prove every polar topology may be embedded in a linear polar topology with the extended scalars such that the embedding is an isomorphism.

A *cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a + b$ and scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a+\beta a$, $\alpha(a+b) = \alpha a+\alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. Equality is such an order, hence cones without an explicit order structure are also included. Note that anti-symmetry is note required for the relation $\leq$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is an ordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars. The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} : b \leq a + v\}$ (respectively, $(a)v = \{b \in \mathcal{P} : a \leq b + v\}$), creating the *upper*, respectively *lower*

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topologies on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$.

For a locally convex cone $(\mathcal{P}, \mathcal{V})$ the collection of all sets $\widetilde{v} \subseteq \mathcal{P}^2$, where $\widetilde{v} = \{(a, b) : a \leq b + v\}$ for all $v \in \mathcal{V}$, defines a *convex quasi-uniform structure* on $\mathcal{P}$. On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including $\mathcal{P}$ as a subcone and induces the same convex quasi-uniform structure. For details see [1, Ch I, 5.2]

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $t : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator*, if $t(a + b) = t(a) + t(b)$ and $t(\alpha a) = \alpha t(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If $\mathcal{V}$ and $\mathcal{W}$ are abstract neighborhood systems on $\mathcal{P}$ and $\mathcal{Q}$, a linear operator $t : \mathcal{P} \to \mathcal{Q}$ is called *uniformly continuous* (*u-continuous*), if for every $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $t(a) \leq t(b) + w$ whenever $a \leq b + v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$. Endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. The set of all u-continuous linear functionals $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is a cone called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$. In a locally convex cone $(\mathcal{P}, \mathcal{V})$ the *polar* v° of $v \in \mathcal{V}$ is defined by $v^\circ = \{\mu \in \mathcal{P}^* : a \leq b + v \text{ implies } \mu(a) \leq \mu(b) + 1\}$. Obviously we have $\mathcal{P}^* = \cup_{v \in \mathcal{V}} v^\circ$.

A linear mapping $\Phi : \mathcal{P} \to \mathcal{Q}$ is called an *embedding* of $(\mathcal{P}, \mathcal{V})$ into $(\mathcal{Q}, \mathcal{W})$ if it can be extended to a mapping $\Phi : \mathcal{P} \cup \mathcal{V} \to \mathcal{Q} \cup \mathcal{W}$ such that $\Phi(\mathcal{V}) = \mathcal{W}$ and

$$a \leq b + v \quad \text{holds if and only if} \quad \Phi(a) \leq \Phi(b) + \Phi(v)$$

for all $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$. This condition implies that Φ is u-continuous, and in case that Φ is one to one, the inverse operator $\Phi^{-1} : \Phi(\mathcal{P}) \to \mathcal{P}$ is also u-continuous. Embeddings are meant to preserve not just the topological structure, but also the particular neighborhood system of a locally convex cone. An embedding Φ of $(\mathcal{P}, \mathcal{V})$ into $(\mathcal{Q}, \mathcal{W})$ is an *isomorphism* if the mapping $\Phi : \mathcal{P} \cup \mathcal{V} \to \mathcal{Q} \cup \mathcal{W}$ is invertible. Then Φ^{-1} is an embedding of $(\mathcal{Q}, \mathcal{W})$ into $(\mathcal{P}, \mathcal{V})$ [11, Ch I, 2.2].

2. Scalar Extensions and Polar Topologies

A *dual pair* $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping $(a, x) \longmapsto \langle a, x \rangle : \mathcal{P} \times \mathcal{Q} \longrightarrow \overline{\mathbb{R}}$, i.e., $\langle \alpha a + \beta b, x \rangle = \alpha \langle a, x \rangle + \beta \langle b, x \rangle$ and $\langle a, \alpha x + \beta y \rangle = \alpha \langle a, x \rangle + \beta \langle a, y \rangle$ for all $a, b \in \mathcal{P}$, $x, y \in \mathcal{Q}$ and $\alpha, \beta \geq 0$. For example, if $\mathcal{P}$ is a cone and $\mathcal{L}(\mathcal{P}, \overline{\mathbb{R}})$ is the cone of all $\overline{\mathbb{R}}$-valued linear mappings on $\mathcal{P}$, then with evaluation as the bilinear mapping on $\mathcal{P} \times \mathcal{L}(\mathcal{P}, \overline{\mathbb{R}})$, $(\mathcal{P}, \mathcal{L}(\mathcal{P}, \overline{\mathbb{R}}))$ forms a dual pair; in particular, for a locally convex cone $(\mathcal{P}, \mathcal{V})$ the pair $(\mathcal{P}, \mathcal{P}^*)$ is a dual pair, where $\mathcal{P}^*$ is the dual cone of $\mathcal{P}$. Suppose $(\mathcal{P}, \mathcal{Q})$ is a dual pair and X a collection of subsets of $\mathcal{Q}$ such that:

(p_0) $\inf \langle A, x \rangle > -\infty$ for all $A \in X$ and $x \in \mathcal{P}$.
(p_1) $\lambda A \in X$ for all $A \in X$ and $\lambda > 0$.
(p_2) for all $A, B \in X$ there is $C \in X$ with $A \cup B \subseteq C$.

