Filomat 34:11 (2020), 3529–3552
https://doi.org/10.2298/FIL2011529B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Fiedler Pencils with Repetition for Rational Matrix Functions

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Abstract. We introduce generalized Fiedler pencil with repetition(GFPR) for an $n \times n$ rational matrix function $G(\lambda)$ relative to a realization of $G(\lambda)$. We show that a GFPR is a linearization of $G(\lambda)$ when the realization of $G(\lambda)$ is minimal and describe recovery of eigenvectors of $G(\lambda)$ from those of the GFPRs. In fact, we show that a GFPR allows operation-free recovery of eigenvectors of $G(\lambda)$. We describe construction of a symmetric GFPR when $G(\lambda)$ is symmetric. We also construct GFPR for the Rosenbrock system matrix $\mathcal{S}(\lambda)$ associated with an linear time-invariant (LTI) state-space system and show that the GFPR are Rosenbrock linearizations of $\mathcal{S}(\lambda)$. We also describe recovery of eigenvectors of $\mathcal{S}(\lambda)$ from those of the GFPR for $\mathcal{S}(\lambda)$. Finally, We analyze operation-free Symmetric/self-adjoint structure Fiedler pencils of system matrix $\mathcal{S}(\lambda)$ and rational matrix $G(\lambda)$. We show that structure pencils are linearizations of $G(\lambda)$.

1. Introduction

Consider an $n \times n$ rational matrix $G(\lambda)$, that is, the entries of $G(\lambda)$ are of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials. Zeros (eigenvalues) and poles (defined later in this section) of rational matrix play an important role in many applications such as in acoustic emissions of high speed trains, calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid-solid structures see [16, 19, 22, 23], Linear Systems Theory [15, 18], and references therein. Recently, by considering realization [15] of $G(\lambda)$, new classes of Fiedler-like pencils of $G(\lambda)$ such as Fiedler Pencil (FP), Generalized Fiedler (GF) Pencils have been introduced in [1, 3] to compute zeros, poles and eigenvectors of $G(\lambda)$.

Consider a minimal realization of $G(\lambda)$ of the form

$$G(\lambda) = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1}B =: P(\lambda) + C(\lambda E - A)^{-1}B, \tag{1}$$

where A, E, C, B are constant matrices of appropriate dimensions. Considering a realization of $G(\lambda)$ given in (1), it is shown in [19] that the eigenvalues and the eigenvectors of $G(\lambda)$ can be computed by solving the

2010 *Mathematics Subject Classification*. 65F15, 15A57, 15B57, 15A18, 65F35
Keywords. Rational matrix function, system matrix, matrix polynomial, eigenvalue, eigenvector, minimal realization, matrix pencil, linearization, Fiedler pencil.
Received: 11 December 2019; Accepted: 26 January 2020
Communicated by Yimin Wei
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generalized eigenvalue problem for the pencil

$$C_1(\lambda) := \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] + \left[\begin{array}{cccc|c} A_{m-1} & A_{m-2} & \cdots & A_0 & C \\ -I_n & 0 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & -I_n & 0 & \\ \hline & & & B & A \end{array}\right], \tag{2}$$

where the void entries represent zero entries. The pencil $C(\lambda)$ referred to as a *companion linearization* of $G(\lambda)$ in [19], where

$$C_1(\lambda) := \lambda \begin{bmatrix} A_m & 0 & \cdots & 0 \\ 0 & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_n \end{bmatrix} + \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix} \tag{3}$$

of the matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$.

For computing zeros (eigenvalues) and poles of rational matrix, linearizations of rational matrix have been introduced recently in [1, 6] via matrix-fraction descriptions (MFD) of rational matrix. Let $G(\lambda) = N(\lambda)D(\lambda)^{-1}$ be a *right coprime* MFD of $G(\lambda)$, where $N(\lambda)$ and $D(\lambda)$ are matrix polynomials with $D(\lambda)$ being regular. Then the *zero structure* of $G(\lambda)$ is the same as the *eigenstructure* of $N(\lambda)$ and the *pole structure* of $G(\lambda)$ is the same as the *eigenstructure* of $D(\lambda)$, see [15]. Also $G(\lambda)$ can be uniquely written as $G(\lambda) = P(\lambda) + Q(\lambda)$, where $P(\lambda)$ is a matrix polynomial and $Q(\lambda)$ is strictly proper [15]. We define $\deg(G) := \deg(P)$, the degree of the polynomial part of $G(\lambda)$.

A realization of $G(\lambda)$ of the form (1) is associated with a linear time-invariant (LTI) system Σ in *state-space-form* (SSF) given by [18, 20]

$$\Sigma : \quad \begin{array}{l} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + P(\frac{d}{dt})u(t) \end{array} \tag{4}$$

for which $G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B$ is the *transfer function*, where $x(t)$ is the state vector and $u(t)$ is the control vector of the system. The *Rosenbrock system polynomial* (also referred to as the Rosenbrock system matrix) associated with the LTI system Σ in (4) is an $(n+r) \times (n+r)$ matrix polynomial $\mathcal{S}(\lambda)$ given by [15, 18]

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right]. \tag{5}$$

The eigenvalues of $\mathcal{S}(\lambda)$ are called *invariant zeros* of the LTI system Σ and the associated eigenvectors are called *invariant zero directions* [15, 18]. The spectrum of $\mathcal{S}$ is the set of invariant zeros of the LTI system Σ, see [1]. The invariant zeros of LTI systems play an important role in Linear Systems Theory [15, 18, 20].

Definition 1.1 (Linearization, [1]). *Let $G(\lambda)$ be an $n \times n$ rational matrix function (regular or singular) and let $G(\lambda) = N(\lambda)D(\lambda)^{-1}$ be a* right coprime *MFD of $G(\lambda)$. Set $r := \deg(\det(D(\lambda)))$, $p := \max(n, r)$ and $m := \deg(G(\lambda))$. If $m \geq 1$ then an $(mn + r) \times (mn + r)$ matrix pencil $\mathbb{L}(\lambda)$ of the form*

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} X - \lambda Y & C \\ \hline \mathcal{B} & A - \lambda E \end{array}\right] \tag{6}$$

is said to be a linearization of $G(\lambda)$ provided that there are $(mn + r) \times (mn + r)$ unimodular matrix polynomials $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$, and $p \times p$ unimodular matrix polynomials $Z(\lambda)$ and $W(\lambda)$ such that $\mathcal{U}(\lambda)\mathrm{diag}(I_{s-(mn+r)}, \mathbb{L}(\lambda))\mathcal{V}(\lambda) = \mathrm{diag}(I_{s-n}, N(\lambda))$ and $Z(\lambda)\mathrm{diag}(I_{p-r}, A - \lambda E)W(\lambda) = \mathrm{diag}(I_{p-n}, D(\lambda))$ for $\lambda \in \mathbb{C}$, where $A - \lambda E$ is an $r \times r$ pencil with E being nonsingular and $s := \max(mn + r, 2n)$.

Thus the zeros and poles of $G(\lambda)$ are the eigenvalues of $\mathbb{L}(\lambda)$ and $A - \lambda E$, respectively.

In [11] they consider a class of GFPRs of matrix polynomial and describe the operation-free recovery of eigenvectors and minimal bases of matrix polynomial from those GFPRs. Though they have derived operataion-free recovery formulas for eigenvector and minimal bases but the explicit maps (forward maps) are open problem which are important for the study of sensitivity and backward error analysis. Those explicit maps are studied in this paper.

Next, note that one significant drawback of the first companion form is that, it usually does not reflect any structure that may be present in the original rational matrix function $G(\lambda)$. Recently, in [1, 6] it has been studied a new class of linearizations generalizing the first companion form referred as Fiedler linearizations of rational matrix. But the drawback is that all those Fiedler pencils do not preserve any structure that the original rational matrix function does have. In this paper, we describe construction of a symmetric GFPR when $G(\lambda)$ is symmetric.

The main contributions of this paper are as follows. First, we introduce operation-free product of Fiedler matrices to study generalized Fiedler pencils with repetitions (GFPR). Then we introduce GFPR of a rational matrix $G(\lambda)$ relative to a realization of $G(\lambda)$ as given in (1) for computing eigenvalues and poles of $G(\lambda)$. In [3] it is shown that the eigenvectors of $\mathcal{S}(\lambda)$ and $G(\lambda)$ can be easily recovered from those of the GF pencils and the recovery is operation-free for the PGF pencils. Secondly, we obtain explicit formulas for the eigenvectors of the generalized Fiedler linearizations of rational matrix in terms of the eigenvectors of $G(\lambda)$. Also, We determine explicit formula of eigenvectors of GFPRs and describe recovery of eigenvectors of $G(\lambda)$ and $\mathcal{S}(\lambda)$ from those of GFPR $\mathbb{L}(\lambda)$ without performing any arithmetic operations. Thus we show that a GFPR of $G(\lambda)$ allows an easy operation-free recovery of eigenvectors of $G(\lambda)$ from those of the GFPR pencil. Note that the explicit formula of eigenvectors of the linearizations will be useful for defining and comparing condition numbers of the eigenvalues of the linearziations with the condition number of eigenvalues of the rational matrix. Lastly, since structure eigenproblems often implies some symmetries in its spectrum, which are meaningful in some sense in physical applications and that can be destroyed when we ignore the structure. Also, if we consider structure preserving pencils then the storage and computational cost can be reduced. Hence, Finally, we study linearizations that preserve the structure of the original problem. In this paper, we discuss structure preserving, in particular, symmetric/self-adjoint linearizations which can be constructed from GFPR.

The rest of the paper is organized as follows. Section 2 contains some basic definitions and results on index tuples and Fiedler matrices which we need throughout this paper. Section 3 introduces operation-free product of Fiedler matrices to study GFPR. Section 4 presents the Generalized Fiedler pencils for rational matrix and eigenvector formula for GF pencils. Section 5 introduces the Generalized Fiedler Pencils with Repetitions (GFPR) and study the eigenvector recovery property of rational matrix. Finally, in the same section we describes the Symmetric/self-adjoint GFPR linearizations for rational matrix and for system matrix.

Notation. We denote by $\mathbb{C}[\lambda]$ the polynomial ring over the complex field $\mathbb{C}$. Further, we denote by $\mathbb{C}^{m\times n}$ and $\mathbb{C}[\lambda]^{m\times n}$, respectively, the vector spaces of $m \times n$ matrices and matrix polynomials over $\mathbb{C}$. An $m \times n$ rational matrix function $G(\lambda)$ is an $m \times n$ matrix whose entries are rational functions of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials in $\mathbb{C}[\lambda]$. An $n \times n$ rational matrix function $G(\lambda)$ is said to be *regular* if $\mathrm{rank}(G(\lambda)) = n$ for some $\lambda \in \mathbb{C}$. If $G(\lambda)$ is regular then $\mu \in \mathbb{C}$ is said to be an *eigenvalue* of $G(\lambda)$ if $\mathrm{rank}(G(\mu)) < n$. An $n \times n$ matrix polynomial $U(\lambda)$ is said to be *unimodular* if $\det(U(\lambda))$ is a nonzero constant independent of λ. We denote the j-th column of the $n \times n$ identity matrix I_n by e_j and the transpose of a matrix A by A^T. We denote the Kronecker product of matrices A and B by $A \otimes B$. The right and the left null spaces of an $m \times n$ matrix A are given by $\mathcal{N}_r(A) := \{x \in \mathbb{C}^n : Ax = 0\}$ and $\mathcal{N}_l(A) := \{y \in \mathbb{C}^m : y^T A = 0\}$.

2. Basic results

Definition 2.1. *[9] An ordered tuple of indices consisting of consecutive integers is called a* string *and denoted by*

$(t:p)$ *for the string of integers from t to p, i.e.,*

$$(t:p) := \begin{cases} (t, t+1, \ldots, p), & \text{if } t \le p \\ \emptyset, & \text{if } t > p. \end{cases}$$

Remark 2.2. *In the above definition, if $t_1 > p$ and $t_2 > p$, then $(t_1 : p)$ and $(t_2 : p)$ correspond to the empty index tuple. To avoid this notation, we will adapt the notation $(\infty : p)$ for any tuple of the form $(t : p)$ having $t > p$ where applicable.*

We use the boldface small letters, such as $\mathbf{t}, \mathbf{q}, \mathbf{s}, \ldots$ for index tuples (that is, ordered tuples of indices). If $\mathbf{q} = (i_1, i_2, \ldots, i_s)$ is an index tuple, then the *reverse* of $\mathbf{q}$, denoted by *rev* $\mathbf{q}$, is defined by *rev* $\mathbf{q} := (i_s, i_{s-1}, \ldots, i_1)$. Let $\mathbf{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple and d be an integer. Then we define

$$-\mathbf{q} = (-i_1, -i_2, \ldots, -i_s) \text{ and } d + \mathbf{q} = (d + i_1, d + i_2, \ldots, d + i_s).$$

Definition 2.3. *[9] Let $\boldsymbol{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, m, -0, -1, \ldots, -m\}$. Then i_j is said to be a simple index of $\boldsymbol{q}$ if $i_j \neq i_k$ for $k = 1 : s$ and $k \neq j$. We say that $\boldsymbol{q}$ is a simple index tuple if each index i_j, $j = 1 : s$, is a simple index of $\boldsymbol{q}$.*

Definition 2.4. *[3] Let $d \ge 1$ be an integer and $\boldsymbol{q} = (i_1, \ldots, i_s)$ be a simple index tuple containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$.*

- *(a) We say that $\boldsymbol{q}$ has a consecution at k if $k, k+1 \in \boldsymbol{q}$ and $\boldsymbol{q}$ is of the form $\boldsymbol{q} = (\cdots, k, \cdots, k+1, \cdots)$. We say that $\boldsymbol{q}$ has an inversion at k if $k, k+1 \in \boldsymbol{q}$ and $\boldsymbol{q}$ is of the form $\boldsymbol{q} = (\cdots, k+1, \cdots, k, \cdots)$.*
- *(b) We say that $\boldsymbol{q}$ has c_k (resp., i_k) consecutions (resp., inversions) at k if $\boldsymbol{q}$ has consecutions (resp., inversions) at $k, k+1, \ldots, k+c_k-1$ (resp., at $k, k+1, \ldots, k+i_k-1$) and it does not have a consecution (resp., inversion) at $k+c_k$ (resp., $k+i_k$).*
- *(c) If $0 \in \boldsymbol{q}$ then we refer to CIP$(\boldsymbol{q}) := (c_0, i_0)$ as the consecution-inversion pair of $\boldsymbol{q}$ at 0, where c_0 (resp., i_0) is number of consecutions (resp., inversions) of $\boldsymbol{q}$ at 0.*

Note that if CIP$(\mathbf{q}) = (c_0, i_0)$ then either CIP$(\mathbf{q}) = (0, 0)$ or CIP$(\mathbf{q}) = (c_0, 0)$ with $c_0 > 0$ or CIP$(\mathbf{q}) = (0, i_0)$ with $i_0 > 0$.