If, for each $A \in X$, we define

$$\mathrm{U}_A = \{(a, b) \in \mathcal{P} \times \mathcal{P} : \langle a, x \rangle \leq \langle b, x \rangle + 1 \text{ for all } x \in A\},$$

then the set of all $\mathrm{U}_A, A \in X$ forms a convex quasi-uniform structure with property (U5) in [1, Ch I, 5.2]. For every $A \in X$, we set $a \leq b + v_A$ for all $a, b \in \mathcal{P}$ if and only if $(a, b) \in \mathrm{U}_A$ and put $\mathcal{V}_X = \{v_A : A \in X\}$. Then, according to [1, Ch I, 5.4], there exists a full cone $\mathcal{P} \oplus \mathcal{V}_{X_0}$ with abstract neighborhood system $\mathrm{V}_X = \{0\} \oplus \mathcal{V}_X$, whose neighborhoods yield the same convex quasi-uniform structure on $\mathcal{P}$. The elements $v_A \in \mathcal{V}_X$ for all $A \in X$ form a basis for V_X in the following sense: For every $A \in X$, $a \leq b + v_A$ for $a, b \in \mathcal{P}$ implies that $a \leq b \oplus \mathrm{v}_A$. The locally convex cone topology on $\mathcal{P}$ induced by $\mathcal{V}_X$ is called the *X-topology* or (*polar topology*) on $\mathcal{P}$.

A cone $\mathcal{P}$ is said to be *linear over* $\mathbb{K}$ with respect to the multiplication operation $\odot$, if $\odot$ extends the scalar multiplication operation of $\mathcal{P}$ over $\mathbb{K}$ and to the requirements for a cone satisfies $\alpha \odot (a + b) = \alpha \odot a + \alpha \odot b$, $\alpha \odot (\beta \odot a) = (\alpha\beta) \odot a$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \in \mathbb{K}$ [12]. We say that the dual pair $(\mathcal{P}, \mathcal{Q})$ is *linear over* $\mathbb{K}$, if the following conditions hold:

($\mathbb{K}_1$) $\mathcal{P}$ and $\mathcal{Q}$ are linear cones over $\mathbb{K}$.
($\mathbb{K}_2$) $\langle \alpha \odot a, x \rangle = \langle a, \alpha \odot x \rangle$ for all $a \in \mathcal{P}, x \in \mathcal{Q}$ and $\alpha \in \mathbb{K}$.
($\mathbb{K}_3$) for all $a \in \mathcal{P}$ and $x \in \mathcal{Q}$,

$$\langle \alpha \odot a, x \rangle = \begin{cases} \Re(\alpha)\langle a, x\rangle + \Im(\alpha)\langle i \odot a, x\rangle & \text{if} \quad \langle a, x\rangle < +\infty,\ \alpha \in \mathbb{K}, \\ +\infty & \text{if} \quad \langle a, x\rangle = +\infty,\ 0 \neq \alpha \in \mathbb{K}, \end{cases}$$

where $\Re(\alpha)$, $\Im(\alpha)$ denote the real and imaginary parts of α, respectively. We note that a linear dual pair over $\mathbb{K}$ carries a *weakened version* of bilinear mapping, i.e., the bilinear mapping is not necessarily homogeneous for all scalars in $\mathbb{K}$, nor does it have the distribution property.

Proposition 2.1. *If $(\mathcal{P}, \mathcal{Q})$ is a linear dual pair over $\mathbb{K}$, then*

$$\langle (\alpha + \beta) \odot a, x \rangle \leq \langle \alpha \odot a, x \rangle + \langle \beta \odot a, x \rangle$$

for all $\alpha, \beta \in \mathbb{K}, a \in \mathcal{P}, x \in \mathcal{Q}$.

Proof. If $\langle a, x\rangle < +\infty$ then $\langle (\alpha + \beta) \odot a, x \rangle = \langle \alpha \odot a, x \rangle + \langle \beta \odot a, x \rangle$ by ($\mathbb{K}_3$). If $\langle a, x\rangle = +\infty$ then from ($\mathbb{K}_3$) we have $\langle \alpha \odot a, x \rangle = \langle \beta \odot a, x \rangle = \langle (\alpha + \beta) \odot a, x \rangle = +\infty$. Then for α, β, if $\alpha = -\beta$, then $0 = \langle (\alpha + \beta) \odot a, x \rangle < \langle \alpha \odot a, x \rangle + \langle \beta \odot a, x \rangle = +\infty$ and if $\alpha \neq -\beta$, then $\langle (\alpha + \beta) \odot a, x \rangle = \langle \alpha \odot a, x \rangle + \langle \beta \odot a, x \rangle = +\infty$. □

Let $(\mathcal{P}, \mathcal{Q})$ be linear over $\mathbb{K}$, X a collection of subsets of $\mathcal{Q}$ satisfying (p_0), (p_1), (p_2) with respect to $(\mathcal{P}, \mathcal{Q})$ and $\Gamma = \{\gamma \in \mathbb{K} : |\gamma| = 1\}$. The $\mathbb{K}$-*modular collection* $\Gamma \odot X$ *of* X, consisting of the all $\mathbb{K}$-*modular sets* $\Gamma \odot A = \{\gamma \odot x : \gamma \in \Gamma, x \in A\}$ for all $A \in X$, satisfies $(p_1), (p_2)$ with respect to $(\mathcal{P}, \mathcal{Q})$. If $\Gamma \odot X$ also satisfies in (p_0), then the set of all

$$\mathrm{U}_{\Gamma\odot A} = \{(a, b) \in \mathcal{P} \times \mathcal{P} : \langle a, \gamma \odot x\rangle \leq \langle b, \gamma \odot x\rangle + 1 \text{ for all } x \in A, \gamma \in \Gamma\},$$

for all $A \in X$ forms a convex quasi-uniform structure with property (U5) in [1, Ch I, 5.2]. For every $A \in X$, we set $a \leq b + v_{\Gamma\odot A}$ for all $a, b \in \mathcal{P}$ if and only if $(a, b) \in \mathrm{U}_{\Gamma\odot A}$ and put $\mathcal{V}_{\Gamma\odot X} = \{v_{\Gamma\odot A} : A \in X\}$. Then, according to [1, Ch I, 5.4], there exists a full cone $\mathcal{P} \oplus \mathcal{V}_{\Gamma\odot X_0}$ with abstract neighborhood system $\mathrm{V}_{\Gamma\odot X} = \{0\} \oplus \mathcal{V}_{\Gamma\odot X}$, whose neighborhoods yield the same convex quasi-uniform structure on $\mathcal{P}$. The elements $v_{\Gamma\odot A} \in \mathcal{V}_{\Gamma\odot X}$ for all $A \in X$ form a basis for $\mathrm{V}_{\Gamma\odot X}$ in the following sense: For every $A \in X$, a $\leq b + v_{\Gamma\odot A}$ for $a, b \in \mathcal{P}$ implies that $a \leq b \oplus \mathrm{v}_{\Gamma\odot A}$. In this case, we say that $(\mathcal{P}, \mathcal{V}_X)$ is a *linear polar topology over* $\mathbb{K}$ and call $(\mathcal{P}, \mathcal{V}_{\Gamma\odot X})$ the $\mathbb{K}$-*modular topology* of $(\mathcal{P}, \mathcal{V}_X)$.

Lemma 2.2. *If $\mathcal{P}$ is a cone, then*

(a) $\mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ and $\mathcal{P}^2 + i \odot \mathcal{P}^2$ with the usual addition and scalar multiplication are cones; where $\mathcal{P}^2 + i \odot \mathcal{P}^2 = \{a + i \odot b : a, b \in \mathcal{P}^2\}$,
(b) if for elements $a \in \mathcal{P}^2$, $a = (a_1, a_2)$ and negative reals $\alpha < 0$, we extend the scalar multiplication to $\mathbb{R}$ by $\alpha \odot a = (-\alpha)\breve{a}$; where $\breve{a} = (a_2, a_1)$ then $\mathcal{P}^2$ is a linear cone over $\mathbb{R}$,
(c) if for all $\alpha \in \mathbb{C}$ and $a + i \odot b \in \mathcal{P}^2 + i \odot \mathcal{P}^2$, we extend the scalar multiplication to $\mathbb{C}$ by

$$\alpha \odot (a + i \odot b) = \Re(\alpha) \odot a + (-1)\Im(\alpha) \odot b + i \odot (\Re(\alpha) \odot b + \Im(\alpha) \odot a),$$

then $\mathcal{P}^2 + i \odot \mathcal{P}^2$ is a liner cone over $\mathbb{C}$.

Proof. The proof is clear. □

Lemma 2.3. *If $(\mathcal{P}, \mathcal{Q})$ is a dual pair, then*

(a) with the bilinear mapping $(a, x) \to \langle a, x\rangle_{\mathbb{R}} : \mathcal{P}^2 \times \mathcal{Q}^2 \longrightarrow \overline{\mathbb{R}}$ for all $a \in \mathcal{P}^2, x \in \mathcal{Q}^2$, $a = (a_1, a_2)$, $x = (x_1, x_2)$ such that

$$\langle a, x\rangle_{\mathbb{R}} = \begin{cases} \langle a_1, x_1\rangle + \langle a_2, x_2\rangle - \langle a_1, x_2\rangle - \langle a_2, x_1\rangle & \textit{if } \langle a_2, x_1\rangle, \langle a_1, x_2\rangle < +\infty; \\ +\infty & \textit{otherwise;} \end{cases}$$

$(\mathcal{P}^2, \mathcal{Q}^2)$ is a linear dual pair over $\mathbb{R}$,

(b) *with the bilinear mapping*

$$(a + i \odot b, x + i \odot y) \longmapsto \langle a + i \odot b, x + i \odot y\rangle_{\mathbb{C}} : (\mathcal{P}^2 + i \odot \mathcal{P}^2) \times (Q^2 + i \odot Q^2) \to \overline{\mathbb{R}}, \textit{ where}$$

$$\langle a + i \odot b, x + i \odot y\rangle_{\mathbb{C}} = \langle a + b, x + y\rangle_{\mathbb{R}}$$

$(\mathcal{P}^2 + i \odot \mathcal{P}^2, Q^2 + i \odot Q^2)$ *is a linear dual pair over* $\mathbb{C}$.

Proof. (a) It is easy to see that $\langle a, x\rangle_{\mathbb{R}} = \langle \breve{a}, \breve{x}\rangle_{\mathbb{R}}$ for all $a \in \mathcal{P}^2$, $x \in Q^2$ and

$$\langle a, \breve{x}\rangle_{\mathbb{R}} = \langle \breve{a}, x\rangle_{\mathbb{R}} = \begin{cases} -\langle a, x\rangle_{\mathbb{R}} & \text{if} \quad \langle a, x\rangle_{\mathbb{R}} < +\infty, \\ +\infty & \text{if} \quad \langle a, x\rangle_{\mathbb{R}} = +\infty. \end{cases}$$

The condition $(\mathbb{R}_1)$ is clear for $(\mathcal{P}^2, Q^2)$. For $(\mathbb{R}_2)$, let $\alpha < 0$. If $\langle a, x\rangle_{\mathbb{R}} < +\infty$ then

$$\langle \alpha \odot a, x\rangle_{\mathbb{R}} = \langle (-\alpha)\breve{a}, x\rangle_{\mathbb{R}} = (-\alpha)\langle \breve{a}, x\rangle_{\mathbb{R}} = (-\alpha)\langle a, \breve{x}\rangle_{\mathbb{R}} = \langle a, \alpha \odot x\rangle_{\mathbb{R}},$$

and if $\langle a, x\rangle_{\mathbb{R}} = +\infty$ then

$$\langle \alpha \odot a, x\rangle_{\mathbb{R}} = \langle (-\alpha)\breve{a}, x\rangle_{\mathbb{R}} = (-\alpha)\langle a, x\rangle_{\mathbb{R}} = +\infty,$$

$$\langle a, \alpha \odot x\rangle_{\mathbb{R}} = \langle a, (-\alpha)\breve{x}\rangle_{\mathbb{R}} = (-\alpha)\langle a, x\rangle_{\mathbb{R}} = +\infty.$$