Definition 2.5. *[21] Let $\boldsymbol{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple. Then $\boldsymbol{q}$ is said to satisfy the* Successor Infix Property (SIP) *if for every pair of indices $i_a, i_b \in \boldsymbol{q}$ with $1 \le a < b \le s$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ such that $a < c < b$.*

Definition 2.6. *[21] Let d be a non-negative integer and $\boldsymbol{q}$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$. Then $\boldsymbol{q}$ is said to be in column standard form if $\boldsymbol{q} = (a_p : b_p, a_{p-1} : b_{p-1}, \ldots, a_2 : b_2, a_1 : b_1)$, with $0 \le b_1 < b_2 < \ldots < b_{p-1} < b_p \le d$ and $0 \le a_j \le b_j$, for all $j = 1, \ldots, p$. Let $\boldsymbol{t}$ be an index tuple containing indices from $\{-d, -d+1, \ldots, -1\}$. Then $\boldsymbol{t}$ is said to be in column standard form if $d + \boldsymbol{t}$ is in column standard form.*

Lemma 2.7. *[9] Let $\boldsymbol{q} = (i_1, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$, for some $d \ge 1$, then $\boldsymbol{q}$ satisfies the SIP if and only if $\boldsymbol{q}$ is equivalent to a (unique) tuple in column standard form.*

Definition 2.8. *[9] Let $\boldsymbol{q} = (i_1, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$, for some $d \ge 1$ and satisfying the SIP. The unique index tuple in column standard form equivalent to an index tuple $\boldsymbol{q}$ satisfying the SIP is called the column standard form of $\boldsymbol{q}$. We denote this tuple by $csf(\boldsymbol{q})$.*

2.1. Fiedler Matrices

Consider the system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right]$$

and the associated transfer function

$$G(\lambda) = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1} B =: P(\lambda) + C(\lambda E - A)^{-1} B. \tag{7}$$

Define $(mn + r) \times (mn + r)$ matrices $\mathbb{M}_0, \ldots, \mathbb{M}_{-m}$ by

$$\mathbb{M}_0 := \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \quad \mathbb{M}_{-m} := \left[\begin{array}{c|c} M_{-m} & 0 \\ \hline 0 & -E \end{array}\right], \tag{8}$$

$$\mathbb{M}_{-i} := \mathbb{M}_i^{-1}, \text{ for } i = 0, 1, \ldots, m-1 \text{ and } \mathbb{M}_m := \mathbb{M}_{-m}^{-1},$$

with $\mathbb{M}_i^{-1} = \left[\begin{array}{c|c} M_i^{-1} & 0 \\ \hline 0 & I_r \end{array}\right]$, $i = 1 : m-1$, where

$$M_0 := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \quad M_{-m} := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \tag{9}$$

$$M_{-i} := M_i^{-1} \text{ for } i = 0, 1, \ldots, m-1 \text{ and } M_m := M_{-m}^{-1}, \tag{10}$$

where

$$M_i := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ and } M_i^{-1} = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}$$

for $i = 1 : m-1$ are the Fiedler matrices of $P(\lambda)$, see [14, 21]. We refer to the matrices $\mathbb{M}_0, \mathbb{M}_1, \ldots, \mathbb{M}_{-m}$ as the Fiedler matrices of $\mathcal{S}(\lambda)$ or $G(\lambda)$.

Observe that M_{-m} and M_0 are invertible if and only if A_m and A_0 are invertible. It follows that $M_i M_j = M_j M_i$ if $||i| - |j|| > 1$. It also follows that $\mathbb{M}_i \mathbb{M}_j = \mathbb{M}_j \mathbb{M}_i$ for $\big||i| - |j|\big| \neq 1$ except for $\big||i| - |j|\big| = m$.

Let $\mathbf{q} = (i_1, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$, for some $d \geq 1$, then $\mathbb{M}_{\mathbf{q}} := \mathbb{M}_{i_1} \mathbb{M}_{i_2} \cdots \mathbb{M}_{i_s}$. If $\mathbf{q} = \phi$ then $\mathbb{M}_{\mathbf{q}} = I_{nm+r}$ [9]. Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be two index tuples containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$, for some $d \geq 1$. We say that $\mathbf{q}_1$ is *equivalent* to $\mathbf{q}_2$, and we will write $\mathbf{q}_1 \sim \mathbf{q}_2$, if $\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_2}$. Note that $\sim$ is an equivalence relation. Observe that if $M_{\mathbf{q}_2}$ is obtained from $M_{\mathbf{q}_1}$ by applying the commutativity relations then $\mathbf{q}_1$ is equivalent to $\mathbf{q}_2$.

3. Operation-free products of Fiedler matrices

In this section we define operation-free product of Fiedler matrices of $G(\lambda)$ to study the generalized Fiedler pencils with repetitions (GFPR).

Definition 3.1. *A product $\mathbb{M}_{\boldsymbol{q}}$ corresponding to the index tuple $\boldsymbol{q} = (i_1, i_2, \ldots, i_m)$ is said to be* operation-free *if the block entries (up to sign) of $\mathbb{M}_{\boldsymbol{q}}$ consist of matrices from $0, I_n, I_r, C, B, A, E$ and $A_0, A_1, \ldots, A_m$.*

For example, when $\sigma = (0, 1, \ldots, m-1)$, the product $\mathbb{M}_\sigma = \mathbb{M}_{\sigma(0)} \cdots \mathbb{M}_{\sigma(m-1)}$ is operation-free. Thus a Fiedler pencil associated with a bijection σ is also operation-free. So the question is: if we allow to repeat the Fiedler matrices, will that product still be an operation-free product?

Lemma 3.2. *The product $\mathbb{M}_i \mathbb{M}_i$, is not operation-free for $i = 0, \pm 1, \ldots, \pm(m-1), -m$.*

Proof. For $i = 0$,

$$\mathbb{M}_0\mathbb{M}_0 = \left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right]\left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right] = \left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & A_0^2 + CB & A_0C + CA \\ \hline & BA_0 + AB & BC + A^2 \end{array}\right]$$

is not operation-free. For $1 \le i \le m-1$, $\mathbb{M}_i\mathbb{M}_i = \left[\begin{array}{c|c} M_iM_i & \\ \hline & I_r \end{array}\right]$ is not operation-free, since M_iM_i is not operation-free, see [21]. Similarly, for $-(m-1) \le i \le -1$, $\mathbb{M}_i\mathbb{M}_i$ is not operation-free, since $\mathbb{M}_{-i}\mathbb{M}_{-i} = (\mathbb{M}_i\mathbb{M}_i)^{-1} = \left[\begin{array}{c|c} (M_iM_i)^{-1} & \\ \hline & I_r \end{array}\right]$ and $(M_iM_i)^{-1}$ is not operation-free, see [21]. Again for $i = -m$,

$$\mathbb{M}_{-m}\mathbb{M}_{-m} = \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right]\left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right] = \left[\begin{array}{cc|c} A_m^2 & & \\ & I_{(m-1)n} & \\ \hline & & E^2 \end{array}\right]$$

is not operation-free. □

Lemma 3.3. *The product $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i$ is operation-free for $i = 1 : m-2$ but $\mathbb{M}_0\mathbb{M}_1\mathbb{M}_0$ is not operation-free. Further, $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1}$ is not operation-free for $i = 0 : m-2$.*

Proof. For $i = 0$, we have $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i = \mathbb{M}_0\mathbb{M}_1\mathbb{M}_0 = \left[\begin{array}{ccc|c} I_{(m-2)n} & & & \\ & -A_1 & -A_0 & -C \\ & -A_0 & CB & CA \\ \hline & -B & AB & A^2 \end{array}\right]$ is not operation-free. For $1 \le i \le m-2$, we have $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i = \left[\begin{array}{c|c} M_iM_{i+1}M_i & \\ \hline & I_r \end{array}\right]$. So $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i$ is operation-free, since $M_iM_{i+1}M_i$ is operation-free, see [21]. Note that $\mathbb{M}_1\mathbb{M}_0\mathbb{M}_1 = \left[\begin{array}{ccc|c} I_{(m-2)n} & & & \\ & A_1^2 - A_0 & -A_1 & -C \\ & -A_1 & I_n & 0 \\ \hline & -B & 0 & -A \end{array}\right]$ is not operation-free. For $1 \le i \le m-2$, we have $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1} = \left[\begin{array}{c|c} M_{i+1}M_iM_{i+1} & \\ \hline & I_r \end{array}\right]$ is not operation-free, since $M_{i+1}M_iM_{i+1}$ is not operation-free, see [21]. □

Corollary 3.4. *The product $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i$ is operation-free for $i = -2, -3, \ldots, -(m-1)$ but $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$ is not operation-free. If $E = I_r$, then $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$ is operation-free. Further, $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1}$ is not operation-free for $i = -2, -3, \ldots, -(m-1)$.*

Proof. We have $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$

$$= \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right]\left[\begin{array}{ccc|c} 0 & I_n & & \\ I_n & A_{m-1} & & \\ & & I_{(m-2)n} & \\ \hline & & & I_r \end{array}\right]\left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right] = \left[\begin{array}{ccc|c} 0 & A_m & & \\ A_m & A_{m-1} & & \\ & & I_{(m-2)n} & \\ \hline & & & E^2 \end{array}\right]$$

is not operation-free. If $E = I_r$, then $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$ is operation-free. The proof for $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1}$ is similar. □

Lemma 3.5. *Let $\boldsymbol{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple of indices from $\{0, 1, \ldots, m-1\}$ such that $\mathbb{M}_{\boldsymbol{q}}$ is not operation-free. Then for any two other index tuples τ and σ from $\{0, 1, \ldots, m-1\}$ the product $\mathbb{M}_\tau\mathbb{M}_{\boldsymbol{q}}\mathbb{M}_\sigma$ is not operation-free.*

Proof. Proof directly follows from Lemma 4, [21]. □

Lemma 3.6. *The product $\mathbb{M}_{(i:j)}$ is operation-free and is given by*

$$\mathbb{M}_{(i:j)} = \left[\begin{array}{c|c|c|c|c} I_{(m-j-1)n} & & & & \\ \hline & \begin{matrix} -A_j \\ -A_{j-1} \\ \vdots \\ -A_i \end{matrix} & I_{(j-i+1)n} & & \\ \hline & I_n & 0_{n\times(j-i+1)n} & & \\ \hline & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{(i:j)} & \\ \hline & I_r \end{array}\right] \tag{11}$$

for $i > 0$, $i \leq j \leq m-1$ and

$$\mathbb{M}_{(0:j)} = \left[\begin{array}{c|c|c|c} I_{(m-j-1)n} & & & \\ \hline & \begin{matrix} -A_j \\ \vdots \\ -A_1 \end{matrix} & I_{jn} & \\ \hline & -A_0 & 0_{n\times jn} & -C \\ \hline & -B & & -A \end{array}\right] = \left[\begin{array}{c|c} M_{(0:j)} & -e_m \otimes C \\ \hline -e_{m-j}^T \otimes B & -A \end{array}\right]. \tag{12}$$

Proof. We prove the result by induction on j. Suppose that $i > 0$. Then for $j = i$, we have $\mathbb{M}_{(i:i)} = \mathbb{M}_i$ which is operation free. So assume that (11) is true for $j = s$. We have to show that (11) is true for $j = s+1$. Now

$$\mathbb{M}_{(i:s+1)} = \mathbb{M}_{(i:s)}\mathbb{M}_{s+1} = \left[\begin{array}{c|c|ccccc|c|c} I_{(m-s-1)n} & & & & & & & & \\ \hline & -A_s & I_n & & & & & & \\ & -A_{s-1} & 0 & I_n & & & & & \\ & \vdots & \vdots & & \ddots & & & & \\ & -A_i & 0 & 0 & \cdots & I_n & & & \\ \hline & I_n & 0 & 0 & \cdots & 0 & & 0 & \\ \hline & & & & & & & & I_r \end{array}\right] \left[\begin{array}{ccc|c} I_{(m-s-2)n} & & & \\ & -A_{(s+1)} & I_n & \\ & I_n & 0 & \\ & & & I_{sn} \\ \hline & & & & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c|ccccc|c|c} I_{(m-s-2)n} & & & & & & & & \\ \hline & -A_{s+1} & I_n & & & & & & \\ & -A_s & 0 & I_n & & & & & \\ & \vdots & \vdots & & \ddots & & & & \\ & -A_i & 0 & 0 & \cdots & I_n & & & \\ \hline & I_n & 0 & 0 & \cdots & 0 & & 0 & \\ \hline & & & & & & & & I_r \end{array}\right] = \left[\begin{array}{c|c|c|c|c} I_{(m-s-2)n} & & & & \\ \hline & \begin{matrix} -A_{s+1} \\ -A_s \\ \vdots \\ -A_i \end{matrix} & I_{(s-i+2)n} & & \\ \hline & I_n & 0 \quad 0 \quad \cdots \quad 0 & & \\ \hline & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right],$$

which satisfies (11). Clearly this is operation-free. This proves the case for $i > 0$.