For $(\mathbb{R}_3)$, if $\langle a, x\rangle_{\mathbb{R}} < +\infty$ then

$$\langle \alpha \odot a, x\rangle_{\mathbb{R}} = \langle (-\alpha)\breve{a}, x\rangle_{\mathbb{R}} = (-\alpha)\langle \breve{a}, x\rangle_{\mathbb{R}} = \alpha\langle a, x\rangle_{\mathbb{R}}$$

and if $\langle a, x\rangle_{\mathbb{R}} = +\infty$ then $\langle \alpha \odot a, x\rangle_{\mathbb{R}} = \langle (-\alpha)\breve{a}, x\rangle_{R} = (-\alpha)\langle \breve{a}, x\rangle_{\mathbb{R}} = +\infty$. Thus $(\mathcal{P}^2, Q^2)$ is a linear dual pair over $\mathbb{R}$.

(b) The condition $(\mathbb{C}_1)$ is clear. For $(\mathbb{C}_2)$, if $a + i \odot b \in \mathcal{P}^2 + i \odot \mathcal{P}^2$, $x + i \odot y \in Q^2 + i \odot Q^2$ and $\alpha \in \mathbb{C}$, then

$$\begin{aligned}
& \langle \alpha \odot (a + i \odot b), (x + i \odot y)\rangle_{\mathbb{C}} \\
= \ & \langle \Re(\alpha) \odot a + (-1)\Im(\alpha) \odot b + i \odot (\Re(\alpha) \odot b + \Im(\alpha) \odot a),\ x + iy\rangle_{\mathbb{C}} \\
= \ & \langle \Re(\alpha) \odot a + (-1)\Im(\alpha) \odot b + \Re(\alpha) \odot b + \Im(\alpha) \odot a,\ x + y\rangle_{\mathbb{R}} \\
= \ & \langle a + b, \Re(\alpha) \odot x + (-1)\Im(\alpha) \odot y + \Re(\alpha) \odot y + \Im(\alpha) \odot x\rangle_{\mathbb{R}} \\
= \ & \langle a + i \odot b, \Re(\alpha) \odot x + (-1)\Im(\alpha) \odot y + i \odot (\Re(\alpha) \odot y + \Im(\alpha) \odot x)\rangle_{\mathbb{C}} \\
= \ & \langle (a + i \odot b), \alpha \odot (x + i \odot y)\rangle_{\mathbb{C}}
\end{aligned}$$

For $(\mathbb{C}_3)$, if $\langle a + i \odot b, x + i \odot y\rangle_{\mathbb{C}} < +\infty$, then $\langle a + b, x + y\rangle_{\mathbb{R}} < +\infty$, so

$$\begin{aligned}
& \langle \alpha \odot (a + i \odot b),\ x + i \odot y\rangle_{\mathbb{C}} \\
= \ & \langle \Re(\alpha) \odot a + (-1)\Im(\alpha) \odot b + \Re(\alpha) \odot b + \Im(\alpha) \odot a,\ x + y\rangle_{\mathbb{R}} \\
= \ & \Re(\alpha)\langle a + b, x + y\rangle_{\mathbb{R}} + \Im(\alpha)\langle a + (-1) \odot b, x + y\rangle_{\mathbb{R}} \\
= \ & \Re(\alpha)\langle a + i \odot b, x + i \odot y\rangle_{\mathbb{C}} + \Im(\alpha)\langle i \odot (a + i \odot b), x + i \odot y\rangle_{\mathbb{C}}
\end{aligned}$$

and if $\langle a + i \odot b, x + i \odot y\rangle_{\mathbb{C}} = +\infty$ then, either

$$\langle \Re(\alpha) \odot b + \Im(\alpha) \odot a, x + i \odot y\rangle_{\mathbb{C}} = \langle a + i \odot b, \Re(\alpha) \odot y + \Im(\alpha) \odot x\rangle_{\mathbb{C}} = +\infty$$

or

$$\langle \Re(\alpha) \odot a + (-1)\Im(\alpha) \odot b, x + i \odot y\rangle_{\mathbb{C}} = \langle a + i \odot b, \Re(\alpha) \odot x + (-1)\Im(\alpha) \odot y\rangle_{\mathbb{C}} = +\infty.$$

So $\langle \alpha \odot (a + i \odot b),\ x + i \odot y\rangle_{\mathbb{C}} = +\infty$. Thus $(\mathcal{P}^2 + i \odot \mathcal{P}^2, Q^2 + i \odot Q^2)$ is a linear dual pair over $\mathbb{C}$. □

Theorem 2.4. *If* $(\mathcal{P}, Q)$ *is a dual pair and* $\mathcal{P}$ *has an X-topology with respect to* $(\mathcal{P}, Q)$, *then* *(a)* $\widehat{X}$ *and* $\widehat{X} + i \odot \widehat{X}$ *satisfy* $(p_0), (p_1), (p_2)$ *with respect to the dual pairs* $(\mathcal{P}^2, Q^2)$ *and* $(\mathcal{P}^2 + i \odot \mathcal{P}^2, Q^2 + i \odot Q^2)$*; respectively, where* $\widehat{X} = \{\hat{A} : A \in X\}$, $\hat{A} = A \times \{0\}$ *for all* $A \in X$,

(b) $(\mathcal{P}^2, \mathcal{V}_{\widehat{X}})$ *and* $(\mathcal{P}^2 + i \odot \mathcal{P}^2, \mathcal{V}_{\widehat{X}+i\odot\widehat{X}})$ *are linear polar topologies over* $\mathbb{R}$ *and* $\mathbb{C}$*, respectively,*

(c) $(\mathcal{P}, \mathcal{V}_X)$ *may be embedded in both the* $(\mathcal{P}^2, \mathcal{V}_{\widehat{X}})$ *and* $(\mathcal{P}^2 + i\odot\mathcal{P}^2, \mathcal{V}_{\widehat{X}+i\odot\widehat{X}})$ *and the embeddings are isomorphism.*

Proof. (a) If $a \in \mathcal{P}^2$, $a = (a_1, a_2)$ and $\hat{A} \in \widehat{X}$, $\hat{A} = A \times \{0\}$, then

$$\inf\langle a, \hat{A}\rangle_{\mathbb{R}} \geq \inf\langle (a_1, a_2), A_a \times \{0\}\rangle_{\mathbb{R}} = \inf\langle a_1, A_a\rangle - \inf\langle a_2, A_a\rangle > -\infty,$$

where $A_a = A \cap (a_1^{-1}(\mathbb{R}) \cup a_2^{-1}(\mathbb{R}))$, i.e., (p_0) holds for $\widehat{X}$. If $a + i \odot b \in \mathcal{P}^2 + i \odot \mathcal{P}^2$, $a = (a_1, a_2), b = (b_1, b_2)$ then

$$\begin{aligned} \inf\langle a + i \odot b, \hat{A} + i \odot \hat{A}\rangle_{\mathbb{C}} &= \inf\langle (a + i \odot b, A \times \{0\} + i \odot (A \times \{0\})\rangle_{\mathbb{C}} \\ &= 2\inf\langle a, A \times \{0\}\}_{\mathbb{R}} + 2\inf\{\langle b, A \times \{0\}\rangle_{\mathbb{R}} \\ &> -\infty, \end{aligned}$$

i.e., $\widehat{X} + i \odot \widehat{X}$ also satisfies in (p_0). The properties $(p_1), (p_2)$ are clear for $\widehat{X}$ and $\widehat{X} + i \odot \widehat{X}$.

(b) If $a \in \mathcal{P}^2$ and $\hat{A} \in \widehat{X}$, then

$$\begin{aligned} \inf\langle a, \Gamma \odot \hat{A}\rangle_{\mathbb{R}} &= \inf\langle a, \{\pm 1\} \odot \hat{A}\rangle_{\mathbb{R}} \\ &= \min\left\{\inf\langle a, (-1) \odot \hat{A}\rangle_{\mathbb{R}},\ \inf\langle a, \hat{A}\rangle_{\mathbb{R}}\right\} \\ &\geq \min\{\inf\langle \breve{a}, A_a \times \{0\}\rangle,\ \inf\langle a, A_a \times \{0\}\rangle\} > -\infty, \end{aligned}$$

and if $a + i \odot b \in \mathcal{P}^2 + i \odot \mathcal{P}^2$ then

$$\begin{aligned} &\inf\langle a + i \odot b, \Gamma \odot (\hat{A} + i \odot \hat{A})\rangle_{\mathbb{C}} \\ =\ &\inf\langle a + i \odot b,\ \Re(\Gamma) \odot \hat{A} + (-1)\Im(\Gamma) \odot \hat{A} + \Re(\Gamma) i \odot \hat{A} + \Im(\Gamma) i \odot \hat{A}\rangle_{\mathbb{C}} \\ =\ &\inf\langle a + b,\ \Re(\Gamma) \odot \hat{A} + (-1)\Im(\Gamma) \odot \hat{A} + \Re(\Gamma) \odot \hat{A} + \Im(\Gamma) \odot \hat{A}\rangle_{\mathbb{R}} \\ =\ &2\inf\langle a, \Re(\Gamma) \odot \hat{A}\rangle + \inf\langle \breve{a}, \Im(\Gamma) \odot \hat{A}\rangle + \inf\langle a, \Im(\Gamma) \odot \hat{A}\rangle + \\ &2\inf\langle b, \Re(\Gamma) \odot \hat{A}\rangle + \inf\langle b, \Im(\Gamma) \odot \hat{A}\rangle + \inf\langle \breve{b}, \Im(\Gamma) \odot \hat{A}\rangle \\ =\ &2\inf\left(|\Re(\Gamma)|\, \langle a, \{\pm 1\} \odot \hat{A}\rangle\right) + \inf\left(|\Im(\Gamma)|\, \langle \breve{a}, \{\pm 1\} \odot \hat{A}\rangle\right) + \inf\left(|\Im(\Gamma)|\, \langle a, \{\pm 1\} \odot \hat{A}\rangle\right) + \\ &2\inf\left(|\Re(\Gamma)|\, \langle b,\ \{\pm 1\} \odot \hat{A}\rangle\right) + \inf\left(|\Re(\Gamma)|\, \langle b,\ (\pm 1) \odot \hat{A}\rangle\right) + \inf\left(|(\Re(\Gamma))|\, \langle \breve{b},\ \{\pm 1\} \odot \hat{A}\rangle\right) \\ >\ &-\infty, \end{aligned}$$

where $\Re(\Gamma) = \{\Re(\gamma) : \gamma \in \Gamma\}$ and $|\Re(\Gamma)| = \{|\Re(\gamma)| : \gamma \in \Gamma\}$. That is, $\Gamma \odot \widehat{X}$ and $\Gamma \odot (\widehat{X} + i \odot \widehat{X})$ satisfy in (p_0).