For the case $i = 0$ multiply $\mathbb{M}_0$ with the case $i > 0$. So we have $\mathbb{M}_{(0:j)} = \mathbb{M}_0\mathbb{M}_{(1:j)}$, which is equal to

$$\left[\begin{array}{ccc|c} I_{(m-j-1)n} & & & \\ & I_{jn} & & \\ & & -A_0 & -C \\ \hline & & -B & -A \end{array}\right] \left[\begin{array}{c|c|ccccc|c} I_{(m-j-1)n} & & & & & & & \\ \hline & -A_j & I_n & & & & & \\ & -A_{j-1} & 0 & I_n & & & & \\ & \vdots & \vdots & & \ddots & & & \\ & -A_1 & 0 & 0 & \cdots & I_n & & \\ \hline & I_n & 0 & 0 & \cdots & 0 & 0 & \\ \hline & & & & & & & I_r \end{array}\right] = \left[\begin{array}{c|c|c|c} I_{(m-j-1)n} & & & \\ \hline & \begin{matrix} -A_j \\ \vdots \\ -A_1 \end{matrix} & I_{jn} & \\ \hline & -A_0 & 0 \quad \ldots \quad 0 & -C \\ \hline & -B & & -A \end{array}\right]$$

is operation-free. □

Remark 3.7. *Similarly the product $\mathbb{M}_{(-i:-j)}$, where $1 \leq j \leq i \leq m$ is operation-free and is given by*

$$\mathbb{M}_{(-i:-j)} = \left[\begin{array}{c|c|c|c|c} I_{(m-i-1)n} & & & & \\ \hline & 0_{n\times(i-j+1)n} & I_n & & \\ \hline & I_{(i-j+1)n} & \begin{matrix} A_i \\ A_{i-1} \\ \vdots \\ A_j \end{matrix} & & \\ \hline & & & I_{(j-1)n} & \\ \hline & & & & I_r \end{array}\right], \qquad j \leq i < m,$$

$$\mathbb{M}_{(-m:-j)} = \left[\begin{array}{c|c|c|c} 0_{n\times(m-j)n} & A_m & & \\ \hline I_{(m-j)n} & \begin{matrix} A_{m-1} \\ \vdots \\ A_j \end{matrix} & & \\ \hline & & I_{(j-1)n} & \\ \hline & & & -E \end{array}\right].$$

The next theorem provides a canonical form of operation-free product, which helps us to construct structure preserving linearizations of system matrix.

Theorem 3.8. *Each product of the form*

$$\prod_{i=m-1}^{1} \mathbb{M}_{(c_i:i)}, \text{ for } c_i \in (1:i) \cup \{\infty\}, \tag{13}$$

is operation-free. The product in (13) is in column standard form.

Proof. We have $\prod_{i=m-1}^{1} \mathbb{M}_{(c_i:i)} = \prod_{i=m-1}^{1} \left[\begin{array}{c|c} M_{(c_i:i)} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} \prod_{i=m-1}^{1} M_{(c_i:i)} & \\ \hline & I_r \end{array}\right]$. Since by Theorem 1, [21], $\prod_{i=m-1}^{1} M_{(c_i:i)}$ is operation-free for $c_i \in (1:i) \cup \{\infty\}$, hence the result follows. □

Theorem 3.9. *Let $\mathbf{q}$ be an index tuple of indices from $\{0, 1, \ldots, m-1\}$.*
Case I : Suppose that $0 \notin \mathbf{q}$. Then the following are equivalent.

(a) $\mathbb{M}_{\mathbf{q}}$ is operation-free.

(b) $\mathbf{q}$ satisfies SIP.

(c) $\mathbb{M}_{\mathbf{q}}$ can be written in the column standard form given in (13).

Case II : Suppose that $0 \in \mathbf{q}$ is a simple index of $\mathbf{q}$. Then the following are equivalent.

(a) $\mathbb{M}_{\mathbf{q}}$ is operation-free.

(b) $\mathbf{q}$ satisfies SIP.

(c) $\mathbb{M}_{\mathbf{q}}$ can be written in the column standard form $\prod_{i=m-1}^{1} \mathbb{M}_{(c_i:i)}$, for $c_i \in (0:i) \cup \{\infty\}$.

Proof. Case I : If $0 \notin \mathbf{q}$ then $\mathbb{M}_{\mathbf{q}} = \left[\begin{array}{c|c} M_{\mathbf{q}} & \\ \hline & I_r \end{array}\right]$. Consequently, $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 2, [21].

Case II : Suppose that $0 \in \mathbf{q}$. Since 0 is a simple index of $\mathbf{q}$, we have $\mathbb{M}_{\mathbf{q}} = \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$ for some index tuples σ_1 and σ_2 from $\{1, \ldots, m-1\}$. Now

$$\mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right]\left[\begin{array}{c|c} M_{\sigma_2} & \\ \hline & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & -M_{\sigma_1}(e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right]\left[\begin{array}{c|c} M_{\mathbf{q}} & -M_{\sigma_1}(e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right].$$

This shows that $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}, M_{\sigma_1}(e_m \otimes C)$, and $(e_m^T \otimes B)M_{\sigma_2}$ are operation-free. Now $M_{\mathbf{q}} = M_{\sigma_1}M_0M_{\sigma_2}$ is operation free implies that M_{σ_1} and M_{σ_2} are operation-free. Hence σ_1 and σ_2 satisfies SIP. Since σ_1 and σ_2 are tuples from $\{1, 2, \ldots, m-1\}$, by Theorem 2, [21], M_{σ_1} and M_{σ_2} can be written in the form $\prod_{i=m-1}^{1} M_{(c_i:i)}$, for $c_i \in (1:i) \cup \{\infty\}$. Hence by Lemma 3.6, it follows that the block entries of $M_{\sigma_1}(e_m \otimes I_n)$ and $(e_m^T \otimes I_n)M_{\sigma_2}$ are either 0 or I_n. In fact, it is easy to see that $M_{\sigma_1}(e_m \otimes I_n) = e_k \otimes I_n$ and $(e_m^T \otimes I_n)M_{\sigma_2} = e_j^T \otimes I_n$ for some k and j. Hence $M_{\sigma_1}(e_m \otimes C)$ and $(e_m^T \otimes B)M_{\sigma_2}$ are operation-free. This shows that $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 2, [21]. □

Remark 3.10. *The assumption that 0 is a simple index can not be relaxed in Theorem 3.9. For example, $\prod_{i=3}^{0} \mathbb{M}_{(c_i:i)}$, for $c_i \in (0:i) \cup \{\infty\}$ may not be operation-free. Indeed, by Lemma 3.5, the product $\mathbb{M}_{0:3}\mathbb{M}_{0:2}\mathbb{M}_{0:1}\mathbb{M}_{0:0}$. is not operation-free since $\mathbb{M}_{0:1}\mathbb{M}_{0:0}$ is not operation-free.*

Considering index tuple from the set $\{-m, -(m-1), \ldots, -1\}$ and using similar arguments as above, we have the following result.

Theorem 3.11. *Let $\boldsymbol{q} = (i_1, i_2, \ldots, i_p)$ be an index tuple from the set $\{-m, -(m-1), \ldots, -1\}$.*
Case I : If $-m \notin \boldsymbol{q}$ then the following are are equivalent.

(a) $\mathbb{M}_{\boldsymbol{q}}$ is operation-free.

(b) $\boldsymbol{q}$ satisfies the SIP.

(c) $\mathbb{M}_{\boldsymbol{q}}$ can be written in the column standard form $\prod_{i=-1}^{-(m-1)} \mathbb{M}_{(c_i:i)}$, for $c_i \in (-(m-1) : i) \cup \{\infty\}$.

Case II : If $-m \in \boldsymbol{q}$ is a simple index of $\boldsymbol{q}$ then the following are are equivalent.

(a) $\mathbb{M}_{\boldsymbol{q}}$ is operation-free.

(b) $\boldsymbol{q}$ satisfies the SIP.

(c) $\mathbb{M}_{\boldsymbol{q}}$ can be written in the column standard form $\prod_{i=-1}^{-(m-1)} \mathbb{M}_{(c_i:i)}$, for $c_i \in (-m : i) \cup \{\infty\}$.

Proof. Case I : If $-m \notin \mathbf{q}$ then $\mathbb{M}_{\mathbf{q}} = \left[\begin{array}{c|c} M_{\mathbf{q}} & \\ \hline & I_r \end{array}\right]$. Consequently, $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 3, [21].

Case II : Suppose that $-m \in \mathbf{q}$. Since $-m \in \mathbf{q}$ is a simple index of $\mathbf{q}$, then we have $\mathbb{M}_{\mathbf{q}} = \mathbb{M}_{\sigma_1}\mathbb{M}_{-m}\mathbb{M}_{\sigma_2}$ for some index tuples σ_1 and σ_2 from $\{-(m-1), \ldots, -1\}$. Now

$$\mathbb{M}_{\sigma_1}\mathbb{M}_{m}\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} M_{-m} & \\ \hline & -E \end{array}\right]\left[\begin{array}{c|c} M_{\sigma_2} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{\sigma_1}M_{-m}M_{\sigma_2} & \\ \hline & -E \end{array}\right].$$

This shows that $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 2, [21]. □

Remark 3.12. *If $E = I_r$, then the product $\prod_{i=-1}^{-m} \mathbb{M}_{(c_i:i)}$, for $c_i \in (-m : i) \cup \{\infty\}$ is operation-free, because $E = I_r$ implies that $\mathbb{M}_{-m}\mathbb{M}_{(-m-1)}\mathbb{M}_{-m}$ is operation-free.*

Remark 3.13. *The assumption that $-m$ is a simple index cannot be relaxed in Theorem 3.11. For example,*

$$\prod_{i=-1}^{-4} \mathbb{M}_{(c_i:i)}, \text{ for } c_i \in (-4 : i) \cup \{\infty\}$$

may not be operation-free. Indeed, by Lemma 3.5, the product $\mathbb{M}_{-4:-1}\mathbb{M}_{-4:-2}\mathbb{M}_{-4:-3}\mathbb{M}_{-4:-4}$ is not operation-free as $\mathbb{M}_{-4:-3}\mathbb{M}_{-4}$ is not operation-free.

4. Eigenvector Formula of Generalized Fiedler (GF) Pencils

Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, m-1\}$. Then the $(mn + r) \times (mn + r)$ matrix pencil $L_{\mathbf{q}}(\lambda)$ given by $L_{\mathbf{q}}(\lambda) = \lambda\mathbb{M}_{-m} - \mathbb{M}_{\mathbf{q}}$ is called the Fiedler pencil of the Rosenbrock system polynomial $\mathcal{S}(\lambda)$ associated with $\mathbf{q}$, see [1]. The pencil $L_{\mathbf{q}}(\lambda)$ is also called the Fiedler pencil of the transfer function $G(\lambda)$ associated with $\mathbf{q}$.

Now, we define the GF pencil of rational matrix $G(\lambda)$.

Definition 4.1. *Let $\mathcal{S}(\lambda)$ be the system matrix given in (5). Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots, m\}$ (C_0 or C_1 may be empty set). Let σ and τ be permutations of C_0 and $-C_1$, respectively. Then the pencil $\mathbb{T}_\omega(\lambda) := \lambda\mathbb{M}_\tau - \mathbb{M}_\sigma$ is said to be a generalized Fiedler (GF) pencil of $\mathcal{S}(\lambda)$ associated with $\omega = (\tau, \sigma)$. If $0 \in C_0$ and $m \in C_1$, then the pencil $\mathbb{T}_\omega(\lambda)$ is said to be a Proper Generalized Fiedler (PGF) pencil of $\mathcal{S}(\lambda)$. We also refer to $\mathbb{T}_\omega(\lambda)$ as the GF (PGF) of the transfer function $G(\lambda)$ associated with $\omega = (\tau, \sigma)$.*

Note that if $C_0 = \emptyset$, then $\mathbb{M}_\sigma = I_{(nm+r)}$ and if $C_1 = \emptyset$, then $\mathbb{M}_\tau = I_{(nm+r)}$. It is clear that any Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ of $\mathcal{S}(\lambda)$ is a special case of a GF pencil with $C_0 = \{0, 1, \ldots, m-1\}$ and $C_1 = \{m\}$.

It has been shown that GF and PGF pencils of $\mathcal{S}(\lambda)$ is a trimmed structured linearization of $\mathcal{S}(\lambda)$ and that the GF and PGF pencils are also a linearization of $G(\lambda)$ whenever the realization (1) of $G(\lambda)$ is minimal, [3]. Also, recovery of eigenvectors of $\mathcal{S}(\lambda)$ from GF pencils follows from Theorem 5.3 given in [3].

Next, consider the system matrix $\mathcal{S}(\lambda)$ and its associated transfer function $G(\lambda)$. It is already established the eigenvector formula for Fiedler pencil of $\mathcal{S}(\lambda)$ and its associated transfer function $G(\lambda)$, see, [2]. Now, we derive the eigenvector formula for PGF and GF pencil of $\mathcal{S}(\lambda)$ and $G(\lambda)$.

Definition 4.2. *Let $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ be a matrix polynomial of degree m. For $0 \le j \le m$, define the polynomial $P_j(\lambda) = A_{m-j} + \lambda A_{m-j+1} + \cdots + \lambda^j A_m$. Then the polynomial is called as the jth Horner shift of $P(\lambda)$. Observe that*

$$P_0(\lambda) = A_m, P_m(\lambda) = P(\lambda), \text{ and } \lambda P_j(\lambda) = P_{j+1}(\lambda) - A_{m-j-1}, \text{ for } 0 \le j \le m-1.$$

Next, consider the block transpose of a block matrix.

Let $H := (H_{ij})$ be a block $m \times n$ matrix with $p \times q$ blocks H_{ij}. The *block transpose* of H, denoted by $H^{\mathcal{B}}$, is the block $n \times m$ matrix with $p \times q$ blocks defined by $(H^{\mathcal{B}})_{ij} := H_{ji}$, see [14].

The following results give the eigenvector formula for Fiedler pencil and generalized Fiedler pencil of $P(\lambda)$, [9].

Theorem 4.3. *[9] Let $P(\lambda)$ be an matrix polynomial of degree m and $P_0, \ldots, P_m$ be the Horner shifts of $P(\lambda)$. Let σ be a permutation of $\{0, 1, \ldots, m-1\}$ with $csf(\sigma) = (\boldsymbol{b}_\beta, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_k = (t_{k-1}+1 : t_k)$, for $k = 1, \ldots, \beta$. Let $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ. Then*

$$E_\sigma(P) := \begin{bmatrix} B_0 \, B_1 \, \ldots \, B_{m-1} \end{bmatrix}^{\mathcal{B}} \tag{14}$$

where, if $\sigma(i) \in \boldsymbol{b}_k$, for some $k = 1, \ldots, \beta$, then

$$B_i = \begin{cases} \lambda^{k-1} I_n, & \text{if } i = m - t_k - 1 \\ \lambda^{k-1} P_i, & \text{otherwise.} \end{cases} \tag{15}$$

Let $H_\sigma(P) := E_{rev\,\sigma}(P^T)$. Then $E_\sigma(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L_\sigma(\lambda))$ and $H_\sigma(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(L_\sigma(\lambda))$ are isomorphisms. Moreover, if σ has c_0 consecutions at 0, then the $(m - c_0)$th block of $E_\sigma(P)$ is equal to I_n, and if σ has i_0 inversions at 0, then the $(m - i_0)$th block of $H_\sigma(P)$ is equal to I_n.

Theorem 4.4. *[9] Let $P(\lambda)$ be an matrix polynomial of degree m. Let P_i for $i = 0, 1, \ldots, m$, be the ith Horner shift of matrix polynomial P. Let $\omega := (\omega_0, \omega_1)$ and $T_\omega(\lambda) = \lambda M_{\omega_1} - M_{\omega_0}$ be a PGF pencil of $P(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(\lambda)$. Assume that ω_1 has c_{-m} consecutions at $-m$, and $csf(\omega_1) = (\tau, -m : -m + c_{-m})$. Set $\xi := csf(-rev\tau, \omega_0) = (\boldsymbol{b}_\alpha, \boldsymbol{b}_{\alpha-1}, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_k = (t_{k-1}+1 : t_k)$, for $k = 1, \ldots, \alpha$. Define $E_{\omega_0,\omega_1}(P) := E_\xi(P)$, if $c_{-m} = 0$ and*

$$E_{\omega_0,\omega_1}(P) := \begin{bmatrix} \lambda^\alpha [P_0 \quad P_1 \quad \ldots \quad P_{c_{-m}-1}] & B_{c_{-m}} & B_{c_{-m}+1} & \ldots & B_{m-1} \end{bmatrix}^{\mathcal{B}}, \tag{16}$$

if $c_{-m} > 0$, where $E_\xi(P)$ is as in Theorem 4.3 and if $\xi(i) \in \boldsymbol{b}_j$, for some $j = 1, 2, \ldots, \alpha$, then the block $B_{i+c_{-m}}$ is as in (15). Then $E_{\omega_0,\omega_1}(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(T_\omega(\lambda))$ is an isomorphism.

Further, set rev $\omega := (rev\,\omega_0, rev\,\omega_1)$ and define $H_{\omega_0,\omega_1}(P) := E_{rev\,\omega_0, rev\,\omega_1}(P^T)$. Then $H_{\omega_0,\omega_1}(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(T_\omega(\lambda))$ is an isomorphism. Furthermore, if ω_0 has c_0 consecutions at 0, then the $(m - c_0)$th block of $E_{\omega_0,\omega_1}(P)$ is equal to I_n, and if ω_0 has i_0 inversions at 0, then the $(m - i_0)$th block of $H_{\omega_0,\omega_1}(P)$ is equal to I_n.

Theorem 4.5. *[6] Let $\mathcal{S}(\lambda)$ and $G(\lambda)$ be as in (5) and (7). Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Define $f : \mathbb{C}^n \to \mathbb{C}^{n+r}$ and $g : \mathbb{C}^n \to \mathbb{C}^{n+r}$ by*

$$f(x) := \begin{bmatrix} x \\ (\lambda_0 E - A)^{-1} B x \end{bmatrix} \text{ and } g(x) := \begin{bmatrix} x \\ (C(\lambda_0 E - A)^{-1})^T x \end{bmatrix}.$$

Then the maps $f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathcal{S}(\lambda_0))$ and $g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathcal{S}(\lambda_0))$ are isomorphisms.

Theorem 4.6. *[2, 6] Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler linearization of $\mathcal{S}(\lambda)$ associated with a bijection σ. Let $\lambda \in \mathbb{C}$ and $E_\sigma(P)$, and $H_\sigma(P)$ be as in Theorem 4.3. Define $\mathbb{E}_\sigma(\mathcal{S}) : \mathbb{C}^{n+r} \to \mathbb{C}^{nm+r}$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathbb{C}^{n+r} \to \mathbb{C}^{nm+r}$ by*

$$\mathbb{E}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} E_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right] \text{ and } \mathbb{H}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} H_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right].$$

Then $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ are isomorphisms.

Remark 4.7. *Since by Theorem 4.5, $f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathcal{S}(\lambda_0))$ and $g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathcal{S}(\lambda_0))$ are isomorphisms and, by Theorem 4.6, $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda_0)) \to \mathcal{N}_r(\mathbb{L}(\lambda_0))$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda_0)) \to \mathcal{N}_l(\mathbb{L}(\lambda_0))$ are isomorphisms, it follows that $\mathbb{E}_\sigma(G) = \mathbb{E}_\sigma(\mathcal{S}) \circ f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathbb{L}(\lambda_0))$ and $\mathbb{H}_\sigma(G) = \mathbb{H}_\sigma(\mathcal{S}) \circ g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathbb{L}(\lambda_0))$ are isomorphisms.*

Next, consider a PGF/GF pencil $\mathbb{T}_\omega(\lambda)$ of the system matrix $\mathcal{S}(\lambda)$. We have to determine the isomorphism between $\mathcal{N}_r(G(\lambda))$ and $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. So, by the remark (4.7), it is sufficient to determine the isomorphism between $\mathcal{N}_r(\mathcal{S}(\lambda))$ and $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

Theorem 4.8 (Eigenvector formula for PGF pencil of system matrix). *Let P_i for $i = 0, 1, \ldots, m$, be the ith Horner shift of matrix polynomial $P(\lambda)$. Let $\omega := (\omega_0, \omega_1)$ and $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be a PGF pencil of $\mathcal{S}(\lambda)$. Let $\lambda \in \mathbb{C}$ be a spectrum of $\mathcal{S}$.*

(a) Define $\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S}) := \left[\begin{array}{c|c} E_{\omega_0,\omega_1}(P) & 0 \\ \hline 0 & I_r \end{array}\right]$, where $E_{\omega_0,\omega_1}(P)$ is as in Theorem 4.4. Then $\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \longrightarrow \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$ is an isomorphism.

(b) Set rev $\omega := (rev\,\omega_0, rev\,\omega_1)$ and define $\mathbb{H}_{\omega_0,\omega_1}(\mathcal{S}) := \mathbb{E}_{(rev\,\omega_0, rev\,\omega_1)}(\mathcal{S}^T) = \left[\begin{array}{c|c} E_{(rev\,\omega_0, rev\,\omega_1)}(P^T) & \\ \hline & I_r \end{array}\right]$. Then $\mathbb{H}_{\omega_0,\omega_1}(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \longrightarrow \mathcal{N}_l(\mathbb{T}_\omega(\lambda))$ is an isomorphism.

Proof. Suppose that ω_1 and ω_0 are in column standard form. Assume that ω_1 has c_{-m} consecutions at $-m$. Then, there exists an index tuple τ such that

$$\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_\tau\mathbb{M}_{(-m:-m+c_{-m})} - \mathbb{M}_{\omega_0}. \tag{17}$$

Case I : If $c_{-m} = 0$, then $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_\tau\mathbb{M}_{-m} - \mathbb{M}_{\omega_0}$ and $\mathbb{L}_\sigma(\lambda) = \mathbb{M}_{-rev\,\tau}\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{-m} - \mathbb{M}_{(-rev\,\tau,\omega_0)}$ is a Fiedler pencil associated with a bijection $\sigma = (-rev\,\tau, \omega_0)$. Hence $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) = \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Observe that the index tuple $(-rev\tau, \omega_0)$ is a permutation of $\{0, 1, \ldots, m-1\}$. Set $\xi = csf(\sigma) = csf(-rev\,\tau, \omega_0) = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$. Then by Theorem 4.6, $\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S}) := \mathbb{E}_\xi(\mathcal{S}) = \left[\begin{array}{c|c} E_\xi(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.
Case II : Suppose that $c_{-m} \neq 0$. Since $c_{-m} \neq 0$, $\mathbb{L}_{\widehat{\sigma}}(\lambda) = \mathbb{M}_{-rev\,\tau}\mathbb{T}_\omega(\lambda)\mathbb{M}_{(m-c_{-m}:m-1)}$

$$= \lambda\mathbb{M}_{-rev\,\tau}\mathbb{M}_\tau\mathbb{M}_{(-m:-m+c_{-m})}\mathbb{M}_{(m-c_{-m}:m-1)} - \mathbb{M}_{-rev\,\tau}\mathbb{M}_{\omega_0}\mathbb{M}_{(m-c_{-m}:m-1)}$$

$$= \lambda\mathbb{M}_{(-m:-m+c_{-m})}\mathbb{M}_{(m-c_{-m}:m-1)} - \mathbb{M}_{-rev\,\tau}\mathbb{M}_{\omega_0}\mathbb{M}_{(m-c_{-m}:m-1)} = \lambda\mathbb{M}_{-m} - \mathbb{M}_{(-rev\,\tau,\omega_0,m-c_{-m}:m-1)}$$

is a Fiedler pencil associated with a bijection $\widehat{\sigma} = (-rev\,\tau, \omega_0, m - c_{-m} : m-1)$. So the map $\mathcal{N}_r(\mathbb{L}_{\widehat{\sigma}}(\lambda)) \to \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, $\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \mathbb{M}_{(m-c_{-m}:m-1)}\begin{bmatrix} u \\ v \end{bmatrix}$ is an isomorphism. By Theorem 4.6, $\mathbb{E}_{\widehat{\sigma}}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\widehat{\sigma}}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{L}_{\widehat{\sigma}}(\lambda))$, where $E_{\widehat{\sigma}}(P)$ is as in Theorem 4.3. Consequently, the map $\mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_{\widehat{\sigma}}(\mathcal{S})\begin{bmatrix} x \\ y \end{bmatrix}$ is an isomorphism. Now our aim is to calculate $\mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_{\widehat{\sigma}}(\mathcal{S})$. Let $\sigma = csf(\widehat{\sigma}) = csf(-rev\,\tau, \omega_0, m - c_{-m} : m-1)$. Then

$$\mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} M_{(m-c_{-m}:m-1)} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} E_\sigma(P) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{(m-c_{-m}:m-1)}E_\sigma(P) & \\ \hline & I_r \end{array}\right].$$

By Theorem 3.3, [9], we have $M_{(m-c_{-m}:m-1)}E_\sigma(P) = E_{\omega_0,\omega_1}(P)$. Hence $\mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_\sigma(\mathcal{S}) = \mathbb{E}_{\omega_0,\omega_1}(\mathcal{S})$. This completes the proof of (*a*).

Next, note that $\mathcal{N}_l(\mathcal{S}(\lambda)) = \mathcal{N}_r(\mathcal{S}(\lambda)^T)$ and $\mathcal{N}_l(\mathbb{T}_\omega(\mathcal{S})) = \mathcal{N}_r(\mathbb{T}_\omega(\mathcal{S})^T) = \mathcal{N}_r(\mathbb{T}_{rev\,\omega}(\mathcal{S}^T))$. This shows that $\mathbb{H}_{\omega_0,\omega_1}(\mathcal{S}) := \mathbb{E}_{rev\,\omega_0,rev\,\omega_1}(\mathcal{S}^T)$ is an isomorphism from $\mathcal{N}_l(\mathcal{S}(\lambda))$ to $\mathcal{N}_l(\mathbb{T}_\omega(\lambda))$. $\square$

The next result directly follows from Remark 4.7

Corollary 4.9 (Eigenvector formula for PGF pencil of $G(\lambda)$). *Let P_i for $i = 0, 1, \ldots, m$, be the ith Horner shift of matrix polynomial P. Let $\omega = (\omega_0, \omega_1)$ and $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be a PGF pencil of $G(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$.*

(a) Define $\mathbb{E}_{\omega_0,\omega_1}(G) := \begin{bmatrix} E_{\omega_0,\omega_1}(P) \\ (\lambda E - A)^{-1}B \end{bmatrix}$, where $E_\omega(P)$ is as in Theorem 4.4. Then $\mathbb{E}_{\omega_0,\omega_1}(G) : \mathcal{N}_r(G(\lambda)) \longrightarrow \mathcal{N}_r(\mathbb{T}(\lambda))$ is an isomorphism.

(b) Set $rev\ \omega := (rev\,\omega_0, rev\,\omega_1)$ and define $\mathbb{H}_{\omega_0,\omega_1}(G) := \mathbb{E}_{(rev\,\omega_0,rev\,\omega_1)}(G^T) := \begin{bmatrix} E_{(rev\,\omega_0,rev\,\omega_1)}(P^T) \\ (C(\lambda E - A)^{-1})^T \end{bmatrix}$. Then $\mathbb{H}_{\omega_0,\omega_1}(G) : \mathcal{N}_l(G(\lambda)) \longrightarrow \mathcal{N}_l(\mathbb{T}_\omega(\lambda))$ is an isomorphism.

Example 4.10. *Consider the system matrix $\mathcal{S}(\lambda)$ with matrix polynomial $P(\lambda)$ of degree $m = 3$ and the associated transfer function $G(\lambda)$. Let $\mathbb{K}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0} = \lambda\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{-1} - \mathbb{M}_0$ be the PGF pencil of $G(\lambda)$. Here ω_1 has 2 consecutions at -3, i.e., $c_{-3} = 2$. Now*

$$\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{-1} = \left[\begin{array}{ccc|c} 0 & 0 & A_3 & \\ I_n & 0 & A_2 & \\ 0 & I_n & A_1 & \\ \hline & & & -E \end{array}\right], \text{ so } \mathbb{K}_\omega(\lambda) = \left[\begin{array}{ccc|c} -I_n & 0 & \lambda A_3 & \\ \lambda I_n & -I_n & \lambda A_2 & \\ 0 & \lambda I_n & \lambda A_1 + A_0 & C \\ \hline & & B & A - \lambda E \end{array}\right].$$

Thus by the Corollary 4.9, $\begin{bmatrix} \lambda A_3 \\ \lambda^2 A_3 + \lambda A_2 \\ I_n \\ (\lambda E - A)^{-1}B \end{bmatrix} x \in \mathcal{N}_r(\mathbb{K}(\lambda))$, *where $x \in \mathcal{N}_r(G(\lambda))$.* $\blacksquare$

The next result describes eigenvector formula for GF pencils of $\mathcal{S}(\lambda)$ for the case when $\mathbb{M}_0^{-1}$ does not appear in the GF pencils. That is, $m \in \sigma$.

Theorem 4.11 (Eigenvector formula for GF pencil of $\mathcal{S}(\lambda)$). *Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\boldsymbol{t}} - \mathbb{M}_\sigma$ be a GF pencil of $\mathcal{S}(\lambda)$ such that $0, m \in \sigma$. Let $\sigma' = \sigma \setminus \{m\}$, $\xi := csf(-rev\boldsymbol{t}, \sigma') = (\boldsymbol{b}_\alpha, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_\alpha = (t_{k-1} + 1 : t_k)$ for $k = 1, 2, \ldots, \alpha$.*
Case I : Suppose that $m - 1$ is to the left of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) If m is to the right of 0 in σ then

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(\mathcal{S}) := \left[\begin{array}{c|c} \begin{bmatrix} A_m B_0 & B_1 & \cdots & B_{m-1} \end{bmatrix}^{\mathcal{B}} & \\ \hline & -E \end{array}\right] \tag{18}$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\xi(i) \in \boldsymbol{b}_j$, for some $j = 1, 2, \ldots, \alpha$, and the block B_i's are as in Theorem 4.3.