(c) The mappings $\varphi_{\mathbb{R}}$ and $\varphi_{\mathbb{C}}$ such that

$$(\mathcal{P}, \mathcal{V}_X) \overset{\varphi_{\mathbb{R}}}{\to} (\mathcal{P} \times \{0\}, \mathcal{V}_{\widehat{X}}) \overset{\varphi_{\mathbb{C}}}{\to} (\mathcal{P}^2 + i \odot \mathcal{P}^2, \mathcal{V}_{\widehat{X}+i\odot\widehat{X}}),$$

$$p \overset{\varphi_{\mathbb{R}}}{\to} (p, 0) \overset{\varphi_{\mathbb{C}}}{\to} (p, 0) + i \odot (p, 0)$$

are linear, one-to-one and for all $p, p' \in \mathcal{P}$ and $A \in X$, we have $p \leq p' + v_A$ if and only if $(p, 0) \leq (p', 0) + v_{\widehat{A}}$ if and only if $(p, 0) + i \odot (p, 0) \leq (p', 0) + i \odot (p', 0) + v_{\widehat{A}+i\odot\widehat{A}}$, that is, $\varphi_{\mathbb{R}}$, $\varphi_{\mathbb{C}}$ and $\varphi_{\mathbb{C}} \circ \varphi_{\mathbb{R}}$ are isomorphism. □

Example 2.5. The usual algebraic operations of $\mathbb{C}$ may be extended on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $\alpha \odot \infty = \infty$ for all $0 \neq \alpha \in \mathbb{C}$ and $0 \odot \infty = 0$ and clearly $\overline{\mathbb{C}}$ is linear over $\mathbb{K}$. Likewise, $\overline{\mathbb{R}}$ is a linear cone over $\mathbb{R}$ with the following extended operation: $\alpha \odot (+\infty) = +\infty$ for all $0 \neq \alpha \in \mathbb{R}$ and $0 \odot (+\infty) = +\infty$. Now, if we set $\mathcal{P} = \overline{\mathbb{K}}$ and $Q = \overline{\mathbb{R}}$ then with the bilinear mapping

$$\langle a, b\rangle = \begin{cases} \Re(ab) & \text{if} \quad a \in \mathbb{K},\ b \in \mathbb{R}, \\ +\infty & \text{if} \quad a = \infty \text{ or } b = +\infty, \end{cases}$$

$(\mathcal{P}, \mathcal{Q})$ is a linear dual pair over $\mathbb{R}$. The collection $X = \{\lambda \mathbf{1} : \lambda > 0\}$ where $\mathbf{1} := \{1\}$ induces a polar topology on $\mathcal{P}$ and we have $a \leq b + \lambda v_{\mathbf{1}}$ for $a, b \in \mathcal{P}$ and $\lambda > 0$ if and only if $\Re(a) \leq \Re(b) + \lambda$ or $b = +\infty$. Evidently, (p_0) holds for the modular collection $\Gamma \odot X = \{\lambda(\Gamma \odot \mathbf{1}) : \lambda > 0\}$, where $\Gamma \odot \mathbf{1} = \{\pm 1\}$. For $a, b \in \mathcal{P}$ and $\lambda > 0$, the modular neighborhoods defined by $a \leq_{\mathbb{R}} b + \lambda v_{\Gamma \odot \mathbf{1}}$ if and only if

$$|\Re(a) - \Re(b)| \leq \lambda \quad \text{or} \quad b = \infty.$$

Example 2.6. The set of all sequences in $\overline{\mathbb{K}}$ with the usual addition and extended scalar multiplication defined by $\alpha \odot (z_n)_{n\in\mathbb{N}} = (\alpha \odot z_n)_{n\in\mathbb{N}}$ for all $\alpha \in \mathbb{K}$ is a linear cone over $\mathbb{K}$. For a sequence $z = (z_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{K}}$, we define the $\overline{\ell}_1$-norm of z by

$$\|z\|_1 = \begin{cases} \sum_{n=1}^{\infty} |z_n| & \text{if} \quad z \subset \mathbb{K}, \\ +\infty & \text{if} \quad \exists n \in \mathbb{N},\ z_n = \infty. \end{cases}$$

Let $\overline{\ell}_1$ be the cone of all sequences $(z_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{K}}$ such that $\|(z_n)_{n\in\mathbb{N}}\|_1 < \infty$ and let $\overline{\ell}_\infty$ be the cone of all $(t_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{K}}$ such that $\|(z_n t_n)_{n\in\mathbb{N}}\|_1 < \infty$ for all $(z_n)_{n\in\mathbb{N}} \in \overline{\ell}_1$; in fact, $\overline{\ell}_\infty$ consists of all sequences $t = (t_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{K}}$ such that $\|t\|_\infty < \infty$, where $\|t\|_\infty$, denotes the $\overline{\ell}_\infty$-norm of t and defined by

$$\|t\|_\infty = \begin{cases} \sup_{n\in\mathbb{N}} |t_n| & \text{if} \quad t \subset \mathbb{K}, \\ +\infty & \text{if} \quad \exists n \in \mathbb{N}, t_n = \infty. \end{cases}$$

With the bilinear mapping $(z, t) \to \langle z, t\rangle : \overline{\ell}_1 \times \overline{\ell}_\infty \to \overline{\mathbb{R}}$ for all $z \in \overline{\ell}_1, t \in \overline{\ell}_\infty, z = (z_n)_{n\in\mathbb{N}}, t = (t_n)_{n\in\mathbb{N}}$ defined by

$$\langle z, t\rangle = \begin{cases} \sum_{n=1}^{\infty} \Re(z_n t_n) & \text{if} \quad z, t \subset \mathbb{K}, \\ +\infty & \text{if} \quad \exists n \in \mathbb{N},\ z_n = \infty \text{ or } t_n = \infty, \end{cases}$$