(b) If m is to the left of 0 in σ and s is the largest index such that $(m - s : m) \in \sigma$ then set $\tau_0 = \sigma \setminus \{m - s : m\}$. Then $\mathbb{E}_{\sigma,\boldsymbol{t}}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\tau_0,\tau_1}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\tau_1 = (-rev(m - s : m), \boldsymbol{t})$ and $E_{\tau_0,\tau_1}(P)$ is as in Theorem 4.4.

Case II : Suppose that $m - 1$ is to the right of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) *If m is to the left of 0 in σ, then $\mathbb{E}_{\sigma,t}(\mathcal{S}) := \mathbb{E}_{\xi}(\mathcal{S})$, is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_\xi(\mathcal{S})$ is given in Theorem 4.6.*

(b) *If m is to the right of 0 in σ and h is the largest index such that $(m, m-1, \ldots, m-h)$ in σ then set $\sigma'' = \sigma \setminus \{m, m-1, \ldots, m-h\}$, $\sigma_4 = (-rev\mathbf{t}, \sigma'')$, $\sigma_3 = (-(m-h), -(m-1), -m) =: (\mathbf{t}_1, -m)$ and $\xi' = csf(-rev\mathbf{t}_1, \sigma_4)$. Then*

$$\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) := \mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)}\mathbb{M}_{-m}\mathbb{E}_{\xi'}(\mathcal{S}) = \left[\begin{array}{c|c} B_1 & 0 \\ B_2 & \\ \vdots & \\ B_h & \\ A_mB_0 + A_{m-1}B_1 + \cdots + A_{m-h}B_h & \\ B_{h+2} & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & -E \end{array}\right].$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_{\xi'}(\mathcal{S})$ is given in Theorem 4.6 and $B_i, i = 1 : m-1$, are given in Theorem 4.3.

Proof. Case *I*: If $m-1$ is to the left of m in $(-rev\mathbf{t}, \sigma)$, then $(-rev\mathbf{t}, \sigma)$ is equivalent to either $(-rev\mathbf{t}, \sigma', m)$ or $(-rev\mathbf{t}, m, \sigma')$, since $\mathbb{M}_0$ and $\mathbb{M}_m$ do not commute, so there is a possibility for m to be either to the left or to the right of 0.

(a) If m is right of 0 then $(-rev\,\mathbf{t}, \sigma) \sim (-rev\mathbf{t}, \sigma', m)$, so $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_\mathbf{t} - \mathbb{M}_\sigma = \lambda\mathbb{M}_\mathbf{t} - \mathbb{M}_{\sigma'}\mathbb{M}_m$ and hence

$$\mathbb{L}_\tau(\lambda) := \mathbb{M}_{-rev\mathbf{t}}\mathbb{T}_\omega(\lambda)\mathbb{M}_{-m} = \mathbb{M}_{-rev\mathbf{t}}(\lambda\mathbb{M}_\mathbf{t} - \mathbb{M}_{\sigma'}\mathbb{M}_m)\mathbb{M}_{-m} = \lambda\mathbb{M}_{-m} - \mathbb{M}_{-rev\mathbf{t}}\mathbb{M}_{\sigma'}$$

is a Fiedler pencil associated with a bijection $\tau = (-rev\mathbf{t}, \sigma')$. So $v \mapsto \mathbb{M}_{-m}v$ is an isomorphism from $\mathcal{N}_r(\mathbb{L}_\tau(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. By Theorem 4.6, we have $\mathbb{E}_\tau(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\tau(\lambda))$ is an isomorphism. Thus for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$ we have $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbb{M}_{-m}\mathbb{E}_\tau(\mathcal{S})\begin{bmatrix} x \\ y \end{bmatrix}$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Since $\xi = csf(\tau) = csf(-rev\,\mathbf{t}, \sigma')$, we have

$$\mathbb{M}_{-m}\mathbb{E}_\xi(\mathcal{S}) = \mathbb{M}_{-m}\left[\begin{array}{c|c} E_\xi(P) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right]\left[\begin{array}{c|c} B_0 & 0 \\ \vdots & 0 \\ B_{m-1} & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} A_mB_0 & B_1 & \cdots & B_{m-1} \end{bmatrix}^{\mathcal{B}} & \\ \hline & -E \end{array}\right].$$

(b) Since m is left of 0 in σ, we have $(-rev\mathbf{t}, \sigma) \sim (-rev\mathbf{t}, m-s : m, \tau_0)$. Consequently, we have $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_\mathbf{t} - \mathbb{M}_\sigma = \lambda\mathbb{M}_\mathbf{t} - \mathbb{M}_{m-s:m-2}\mathbb{M}_{m-1}\mathbb{M}_m\mathbb{M}_{\tau_0}$. This shows that $\mathbb{L}_\tau(\lambda) := \mathbb{M}_{-rev(m-s:m)}\mathbb{T}_\omega(\lambda)$ $= \mathbb{M}_{-rev(m-s:m)}(\lambda\mathbb{M}_\mathbf{t} - \mathbb{M}_{m-s:m-2}\mathbb{M}_{m-1}\mathbb{M}_m\mathbb{M}_{\tau_0}) = \lambda\mathbb{M}_{-rev(m-s:m)}\mathbb{M}_\mathbf{t} - \mathbb{M}_{\tau_0} = \lambda\mathbb{M}_{\tau_1} - \mathbb{M}_{\tau_0}$ is a PGF pencil. Hence $\mathcal{N}_r(\mathbb{T}_\omega(\lambda)) = \mathcal{N}_r(\mathbb{L}_\tau(\lambda))$. By Theorem 4.8, we have $\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\tau_0,\tau_1}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

Case*II*: (a) If $m-1$ is to the right of m and m is to the left of 0 in $(-rev\mathbf{t}, \sigma)$, then $(-rev\mathbf{t}, \sigma)$ is equivalent to $(m, -rev\mathbf{t}, \sigma')$. So $\mathbb{L}_\tau(\lambda) = \mathbb{M}_{-m}\mathbb{M}_{-rev\mathbf{t}}\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{-m} - \mathbb{M}_{-rev\mathbf{t}}\mathbb{M}_{\sigma'}$ is a Fiedler pencil associated with a bijection $\tau = (-rev\mathbf{t}, \sigma')$. Hence $\mathcal{N}_r(\mathbb{L}_\tau(\lambda)) = \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Since $\xi = csf(\tau)$, by Theorem 4.6, $\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) := \mathbb{E}_\xi(\mathcal{S})$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

(b) If $m-1$ is to the right of m in $(-rev\mathbf{t}, \sigma)$ and m is to the right of 0 in σ, then $(-rev\mathbf{t}, \sigma) \sim (-rev\mathbf{t}, \sigma'', m, m-1, \ldots, m-h)$. So

$$\mathbb{L}_\tau(\lambda) = \mathbb{M}_{-rev\mathbf{t}}\mathbb{T}_\omega(\lambda)\mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)}\mathbb{M}_{-m} = \lambda\mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)}\mathbb{M}_{-m} - \mathbb{M}_{-rev\mathbf{t}}\mathbb{M}_{\sigma''} = \lambda\mathbb{M}_{\sigma_3} - \mathbb{M}_{\sigma_4}$$

is a PGF pencil. Hence $\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)}\mathbb{M}_{-m}v$ is an isomorphism from $\mathcal{N}_r(\mathbb{L}_\tau)$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

Since σ_3 has always 0 consecutions at $-m$, i.e., $c_{-m} = 0$, by Theorem 4.8, $\mathbb{E}_{\xi'}(\mathcal{S}) = \left[\begin{array}{c|c} B_0 & \\ B_1 & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{L}_\tau(\lambda))$, where $B_i, i = 0 : m-1$ are as in Theorem 4.3. This shows that

$$\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) = \mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)}\mathbb{M}_{-m}\mathbb{E}_{\xi'}(\mathcal{S})$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Now

$$\mathbb{M}_{-(m-1)}\mathbb{M}_{-m} = \left[\begin{array}{cc|c|c} 0 & I_n & & \\ A_m & A_{m-1} & & \\ \hline & & I_{(m-2)n} & \\ \hline & & & -E \end{array}\right] \text{ and } \mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)}\mathbb{M}_{-m} = \left[\begin{array}{ccccc|c|c} 0 & I_n & & & & & \\ 0 & 0 & I_n & & & & \\ & \vdots & & & & & \\ 0 & \vdots & 0 & I_n & & & \\ A_m & A_{m-1} & \cdots & A_{m-h} & & & \\ \hline & & & & & I_{(m-h-1)n} & \\ \hline & & & & & & -E \end{array}\right].$$

Hence

$$\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) = \left[\begin{array}{cccc|c|c} 0 & I_n & & & & \\ 0 & 0 & I_n & & & \\ & \vdots & & & & \\ 0 & \vdots & 0 & I_n & & \\ A_m & A_{m-1} & \cdots & A_{m-h} & & \\ \hline & & & & I_{(m-h-1)n} & \\ \hline & & & & & -E \end{array}\right] \left[\begin{array}{c|c} B_0 & \\ B_1 & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} B_1 & 0 \\ B_2 & \\ \vdots & \\ B_h & \\ A_mB_0 + A_{m-1}B_1 + \cdots + A_{m-h}B_h & \\ B_{h+2} & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & -E \end{array}\right].$$

□

The next result directly follows from Remark 4.7.

Corollary 4.12 (Eigenvector formula for GF pencil of transfer function). *Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\boldsymbol{t}} - \mathbb{M}_\sigma$ ba GF pencil of a regular $G(\lambda)$ given in (7). Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Assume that $0, m \in \sigma$. Let $\sigma' = \sigma \setminus \{m\}$, $\xi = csf(-rev\boldsymbol{t}, \sigma')$.*
Case I : Suppose that $m-1$ is to the left of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) If m is to the right of 0 then

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(G) := \begin{bmatrix} A_mB_0 \\ B_1 \\ \vdots \\ B_{m-1} \\ -E(\lambda E - A)^{-1}B \end{bmatrix} \tag{19}$$

is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\xi(i) \in \boldsymbol{b}_j$, for some $j = 1, 2, \ldots, \alpha$, and the block B_i is as in Theorem 4.3.

(b) If m is to the left of 0 in σ and s is the largest index such that $(m-s : m) \in \sigma$ then set $\tau_0 = \sigma \setminus \{m-s : m\}$. Then $\mathbb{E}_{\sigma,\boldsymbol{t}}(G) = \begin{bmatrix} E_{\tau_0,\tau_1}(P) \\ (\lambda E - A)^{-1}B \end{bmatrix}$ is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\tau_1 = (-rev(m-s : m), \boldsymbol{t})$ and $E_{\tau_0,\tau_1}(P)$ is as in Theorem 4.4.

Case II : Suppose that $m-1$ is to the right of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) If m is to the left of 0, then $\mathbb{E}_{\sigma,\boldsymbol{t}}(G) := \mathbb{E}_\xi(G)$ is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_\xi(G)$ is given in Remark 4.7.

(b) If m is to the right of 0 in σ and h is the largest index such that $(m, m-1, \ldots, m-h)$ in σ then set $\sigma'' = \sigma \setminus \{m, m-1, \ldots, m-h\}$, $\sigma_4 = (-rev\boldsymbol{t}, \sigma'')$, $\sigma_3 = (-(m-h), -(m-1), -m) =: (\boldsymbol{t}_1, -m)$ and $\xi' = csf(-rev\boldsymbol{t}_1, \sigma_4)$.

Then

$$\mathbb{E}_{\sigma,t}(G) := \mathbb{M}_{-(m-1)}\mathbb{M}_{-m}\mathbb{E}_{\xi'}(\mathcal{S}) = \left[\begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_h \\ A_mB_0 + A_{m-1}B_1 + \cdots + A_{m-h}B_h \\ B_{h+1} \\ \vdots \\ B_{m-1} \\ \hline -E(\lambda E - A)^{-1}B \end{array}\right]$$

is an isomorphism from $\mathcal{N}_r(G(\lambda))$ *to* $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, *where* $\mathbb{E}_{\xi'}(G)$ *is given in Remark 4.7 and* $B_i, i = 1 : m-1$, *are given in Theorem 4.3.*

The following examples illustrate the result in Corollary 4.12.

Example 4.13. *Consider a rational matrix function* $G(\lambda) = \lambda^6 A_6 + \ldots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$ *and the GF pencil* $\mathbb{T}(\lambda) = \lambda\mathbb{M}_{\boldsymbol{t}} - \mathbb{M}_\sigma = \lambda\mathbb{M}_{-1}\mathbb{M}_{-5}\mathbb{M}_{-3} - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_6\mathbb{M}_4$. *Now* $\sigma = (2,0,6,4)$, $\boldsymbol{t} = (-1,-5,-3)$, $-rev\boldsymbol{t} = (3,5,1)$, *and* $(-rev\boldsymbol{t},\sigma) = (3,5,1,2,0,6,4)$. *Note that* 5 *is to the left of* 6 *in* $(-rev\boldsymbol{t},\sigma)$ *and* 6 *is to the right of* 0 *in* σ. *This is the case* $I(a)$ *of Corollary 4.12. Hence by Corollary 4.12, we have* $\sigma' = (2,0,4)$, $\xi = csf(-rev\,\boldsymbol{t},\sigma') = (5,3:4,1:2,0)$, *and*

$$\mathbb{E}_{\sigma,t}(G) = \left[\begin{array}{c} \left[\begin{array}{cccccc} A_6\lambda^3 I_n & \lambda^2 I_n & \lambda^2 P_2 & \lambda I_n & \lambda P_4 & I_n \end{array}\right]^{\mathcal{B}} \\ -E(\lambda E - A)^{-1}B \end{array}\right].$$

Thus $\mathbb{E}_{\sigma,t}(G)x$ *is a right eigenvector of* $\mathbb{T}(\lambda)$.

Now, consider the GF pencil $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{-1}\mathbb{M}_{-5}\mathbb{M}_{-3} - \mathbb{M}_6\mathbb{M}_2\mathbb{M}_0\mathbb{M}_4 = \lambda\mathbb{M}_{\boldsymbol{t}} - \mathbb{M}_\sigma$. *Then* $\sigma = (6,2,0,4)$, $\boldsymbol{t} = (-1,-5,-3)$, *and* $-rev\boldsymbol{t} = (3,5,1)$. *By Corollary 4.12, we have* $\sigma' = (2,0,4)$ *and* $(-rev\boldsymbol{t},\sigma) = (3,5,1,6,2,0,4) \sim (3,5,1,6,2,0,4)$. *Note that this is the Case* $I(b)$ *of Corollary 4.12. Thus* $\tau_1 = (-6,-1,-5,-3)$ *and* $\tau_0 = (2,0,4)$. *Hence by corollary 4.9 we have* $c_{-m} = 1$, $\xi = (3:4,1:2,0)$ *and*

$$\mathbb{E}_{\sigma,t}(G) = \left[\begin{array}{c} \left[\begin{array}{cccccc} \lambda^3 P_0 & \lambda^2 I_n & \lambda^2 P_2 & \lambda I_n & \lambda P_4 & I_n \end{array}\right]^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{array}\right].$$

Thus $\mathbb{E}_{\sigma,t}(G)x$ *is a right eigenvector of* $\mathbb{T}_\omega(\lambda)$.

Finally, consider the GF pencil $\mathbb{T}(\lambda) = \lambda\mathbb{M}_{-1}\mathbb{M}_{-4}\mathbb{M}_{-3} - \mathbb{M}_2\mathbb{M}_6\mathbb{M}_0\mathbb{M}_5$. *Now* $\sigma = (2,6,0,5)$, $\boldsymbol{t} = (-1,-4,-3)$, *and* $-rev\,\boldsymbol{t} = (3,4,1)$. *Thus* $\sigma' = (2,0,5)$ *and* $(-rev\,\boldsymbol{t},\sigma) = (3,4,1,2,6,0,5) \sim (6,3,4,1,2,0,5)$. *Note that this is the case* $II(a)$ *of Corollary 4.12. Hence* $\xi = csf(-rev\,\boldsymbol{t},\sigma') = (3:5,1:2,0)$ *and by Corollary 4.12, we have*

$$\mathbb{E}_{\sigma,t}(G) = \left[\begin{array}{c} \left[\begin{array}{cccccc} \lambda^2 I_n & \lambda^2 P_1 & \lambda^2 P_2 & \lambda I_n & \lambda P_4 & I_n \end{array}\right]^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{array}\right].$$

Thus $\mathbb{E}_{\sigma,t}(G)x$ *is a right eigenvector of* $\mathbb{T}(\lambda)$.

■

5. Generalized Fiedler pencil with repetition

It is shown that a self-adjoint $\mathcal{S}(\lambda)$ does not admit a self-adjoint GF pencil when m is even see, [3]. It turns out that allowing Fiedler matrices to repeat in the products $\mathbb{M}_\sigma$ and $\mathbb{M}_\tau$ enables a pencil $\mathbb{L}(\lambda) = \lambda\mathbb{M}_\tau - \mathbb{M}_\sigma$ to have certain desired properties.

Example 5.1. *Consider* $\mathcal{S}(\lambda)$ *with* $m = 3$. *Let* $\mathbb{L}(\lambda) := \lambda\mathbb{M}_{-3}\mathbb{M}_1 - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1$. *Then*

$$\mathbb{L}(\lambda) = \lambda\left[\begin{array}{ccc|c} A_3 & & & \\ & -A_1 & I_n & \\ & I_n & 0 & \\ \hline & & & -E \end{array}\right] - \left[\begin{array}{ccc|c} -A_2 & -A_1 & I_n & 0 \\ -A_1 & -A_0 & 0 & -C \\ I_n & 0 & 0 & 0 \\ \hline 0 & -B & 0 & -A \end{array}\right].$$

Note that $\mathbb{L}(\lambda) = (\lambda\mathbb{M}_{-3} - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0)\mathbb{M}_1$ *is not a GF pencil. Also note that in this pencil the Fiedler matrix* $\mathbb{M}_1$ *is repeated and at the same time the pencil is operation-free. Further,* $\mathbb{L}(\lambda)$ *is symmetric when* $\mathcal{S}(\lambda)$ *is symmetric.* ■

So we are interested in finding the operation-free pencils which allow repetition of Fiedler matrices. We define generalized Fiedler pencils with repetition (GFPR), which are operation-free and derive explicitly the eigenvector formula for GFPR. Further, we define structure preserving GFPRs.

Definition 5.2 (GFPR). *Let* $\mathcal{S}(\lambda)$ *be the system matrix. Let* $0 \leq h \leq m-1$*, and let* σ *and* τ *be permutations of* $\{0, 1, \ldots, h\}$ *and* $\{-m, -m+1, \ldots, -h-1\}$*, respectively. Let* σ_1 *and* σ_2 *be index tuples with elements from* $\{1, 2, \ldots, h-1\}$ *such that* $(\sigma_1, \sigma, \sigma_2)$ *satisfies the SIP. Similarly, let* τ_1 *and* τ_2 *be index tuples with elements from* $\{-m+1, \ldots, -h-2\}$ *such that* (τ_1, τ, τ_2) *satisfies the SIP. Then the pencil* $\mathbb{L}(\lambda) := \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ *is called a generalized Fiedler pencil with repetition (GFPR) of* $\mathcal{S}(\lambda)$*.*

Remark 5.3. *Note that in the Definition 5.2, 0 is simple index and so* $\mathbb{L}(\lambda)$ *can always be expressed as* $\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}(\lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma})\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$*, where* $\lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma}$ *is a PGF pencil. Thus a GFPR is strictly equivalent to a PGF pencil. If* $\tau_1, \tau_2, \sigma_1, \sigma_2$ *are all the empty index tuples, then* $\mathbb{M}_{\tau_1}, \mathbb{M}_{\tau_2}, \mathbb{M}_{\sigma_2}, \mathbb{M}_{\sigma_1}$ *are all* I_{nm+r} *and* $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma}$ *is a PGF pencil and hence a GF pencil. Note, however, that not all GF pencils are GFPR. For example,* $\mathbb{L}(\lambda) = \lambda\mathbb{M}_2^{-1}\mathbb{M}_1^{-1} - \mathbb{M}_3\mathbb{M}_6^{-1}\mathbb{M}_0\mathbb{M}_5$ *is a GF pencil of* $\mathcal{S}(\lambda)$ *with* $m = 6$*, but not a GFPR.*

We now show that a GFPR for $\mathcal{S}(\lambda)$ is a trimmed structured linearization of $\mathcal{S}(\lambda)$ and that the GFPR pencil is also a linearization of $G(\lambda)$ whenever the realization (1) of $G(\lambda)$ is minimal.

Theorem 5.4. *Let* $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ *be a GFPR of* $\mathcal{S}(\lambda)$*. Then* $\mathbb{L}(\lambda)$ *is a linearization of* $\mathcal{S}(\lambda)$*. If* $G(\lambda)$ *is minimal, then* $\mathbb{L}(\lambda)$ *is also a linearization of* $G(\lambda)$*.*

Proof. We have $\mathbb{L}(\lambda) = \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}(\lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma})\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$. Since $\mathbb{M}_{\tau_i} = \left[\begin{array}{c|c} M_{\tau_i} & \\ \hline & I_r \end{array}\right]$, $\mathbb{M}_{\sigma_i} = \left[\begin{array}{c|c} M_{\sigma_i} & \\ \hline & I_r \end{array}\right]$, $i = 1, 2$, the pencil $\mathbb{L}(\lambda)$ is strictly equivalent to a PGF pencil. Therefore every GFPR $\mathbb{L}(\lambda)$ of $\mathcal{S}(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$. □

Example 5.5. *Consider the system matrix* $\mathcal{S}(\lambda)$ *with* $m = 12$*. Consider* $\sigma = (6, 1 : 5, 0)$*,* $\sigma_2 = (1 : 4), \tau = (-7, -8, -11 : -9, -12), \tau_2 = (-11 : -10), \sigma_1 = \{\emptyset\}, \tau_1 = (-8, -9)$*. Then* $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ *is a GFPR of* $\mathcal{S}(\lambda)$*.* ■

Definition 5.6 (Block transpose [1]). *Let* $\mathcal{A}$ *be an* $(mn + r) \times (mn + r)$ *system matrix given by*

$$\mathcal{A} := \left[\begin{array}{c|c} A & e_i \otimes X \\ \hline e_j^T \otimes Y & Z \end{array}\right],$$

where $A := [A_{ij}]$ *is an* $m \times m$ *block matrix with* $A_{ij} \in \mathbb{C}^{n\times n}$*,* $X \in \mathbb{C}^{n\times r}, Y \in \mathbb{C}^{r\times n}, Z \in \mathbb{C}^{r\times r}$ *and* e_k *is the k-th column of* I_m*. The* block transpose *of* $\mathcal{A}$*, denoted by* $\mathcal{A}^{\mathbb{B}}$*, is defined by*

$$\mathcal{A}^{\mathbb{B}} := \left[\begin{array}{c|c} A^{\mathcal{B}} & e_j \otimes X \\ \hline e_i^T \otimes Y & Z \end{array}\right],$$

where $A^{\mathcal{B}}$ *is the block transpose of A.*

Definition 5.7 (Block-symmetry). *A block* $p \times p$ *matrix A with* $m \times n$ *blocks is said to be* block-symmetric *if* $A^{\mathcal{B}} = A$*.*

Since $M_i^{\mathcal{B}} = M_i$ for $i = 0 : m$, by Definition 5.6, we have $(\mathbb{M}_i)^{\mathbb{B}} = \mathbb{M}_i$ for $i = 0 : m$. Thus the Fiedler matrices are block symmetric.

Lemma 5.8. *[21] Let* $\boldsymbol{q}_1$ *and* $\boldsymbol{q}_2$ *be two index tuples from the set* $\{0, 1, \ldots, m-1\}$*. Let* $M_{\boldsymbol{q}_1}$ *and* $M_{\boldsymbol{q}_2}$ *be two products of Fiedler matrices associated with the matrix polynomial* $P(\lambda)$ *such that* $M_{\boldsymbol{q}_1}M_{\boldsymbol{q}_2}$ *is operation-free. Then* $(M_{\boldsymbol{q}_1}M_{\boldsymbol{q}_2})^{\mathcal{B}} = (M_{\boldsymbol{q}_2})^{\mathcal{B}}(M_{\boldsymbol{q}_1})^{\mathcal{B}}$*. Further, the result also holds for indices from* $\{-1, -2, \ldots - m\}$*.*

We have the following result for Fiedler matrices associated with $\mathcal{S}(\lambda)$.

Lemma 5.9. *Let q_1 and q_2 be two index tuples from the set $\{0, 1, 2, \ldots, m-1\}$. Let $\mathbb{M}_{q_1}$ and $\mathbb{M}_{q_2}$ be two products of Fiedler matrices associated with the system matrix $\mathcal{S}(\lambda)$ such that $\mathbb{M}_{q_1}\mathbb{M}_{q_2}$ is operation-free. Then $(\mathbb{M}_{q_1}\mathbb{M}_{q_2})^{\mathbb{B}} = (\mathbb{M}_{q_2})^{\mathbb{B}}(\mathbb{M}_{q_1})^{\mathbb{B}}$. Further, the result also holds for indices from $\{-1, -2, \ldots -m\}$.*

Proof. Case I : Suppose that $0 \notin \mathbf{q}_1 \cup \mathbf{q}_2$. Then we have

$$\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2} = \left[\begin{array}{c|c} M_{\mathbf{q}_1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} M_{\mathbf{q}_2} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{\mathbf{q}_1}M_{\mathbf{q}_2} & \\ \hline & I_r \end{array}\right].$$

Thus

$$\left[\begin{array}{c|c} M_{\mathbf{q}_1}M_{\mathbf{q}_2} & \\ \hline & I_r \end{array}\right]^{\mathbb{B}} = \left[\begin{array}{c|c} (M_{\mathbf{q}_1}M_{\mathbf{q}_2})^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} M_{\mathbf{q}_1}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] = (\mathbb{M}_{\mathbf{q}_2})^{\mathbb{B}}(\mathbb{M}_{\mathbf{q}_1})^{\mathbb{B}}.$$

Case II : Suppose that $0 \in \mathbf{q}_1 \cup \mathbf{q}_2$. If $0 \in \mathbf{q}_1 \cap \mathbf{q}_2$ then $\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2}$ is not operation-free. Thus either $0 \in \mathbf{q}_1$ or $0 \in \mathbf{q}_2$. Without loss of generality assume that $0 \in \mathbf{q}_1$. Then $\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_p\mathbb{M}_0\mathbb{M}_{p_1}$ for some index tuples p and p_1. Now $\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2} = \mathbb{M}_p\mathbb{M}_0\mathbb{M}_{p_1}\mathbb{M}_{\mathbf{q}_2} = \mathbb{M}_p\mathbb{M}_0\mathbb{M}_{\mathbf{s}} = \left[\begin{array}{c|c} M_pM_0M_{\mathbf{s}} & -M_p(e_m \otimes C) \\ \hline -(e_m^T \otimes B)M_{\mathbf{s}} & -A \end{array}\right]$, where $M_{\mathbf{s}} = \mathbb{M}_{p_1}\mathbb{M}_{\mathbf{q}_2}$. Since $\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2}$ is operation-free, we have $M_{\mathbf{q}_1}M_{\mathbf{q}_2}$ is operation-free. Hence $M_{\mathbf{q}_1}$ and $M_{\mathbf{q}_2}$ are operation-free. Thus M_p is operation-free. Consequently, from the proof of Lemma 3.9, we have $M_p(e_m \otimes I_n) = e_k \otimes I_n$, $(e_m^T \otimes I_n)M_{p_1} = e_j^T \otimes I_n$ and $(e_m^T \otimes I_n)M_{\mathbf{s}} = e_h^T \otimes I_n$ for some k, j and h. Hence we have $\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2} = \left[\begin{array}{c|c} M_pM_0M_{\mathbf{s}} & -e_k \otimes C \\ \hline -e_h^T \otimes B & -A \end{array}\right]$. This shows that $(\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2})^{\mathbb{B}} = \left[\begin{array}{c|c} (M_pM_0M_{\mathbf{s}})^{\mathcal{B}} & -e_h \otimes C \\ \hline -e_k^T \otimes B & -A \end{array}\right]$. Now

$$(\mathbb{M}_{\mathbf{q}_2})^{\mathbb{B}}(\mathbb{M}_{\mathbf{q}_1})^{\mathbb{B}} = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right](\mathbb{M}_p\mathbb{M}_0\mathbb{M}_{p_1})^{\mathbb{B}} = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right]\left(\left[\begin{array}{c|c} M_pM_0M_{p_1} & -M_p(e_m \otimes C) \\ \hline -(e_m^T \otimes B)M_{p_1} & -A \end{array}\right]\right)^{\mathbb{B}}$$

$$= \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right]\left(\left[\begin{array}{c|c} M_pM_0M_{p_1} & -e_k \otimes C \\ \hline -(e_j^T \otimes B) & -A \end{array}\right]\right)^{\mathbb{B}} = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} (M_pM_0M_{p_1})^{\mathcal{B}} & -(e_j \otimes C) \\ \hline -e_k^T \otimes B & -A \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}}(M_pM_0M_{p_1})^{\mathcal{B}} & -M_{\mathbf{q}_2}^{\mathcal{B}}(e_j \otimes C) \\ \hline -e_k^T \otimes B & -A \end{array}\right].$$