$(\overline{\ell}_1, \overline{\ell}_\infty)$ is a linear dual pair over $\mathbb{K}$. If we set $X = \{\lambda e : \lambda > 0\}$, where $e = \{(1, 1, ...)\}$, then X is a collection of subsets in $\overline{\ell}_\infty$ satisfying $(p_0), (p_1), (p_2)$ with respect to $(\overline{\ell}_1, \overline{\ell}_\infty)$, so $(\overline{\ell}_1, \mathcal{V}_X)$ is a polar topology. For elements $z, z' \in \overline{\ell}_1, z = (z_n)_{n\in\mathbb{N}}, z' = (z'_n)_{n\in\mathbb{N}}$ and $\lambda > 0$ we have $z \leq z' + \lambda v_e$ if and only if

$$\sum_{n\in\mathbb{N}} \Re(z_n) \leq \sum_{n\in\mathbb{N}} \Re(z'_n) + \lambda \quad \text{or} \quad \exists n \in \mathbb{N},\ z'_n = \infty.$$

We note that the dual cone of $\overline{\ell}_1$ under $\mathcal{V}_X$ is $\overline{\ell}_\infty$ (cf.[2, 2.10-6]). The modular collection $\Gamma \odot X = \{\Gamma \odot \lambda e : \lambda > 0\}$ satisfies in (p_0), where $\Gamma \odot e = \{(-1, -1, ...), (1, 1, ...)\}$ for $\mathbb{K} = \mathbb{R}$ and $\Gamma \odot e = \{(\gamma, \gamma, ...) : \gamma \in \Gamma\}$ for $\mathbb{K} = \mathbb{C}$ and the corresponding $\mathbb{C}$-modular neighborhoods defined by $z \leq_{\mathbb{C}} z' + \lambda\, v_{\Gamma \odot e}$ if and only if

$$|\sum_{n\in\mathbb{N}} \Re(z_n) - \sum_{n\in\mathbb{N}} \Re(z'_n)| \leq \lambda \quad \text{and} \quad |\sum_{n\in\mathbb{N}} \Im(z_n) - \sum_{n\in\mathbb{N}} \Im(z'_n)| \leq \lambda$$

or $z'_n = \infty$ for some $n \in \mathbb{N}$. Thus $(\overline{\ell}_1, \mathcal{V}_X)$ is a linear polar topology over $\mathbb{K}$ with respect to the dual pair $(\overline{\ell}_1, \overline{\ell}_\infty)$.

Proposition 2.7. *If $(\mathcal{P}, \mathcal{Q})$ is a linear dual pair over $\mathbb{K}$, $\mathrm{P} = Conv(\mathcal{P})$ and $\mathrm{Q} = Conv(\mathcal{Q})$ then with the bilinear mapping $(\mathrm{p}, \mathrm{q}) \to \langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle : \mathrm{P} \times \mathrm{Q} \to \overline{\mathbb{R}}$, where*

$$\langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle = \begin{cases} \inf\langle \mathrm{p}, \mathrm{q}\rangle + \sup\langle \mathrm{p}, \mathrm{q}\rangle & \textit{if} \quad \inf\langle \mathrm{p}, \mathrm{q}\rangle > -\infty, \\ +\infty & \textit{if} \quad \inf\langle \mathrm{p}, \mathrm{q}\rangle = -\infty, \end{cases}$$

(P, Q) is a linear dual pair over $\mathbb{K}$, where $\langle \mathrm{p}, \mathrm{q}\rangle = \{\langle x, y\rangle : x \in \mathrm{p}, y \in \mathrm{q}\}$.

Proof. The condition ($\mathbb{R}_1$) is clear. For ($\mathbb{R}_2$), let $\alpha < 0$. If $\inf\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle = -\infty$ then $\inf\langle \mathrm{p}, \alpha \odot \mathrm{q}\rangle = -\infty$ so $\langle\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle\rangle = \langle\langle \mathrm{p}, \alpha \odot \mathrm{q}\rangle\rangle = +\infty$ and if $\inf\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle > -\infty$ then

$$\begin{aligned}\langle\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle\rangle &= \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \alpha \odot x, y\rangle + \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \alpha \odot x, y\rangle \\ &= \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle x, \alpha \odot y\rangle + \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle x, \alpha \odot y\rangle = \langle\langle \mathrm{p}, \alpha \odot \mathrm{q}\rangle\rangle,\end{aligned}$$

since $\inf\langle \mathrm{p}, \alpha \odot \mathrm{q}\rangle > -\infty$. For ($\mathbb{R}_3$), if $\langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle < +\infty$ then $\inf\langle \mathrm{p}, \mathrm{q}\rangle > -\infty$ and $\sup\langle \mathrm{p}, \mathrm{q}\rangle < +\infty$, whence

$$\langle\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle\rangle = \inf\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle + \sup\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle = \alpha \sup\langle \mathrm{p}, \mathrm{q}\rangle + \alpha \inf\langle \mathrm{p}, \mathrm{q}\rangle = \alpha\langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle.$$

If $\langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle = +\infty$ then $\inf\langle \alpha\mathrm{p}, \mathrm{q}\rangle = \alpha \sup\langle \mathrm{p}, \mathrm{q}\rangle = -\infty$ or $\sup\langle \alpha\mathrm{p}, \mathrm{q}\rangle = \alpha \inf\langle \mathrm{p}, \mathrm{q}\rangle = +\infty$, so $\langle\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle\rangle = +\infty$. That is, (P, Q) is a linear dual pair over $\mathbb{R}$. For ($\mathbb{C}_3$), let $\alpha \in \mathbb{C}$, $\Re(\alpha) < 0$ and $\Im(\alpha) > 0$. If $\langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle < +\infty$, then

$$\begin{aligned}\langle\langle \alpha \odot \mathrm{p}, \mathrm{q}\rangle\rangle &= \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \alpha \odot x, y\rangle + \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \alpha \odot x, y\rangle \\ &= \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \Re(\alpha) \odot x, y\rangle + \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \Im(\alpha) i \odot x, y\rangle + \\ &\quad \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \Re(\alpha) \odot x, y\rangle + \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle \Im(\alpha) i \odot x, y\rangle \\ &= \Re(\alpha) \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle x, y\rangle + \Im(\alpha) \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle i \odot x, y\rangle + \\ &\quad \Re(\alpha) \inf_{x\in \mathrm{p}, y \in \mathrm{q}} \langle x, y\rangle + \Im(\alpha) \sup_{x\in \mathrm{p}, y \in \mathrm{q}} \langle i \odot x, y\rangle \\ &= \Re(\alpha)\langle\langle \mathrm{p}, \mathrm{q}\rangle\rangle + \Im(\alpha)\langle\langle i \odot \mathrm{p}, \mathrm{q}\rangle\rangle.\end{aligned}$$

In the similar way, we can veryfy the three othere cases on $\Re(\alpha)$ and $\Im(\alpha)$. The conditions ($\mathbb{C}_1$) and ($\mathbb{C}_2$) are clear. □

Example 2.8. (i) If we set $\mathrm{P} = Conv(\overline{\mathbb{K}})$ and $\mathrm{Q} = Conv(\overline{\mathbb{R}})$ then $\mathrm{X} = \{\lambda A : \lambda > 0\}$, where $A := \{\{1\}\}$ satisfies $(p_0), (p_1), (p_2)$ with respect to (P, Q) and $(\mathrm{P}, \mathcal{V}_{\mathrm{X}})$ is a polar topology. The collection $\Gamma \odot \mathrm{X} = \{\lambda\, (\Gamma \odot A) : \lambda > 0\}$ also satisfies in (p_0) with respect to (P, Q); where $\Gamma \odot A = \{\{-1\}, \{1\}\}$, so $(\mathrm{P}, \mathcal{V}_{\mathrm{X}})$ is linear over $\mathbb{R}$. For $\mathrm{p}, \mathrm{p}' \in \mathrm{P}$ and $\lambda > 0$, we have $\mathrm{p} \le \mathrm{p}' + \lambda v_A$ if and only if

$$\inf_{x\in\mathrm{p}} \Re(x) + \sup_{x\in\mathrm{p}} \Re(x) \le \inf_{x'\in\mathrm{p}'} \Re(x') + \sup_{x'\in\mathrm{p}'} \Re(x') + \lambda \text{ or } \inf_{x'\in\mathrm{p}'} \Re(x') = -\infty \text{ or } \infty \in \mathrm{p}'$$

and $\mathrm{p} \le_{\mathbb{R}} \mathrm{p}' + \lambda v_{\Gamma\odot A}$ if and only if

$$|\inf_{x\in\mathrm{p}} \Re(x) + \sup_{x\in\mathrm{p}} \Re(x) - \inf_{x'\in\mathrm{p}'} \Re(x') - \sup_{x'\in\mathrm{p}'} \Re(x')| \le \lambda \quad \text{or} \quad \inf_{x'\in\mathrm{p}'} \Re(x') = -\infty \quad \text{or} \quad \infty \in \mathrm{p}'.$$

(ii) For $\mathrm{P} = Conv(\bar{\ell}_1)$ and $\mathrm{Q} = Conv(\bar{\ell}_\infty)$, (P, Q) is a linear dual pair over $\mathbb{K}$ by Proposition 2.7. If we set $\mathrm{X} = \{\lambda\, \mathrm{e} : \lambda > 0\}$, where $\mathrm{e} = \{\{(1, 1, \ldots)\}\}$, then for $\mathrm{p}, \mathrm{p}' \in \mathrm{P}$ we have $\mathrm{p} \le \mathrm{p}' + \lambda v_{\mathrm{e}}$ if and only if

$$\inf_{z\in\mathrm{p}} \sum_{n\in\mathbb{N}} \Re(z_n) + \sup_{z\in\mathrm{p}} \sum_{n\in\mathbb{N}} \Re(z_n) \le \inf_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Re(z'_n) + \sup_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Re(z'_n) + \lambda,$$

or $\inf_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Re(z'_n) = -\infty$ or $\exists z' \in \mathrm{p}', z' = (z'_n)_{n\in\mathbb{N}}$ s.t., $z'_n = \infty$ for some $n \in \mathbb{N}$, and $\mathrm{p} \le_{\mathbb{C}} \mathrm{p}' + \lambda v_{\Gamma\odot\mathrm{e}}$ if and only if

$$\begin{aligned}|\inf_{z\in\mathrm{p}} \sum_{n\in\mathbb{N}} \Re(z_n) + \sup_{z\in\mathrm{p}} \sum_{n\in\mathbb{N}} \Re(z_n) - \inf_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Re(z'_n) - \sup_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Re(z'_n)| \le \lambda, \\ |\inf_{z\in\mathrm{p}} \sum_{n\in\mathbb{N}} \Im(z_n) + \sup_{z\in\mathrm{p}} \sum_{n\in\mathbb{N}} \Im(z_n) - \inf_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Im(z'_n) - \sup_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Im(z'_n)| \le \lambda\end{aligned}$$

or $\inf_{z'\in\mathrm{p}'} \sum_{n\in\mathbb{N}} \Re(z'_n) = -\infty$ or $\exists z' \in \mathrm{p}', z' = (z'_n)_{n\in\mathbb{N}}$ s.t., $z'_n = \infty$ for some $n \in \mathbb{N}$.

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