Since $(e_m^T \otimes I_n)M_{\mathbf{s}} = e_h^T \otimes I_n$ and $(e_m^T \otimes I_n)M_{p_1} = e_j^T \otimes I_n$, we have $(e_j^T \otimes I_n)M_{\mathbf{q}_2} = e_h^T \otimes I_n$. This shows that $M_{\mathbf{q}_2}^{\mathcal{B}}(e_j \otimes I_n) = e_h \otimes I_n$. Hence by Lemma 5.8, we have $(\mathbb{M}_{\mathbf{q}_2})^{\mathbb{B}}(\mathbb{M}_{\mathbf{q}_1})^{\mathbb{B}} =$

$$\left[\begin{array}{c|c} (M_pM_0M_{p_1}M_{\mathbf{q}_2})^{\mathcal{B}} & -M_{\mathbf{q}_2}^{\mathcal{B}}(e_j \otimes C) \\ \hline -e_k^T \otimes B & -A \end{array}\right] = \left[\begin{array}{c|c} (M_pM_0M_{\mathbf{s}})^{\mathcal{B}} & -e_h \otimes C \\ \hline -e_k^T \otimes B & -A \end{array}\right] = (\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2})^{\mathbb{B}}.$$

□

Consider the first companion form $\mathcal{C}_1(\lambda)$ of $\mathcal{S}(\lambda)$ given by

$$\mathcal{C}_1(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2}\cdots\mathbb{M}_1\mathbb{M}_0 = \left[\begin{array}{c|c} C_1(\lambda) & -e_1 \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right],$$

where $C_1(\lambda)$ is the first companion form of $P(\lambda)$ given in (3). Note that $\mathcal{C}_1(\lambda)$ is operation-free. Hence by Lemma 5.9, we have

$$\mathcal{C}_1(\lambda)^{\mathbb{B}} = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_1\cdots\mathbb{M}_{m-2}\mathbb{M}_{m-1} = \left[\begin{array}{c|c} C_2(\lambda) & -e_m \otimes C \\ \hline -e_1^T \otimes B & -A \end{array}\right] = \mathcal{C}_2(\lambda)$$

is the second companion form of $\mathcal{S}(\lambda)$, where $C_2(\lambda)$ is the second companion form of $P(\lambda)$ given in [5].

Let $\mathbf{q}_1$ be the index tuple containing indices from $\{0, 1, \ldots, m-1\}$. Let $\mathbb{M}_{\mathbf{q}_1}$ be the product of Fiedler matrices such that $\mathbb{M}_{\mathbf{q}_1}$ is operation-free. Since $\mathbb{M}_i^{\mathbb{B}} = \mathbb{M}_i$ then by Lemma 5.9, we have $\mathbb{M}_{rev\,\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_1}^{\mathbb{B}}$ is also operation-free. Further, $\mathbb{M}_{\mathbf{q}_1}$ is operation-free and block symmetric if and only if $\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_1}^{\mathbb{B}} = \mathbb{M}_{rev\,\mathbf{q}_1} \Leftrightarrow \mathbf{q}_1 \sim rev\,\mathbf{q}_1$. For example, consider $\mathbf{q}_1 = (0, 2, 3, 2)$. Then

$$\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_0\mathbb{M}_2\mathbb{M}_3\mathbb{M}_2 = \left[\begin{array}{ccccc|c} I_n & & & & & \\ & -A_3 & -A_2 & I_n & & \\ & -A_2 & I_n & 0 & & \\ & I_n & 0 & 0 & & \\ & & & & -A_0 & -C \\ \hline & & & & -B & -A \end{array}\right] = \mathbb{M}_2\mathbb{M}_3\mathbb{M}_2\mathbb{M}_0.$$

Since $\mathbb{M}_{\mathbf{q}_1}$ is operation-free and $(\mathbb{M}_i)^{\mathbb{B}} = \mathbb{M}_i$, we have $(\mathbb{M}_{\mathbf{q}_1})^{\mathbb{B}} = \mathbb{M}_2^{\mathbb{B}}\mathbb{M}_3^{\mathbb{B}}\mathbb{M}_2^{\mathbb{B}}\mathbb{M}_0^{\mathbb{B}} = \mathbb{M}_2\mathbb{M}_3\mathbb{M}_2\mathbb{M}_0 = \mathbb{M}_{rev\,\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_1}$, that is, $rev\,\mathbf{q}_1 \sim \mathbf{q}_1$. Thus $\mathbb{M}_{\mathbf{q}_1}$ is block symmetric.

The following examples illustrate symmetric GFPR of $\mathcal{S}(\lambda)$ when $\mathcal{S}(\lambda)$ is symmetric.

Example 5.10. *Suppose that $\mathcal{S}(\lambda)$ is symmetric with $m = 3$. Consider $\sigma = (1 : 2), \tau = \emptyset, \sigma_2 = 1, \tau_2 = \emptyset, \sigma_1 = \emptyset, \tau_1 = \emptyset$. Then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{-3}\mathbb{M}_1 - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1$ in Example 5.1, is symmetric, since $\mathcal{S}$ is symmetric. Note that $\mathbb{L}(\lambda)$ is block symmetric, since rev $(1, 2, 0, 1) \sim (1, 2, 0, 1)$ and rev $(-3, 1) \sim (-3, 1)$.* ■

Example 5.11. *Suppose that $\mathcal{S}(\lambda)$ is symmetric with $m = 5$. Consider $\sigma = (1 : 2), \tau = (-4 : -3), \sigma_1 = \emptyset, \tau_1 = \emptyset, \sigma_2 = 1, \tau_2 = -4$. Then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{-4}\mathbb{M}_{-3}\mathbb{M}_{-5}\mathbb{M}_1\mathbb{M}_{-4} - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_{-4}$ is operation-free symmetric pencil, since $\mathcal{S}$ is symmetric. Note that $\mathbb{L}(\lambda)$ is block symmetric, since*

$$rev\,(-4, -3, -5, 1, -4) \sim (-4, -3, -5, 1, -4) \text{ and } rev\,(1, 2, 0, 1, -4) \sim (1, 2, 0, 1, -4).$$

■

The next example illustrates a symmetric pencil of a symmetric $\mathcal{S}(\lambda)$, which is not a GFPR.

Example 5.12. *Suppose that $\mathcal{S}(\lambda)$ is symmetric with $m = 4$. Then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_4\mathbb{M}_{-2}\mathbb{M}_{-3} - \mathbb{M}_1\mathbb{M}_0\mathbb{M}_1$ is symmetric, since $(\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_4\mathbb{M}_{-2}\mathbb{M}_{-3})^T = (\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_4\mathbb{M}_{-2}\mathbb{M}_{-3})$, $(\mathbb{M}_1\mathbb{M}_0\mathbb{M}_1)^T = \mathbb{M}_1\mathbb{M}_0\mathbb{M}_1$, and $\mathcal{S}$ is symmetric. Note that $\mathbb{L}(\lambda)$ does not satisfy SIP. Hence $\mathbb{L}(\lambda)$ is not operation-free, but symmetric. Therefore, we have more pencils with repetition of Fiedler matrices which is not operation-free but have symmetric structure.* ■

Next, we recall some definitions given in [9] for recovery of eigenvectors from GFPR and the eigenvector formula of GFPR.

Definition 5.13 (Type 1 indices relative to a simple index tuple [9]). *Let h be a non-negative integer and σ be a permutation of $\{0, 1, \ldots, h\}$. Let s be an index in $\{0, 1, \ldots, h-1\}$. Then s is said to be a* right index of type 1 relative to σ *if there is a string $(t_{d-1} + 1 : t_d)$ in $csf(\sigma)$ such that $s = t_{d-1} + 1 < t_d$.*

Definition 5.14 (Associated simple tuple [9]). *Let h be a non-negative integer and σ be a permutation of $\{0, 1, \ldots, h\}$. Let $csf(\sigma) = (\boldsymbol{b}_{\alpha+1}, \boldsymbol{b}_\alpha, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_i = (t_{i-1} + 1 : t_i), i = 1, \ldots, \alpha + 1$, are the strings of $csf(\sigma)$. Set $\mathfrak{s}(\sigma) := csf(\sigma)$ Then we say that $\mathfrak{s}(\sigma)$ is the* simple tuple associated with σ. *If s is an index of type 1 with respect to σ, say $s = t_{d-1} + 1 < t_d$, then the* simple tuple associated with (σ, s) *is the simple tuple:*

$$\begin{aligned} &\mathfrak{s}(\sigma, s) := (\boldsymbol{b}_{\alpha+1}, \boldsymbol{b}_\alpha, \ldots, \boldsymbol{b}_{d+1}, \tilde{\boldsymbol{b}}_d, \tilde{\boldsymbol{b}}_{d-1}, \boldsymbol{b}_{d-2}, \ldots, \boldsymbol{b}_1), \\ &\qquad \text{where } \tilde{\boldsymbol{b}}_d = (t_{d-1} + 2 : t_d),\ \tilde{\boldsymbol{b}}_{d-1} = (t_{d-2} + 1 : t_{d-1} + 1) \text{ if } s \neq 0 \end{aligned}$$

and $\mathfrak{s}(\sigma, 0) := (\boldsymbol{b}_{\alpha+1}, \boldsymbol{b}_\alpha, \ldots, \tilde{\boldsymbol{b}}_1, \tilde{\boldsymbol{b}}_0)$, *where* $\tilde{\boldsymbol{b}}_1 = (1 : t_1)$, $\tilde{\boldsymbol{b}}_0 = (0)$.

Definition 5.15 (Index tuple of type 1 [9]). *Let h be a non-negative integer and σ be a permutation of $\{0, 1, \ldots, h\}$. Let σ_2 and σ_1 be tuples with indices from $\{0, 1, \ldots, h-1\}$, possibly with repetitions. We say that $\sigma_2 = (s_1, \ldots, s_r)$, where s_i is the ith index of σ_2, is an index tuple of type 1 relative to σ if, for $i = 1, \ldots, r$, s_i is a right index of type 1 with respect to $\mathfrak{s}(\sigma, (s_1, \ldots, s_{i-1}))$, where $\mathfrak{s}(\sigma, (s_1, \ldots, s_{i-1})) := \mathfrak{s}(\mathfrak{s}(\sigma, (s_1, \ldots, s_{i-2})), s_{i-1})$ for $i > 2$.*

5.1. Eigenvector formula for GFPR

We derive eigenvector formula for GFPR, when the tuples are of type 1.

Lemma 5.16. *Let $0 \leq h \leq m-3$, and let σ and τ be permutations of $\{0,1,\ldots,h\}$ and $\{-m,-m+1,\ldots,-h-1\}$ respectively. Assume that σ_1,σ_2 are index tuples with elements from $\{1,2,\ldots,h-1\}$, and τ_1,τ_2 are index tuples with elements from $\{-m+1,-m+2\ldots,-h-2\}$. Then $\mathbb{M}_{\tau_2}$ commutes with $\mathbb{M}_\sigma$ and $\mathbb{M}_{\sigma_2}$, and $\mathbb{M}_\tau$ commutes with $\mathbb{M}_{\sigma_2}$.*

Proof. Since the distance between each pair of indices in τ_2 and σ is greater than 1, by commutativity relation $\mathbb{M}_{\tau_2}$ commute with $\mathbb{M}_\sigma$. Similarly, the other cases follow. □

Lemma 5.17. *Let $\sigma,\tau,\sigma_1,\sigma_2,\tau_1,\tau_2$ be as in Lemma 5.16 such that $(\sigma_1,\sigma,\sigma_2)$ and (τ_1,τ,τ_2) satisfy the SIP. Suppose that τ_2 and σ_2 are type 1 tuples relative to τ and σ, respectively. Let $\mathfrak{s}(\sigma,\sigma_2)$ and $\mathfrak{s}(\tau,\tau_2)$ be the simple tuple associated with (σ,σ_2) and (τ,τ_2). Then $(\sigma,\sigma_2)\sim(\sigma_2,\mathfrak{s}(\sigma,\sigma_2))$ and $(\tau,\tau_2)\sim(\tau_2,\mathfrak{s}(\tau,\tau_2))$.*

Proof. First we prove $(\sigma,\sigma_2)\sim(\sigma_2,\mathfrak{s}(\sigma,\sigma_2))$ by induction on the number of indices of σ_2, (other one follow similarly applying induction on τ_2). Assume that $\sigma_2=(r_1,r_2,\ldots,r_p)$, where r_i denotes the ith index in σ_2, and $\sigma=(\mathbf{b}_\alpha,\mathbf{b}_{\alpha-1},\ldots,\mathbf{b}_1)$, where $\mathbf{b}_i=(t_{i-1}+1,t_i)$, for $i=1,2,\ldots,\alpha$. Since σ_2 is of type 1 relative to σ, we have $r_1=t_{d-1}+1<t_d$ for some $1\leq d\leq\alpha$. So $(\sigma,r_1)\sim(t_{d-1}+1,\mathbf{b}_\alpha,\ldots,\mathbf{b}_{d+1},t_{d-1}+2:t_d,\mathbf{b}_{d-1},t_{d-1}+1,\ldots,\mathbf{b}_1))=(r_1,\mathfrak{s}(\sigma,r_1))$, if $d>1$. We mention that σ_2 does not contain 0. Hence $d\neq 1$. Applying induction on indices of σ_2 we have $(\sigma,\sigma_2)\sim(\sigma_2,\mathfrak{s}(\sigma,\sigma_2))$. Similarly applying induction τ_2 we get $(\tau,\tau_2)\sim(\tau_2,\mathfrak{s}(\tau,\tau_2))$. □

Example 5.18. *Consider $m=12$. Let $\sigma=(6,1:5,0)=(\boldsymbol{b}_3,\boldsymbol{b}_2,\boldsymbol{b}_1),\sigma_2=(1:4)=(r_1,r_2,r_3,r_4)$ such that σ_2 is of type 1 relative to σ. By Lemma 5.17, here $\alpha=3$, $d=2$, and $r_1=t_{d-1}+1=1$. So $(\sigma,r_1)=(6,1:5,0,1)\sim(1,6,2:5,0,1)=(1,\boldsymbol{b}_3,2:5,\boldsymbol{b}_1,1)$, and $\mathfrak{s}(\sigma,r_1)=(6,1:5,0,1)=(1,6,2:5,0,1)=(6,2:5,0,1)$, and $(r_1,\mathfrak{s}(\sigma,r_1))=(1,6,2:5,0,1)$. Therefore $(\sigma,r_1)=(r_1,\mathfrak{s}(\sigma,r_1))$. Again $(\sigma,r_1,r_2)=(6,1:5,0,1,2)\sim(1,2,6,3:5,0,1,2)=(1,2,\boldsymbol{b}_3,3:5,\boldsymbol{b}_1,1,2)$, and $\mathfrak{s}(\sigma,(r_1,r_2))=(6,1:5,0)(1,2)=(1,6,2:5,0,1)(2)=(6,2:5,0,1)(2)=(6,3:5,0,1,2)$, and $((r_1,r_2),\mathfrak{s}(\sigma,(r_1,r_2)))=(1,2,6,3:5,0,1,2)$. Therefore $(\sigma,r_1,r_2)=((r_1,r_2)\mathfrak{s}(\sigma,(r_1,r_2)))$. Similarly we get $(\sigma,\sigma_2)\sim(\sigma_2,\mathfrak{s}(\sigma,\sigma_2))$.* ■

The following result gives eigenvector formula for GFPR and eigenvector recovery from GFPR.

Theorem 5.19 (Eigenvector formula of GFPR). *Let $\sigma,\tau,\sigma_1,\sigma_2,\tau_1,\tau_2$ satisfy the conditions in Lemma 5.16 and Lemma 5.17. Let $\mathbb{L}(\lambda)=\lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_\tau\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}-\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_\sigma\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ be a GFPR of $\mathcal{S}(\lambda)$. Let $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$.*

(a) Set $\omega_0=\mathfrak{s}(\sigma,\sigma_2)$, $\omega_1=\mathfrak{s}(\tau,\tau_2)$ and $\omega=(\omega_0,\omega_1)$. Define $\mathbb{E}_\mathbb{L}(\mathcal{S}):=\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S})$, where $\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S})$ is as in Theorem 4.8. Then $\mathbb{E}_\mathbb{L}(\mathcal{S}):\mathcal{N}_r(\mathcal{S}(\lambda))\to\mathcal{N}_r(\mathbb{L}(\lambda))$ is an isomorphism. Further, assume that CIP$(\omega_0)=(c_0,i_0)$. Define $\mathbb{F}_\mathbb{L}(\mathcal{S}):=\mathbb{F}_{\omega_0,\omega_1}(\mathcal{S})$, where $\mathbb{F}_{\omega_0,\omega_1}(\mathcal{S}):=\left[\begin{array}{c|c}(e^T_{(m-c_0)}\otimes I_n) & \\ \hline & I_r\end{array}\right]$. Then $\mathbb{F}_\mathbb{L}(\mathcal{S}):\mathcal{N}_r(\mathbb{L}(\lambda))\to\mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism.

(b) Assume that rev τ_1 and rev σ_1 are type 1 tuples relative to rev τ and rev σ, respectively. Also suppose that $\mathfrak{s}(rev\sigma,rev\sigma_1)$, $\mathfrak{s}(rev\tau,rev\tau_1)$ are the simple tuples associated with $(rev\sigma,rev\sigma_1)$ and $(rev\tau,rev\tau_1)$ respectively. Set $\xi_1=\mathfrak{s}(rev\tau,rev\tau_1)$, $\xi_0=\mathfrak{s}(rev\sigma,rev\sigma_1)$ and $\xi=(\xi_0,\xi_1)$. Define $\mathbb{H}_\mathbb{L}(\mathcal{S}):=\mathbb{E}_{\xi_0,\xi_1}(\mathcal{S}^T)$, where $\mathbb{E}_{\xi_0,\xi_1}(\mathcal{S}^T)$ is as in Theorem 4.8. Then $\mathbb{H}_\mathbb{L}(\mathcal{S}):\mathcal{N}_l(\mathcal{S}(\lambda))\to\mathcal{N}_l(\mathbb{L}(\lambda))$ is an isomorphism. Further, assume that CIP$(\xi_0)=(c_0,i_0)$. Define $\mathbb{K}_\mathbb{L}(\mathcal{S}):=\mathbb{K}_{\xi_0,\xi_1}(\mathcal{S}^T)$, where $\mathbb{K}_{\xi_0,\xi_1}(\mathcal{S}^T):=\left[\begin{array}{c|c}(e^T_{(m-c_0)}\otimes I_n) & \\ \hline & I_r\end{array}\right]$. Then $\mathbb{K}_\mathbb{L}(\mathcal{S}):\mathcal{N}_l(\mathbb{L}(\lambda))\to\mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism.

Proof. Given that $\mathbb{L}(\lambda)=\lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_\tau\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}-\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_\sigma\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ is a GFPR of $\mathcal{S}(\lambda)$. By Lemma 5.16, $\mathbb{M}_{\tau_2}$ commutes with $\mathbb{M}_\sigma$ and $\mathbb{M}_{\sigma_2}$, and $\mathbb{M}_\tau$ commutes with $\mathbb{M}_{\sigma_2}$. Hence by Lemma 5.17, we have

$$\begin{aligned}\mathbb{L}(\lambda)&=\lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)}-\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}\mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)}\\&=\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}(\lambda\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)}-\mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)})=\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}\mathcal{K}_\omega(\lambda),\end{aligned}$$

where $\mathcal{K}_\omega(\lambda) = \lambda\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)} - \mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)} = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is a PGF pencil. This shows that $\mathcal{N}_r(\mathbb{L}(\lambda)) = \mathcal{N}_r(\mathcal{K}_\omega(\lambda))$. Hence by Theorem 4.8, $\mathbb{E}_\mathbb{L}(\mathcal{S}) := \mathbb{E}_{\omega_0,\omega_1}(\mathcal{S})$ is an isomorphism from $\mathcal{N}_l(\mathcal{S}(\lambda))$ to $\mathcal{N}_l(\mathbb{L}(\lambda))$. Since $\mathcal{K}_\omega(\lambda)$ is a PGF pencil and ω_0 has c_0 consecutions at 0, then by Theorem 5.2, given in [3] $\mathbb{F}_\mathbb{L}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism.

Next, note that $\mathcal{N}_l(\mathcal{S}(\lambda)) = \mathcal{N}_r(\mathcal{S}(\lambda)^T)$ and $\mathcal{N}_l(\mathbb{L}(\lambda)) = \mathcal{N}_r(\mathbb{L}(\lambda)^T)$. Now

$$\begin{aligned}\mathbb{L}(\lambda)^T &= (\lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2})^T - (\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2})^T\\ &= \lambda\mathbb{M}_{\tau_2}^T\mathbb{M}_{\sigma_2}^T\mathbb{M}_{\tau}^T\mathbb{M}_{\sigma_1}^T\mathbb{M}_{\tau_1}^T - \mathbb{M}_{\tau_2}^T\mathbb{M}_{\sigma_2}^T\mathbb{M}_{\sigma}^T\mathbb{M}_{\sigma_1}^T\mathbb{M}_{\tau_1}^T\\ &= \lambda\widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{rev\,\tau_1} - \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\sigma}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{rev\,\tau_1},\end{aligned}$$

where $\widehat{\mathbb{M}}_j = \mathbb{M}_j(\mathcal{S}^T)$ is the Fiedler matrices associated with $\mathcal{S}^T(\lambda)$. Note that by Lemma 5.16, we have $\mathbb{M}_{\tau_1}\mathbb{M}_\sigma = \mathbb{M}_\sigma\mathbb{M}_{\tau_1}$. Taking transpose we have $\widehat{\mathbb{M}}_{rev\,\sigma}\widehat{\mathbb{M}}_{rev\,\tau_1} = \widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma}$. Similarly, by Lemma 5.16, $\widehat{\mathbb{M}}_{rev\,\tau_1}$ commutes $\widehat{\mathbb{M}}_{rev\,\sigma_1}$, and $\widehat{\mathbb{M}}_{rev\,\tau}$ commutes with $\widehat{\mathbb{M}}_{rev\,\sigma_1}$. Since rev τ_1 and rev σ_1 are type 1 tuples relative to rev τ and rev σ, respectively, and $\mathfrak{s}(rev\sigma, rev\sigma_1)$ and $\mathfrak{s}(rev\tau, rev\tau_1)$ are the simple tuples associated with $(rev\sigma, rev\sigma_1)$ and $(rev\tau, rev\tau_1)$, respectively, by Lemma 5.17, we have $(rev\,\sigma, rev\,\sigma_1) \sim (rev\,\sigma_1, \xi_0)$ and $(rev\,\tau, rev\,\tau_1) \sim (rev\,\tau_1, \xi_1)$. Therefore

$$\begin{aligned}\mathbb{L}(\lambda)^T &= \lambda\widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{\xi_1} - \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{\xi_0}\\ &= \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}(\lambda\widehat{\mathbb{M}}_{\xi_1} - \widehat{\mathbb{M}}_{\xi_0}) = \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\left(\lambda\mathbb{M}_{\xi_1}(\mathcal{S}^T) - \mathbb{M}_{\xi_0}(\mathcal{S}^T)\right)\\ &= \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\mathcal{H}_\xi(\mathcal{S}^T(\lambda)),\end{aligned}$$

where $\mathcal{H}_\xi(\mathcal{S}^T(\lambda)) = \lambda\mathbb{M}_{\xi_1}(\mathcal{S}^T) - \mathbb{M}_{\xi_0}(\mathcal{S}^T)$ is a PGF pencil. This shows that $\mathcal{N}_l(\mathbb{L}(\lambda)) = \mathcal{N}_r(\mathbb{L}(\lambda)^T) = \mathcal{N}_r(\mathcal{H}_\xi(\mathcal{S}^T(\lambda)))$. By part (*a*) it follows that $\mathbb{H}_\mathbb{L}(\mathcal{S}) := \mathbb{E}_{\xi_0,\xi_1}(\mathcal{S}^T)$ is an isomorphism from $\mathcal{N}_l(\mathcal{S}(\lambda))$ to $\mathcal{N}_l(\mathbb{L}(\lambda))$. Since $\mathcal{H}_\xi(\mathcal{S}^T(\lambda))$ is a PGF pencil and ξ_0 has c_0 consecutions at 0, then by Theorem 5.2, given in [3], we have $\mathbb{K}_\mathbb{L}(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism. □

Example 5.20. *Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 12$ and the associated transfer function $G(\lambda)$. Consider the GFPR $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_\tau\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_\sigma\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$. Choose σ and σ_2 as in Example 5.18 and consider $\tau = (-7, -8, -11 : -9, -12), \tau_2 = (-11 : -10), \sigma_1 = \{\emptyset\}, \tau_1 = (-8, -9)$. Now $(\sigma, \sigma_2) = (6, 1 : 5, 0 : 4)$ and $\omega_0 = \mathfrak{s}(\sigma, \sigma_2) = (6, 5, 0 : 4)$, and $(\tau, \tau_2) = (-7, -8, -11 : -9, -12 : -10)$, $\omega_1 = \mathfrak{s}(\tau, \tau_2) = (-7, -8, -9, -12 : -10)$. So $\mathcal{K}_\omega(\lambda) = \lambda\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)} - \mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)}$ is a PGF pencil. By Theorem 4.8, we have $c_{-m} = 2$ and $csf(\mathfrak{s}(\tau, \tau_2)) = (\boldsymbol{s}_1, -m : -m + c_{-m}) = (-7, -8, -9, -12 : -10)$, where $\boldsymbol{s}_1 = (-7, -8, -9)$. So $\xi = (9, 8, 7, 6, 5, 0 : 4)$. Hence*

$$\begin{bmatrix} \begin{bmatrix} \lambda^6 P_0 & \lambda^6 P_1 & \lambda^5 I_n & \lambda^4 I_n & \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n & P_8 & P_9 & P_{10} & P_{11} \end{bmatrix}^{\mathcal{B}} \\ (\lambda E - A)^{-1} B \end{bmatrix} x \in \mathcal{N}_r(\mathbb{L}(\lambda)),$$

where $x \in \mathcal{N}_r(G(\lambda))$.

Now $(rev\sigma, rev\sigma_1) = (0, 5, 4, 3, 2, 1, 6) \sim (5 : 6, 4, 3, 2, 0 : 1)$ and $\xi_0 = \mathfrak{s}(rev\sigma, rev\sigma_1) = (5 : 6, 4, 3, 2, 0 : 1)$, $(rev\tau, rev\tau_1) = (-12, -9, -10, -11, -8, -7, -9, -8) \sim (-9 : -7, -10, -12 : -11, -9 : -8)$, and $\xi_1 = \mathfrak{s}(rev\tau, rev\tau_1) = (-7, -10, -12 : -11, -9, -8) \sim (-7, -10 : -8, -12 : -11)$. So $\mathcal{H}_\xi(\mathcal{S}^T(\lambda)) = \lambda\mathbb{M}_{\mathfrak{s}(rev\tau,rev\tau_1)}(\mathcal{S}^T) - \mathbb{M}_{\mathfrak{s}(rev\sigma,rev\sigma_1)}(\mathcal{S}^T)$ is a PGF pencil, and by Theorem 4.8, we have $c_{-m} = 1$, and $csf(\mathfrak{s}(rev\tau, rev\tau_1)) = (\tau_1, -m : -m + c_{-m})$, where $\tau_1 = (-7, -10 : -8)$. So $\xi = csf(-rev\,\tau_1, \mathfrak{s}(rev\,\sigma, rev\sigma_1)) = csf(8, 9, 10, 7, 5 : 6, 4, 3, 2, 0 : 1) = (8 : 10, 7, 5 : 6, 4, 3, 2, 0 : 1)$. Hence

$$\begin{bmatrix} \begin{bmatrix} \lambda^7 P_0^T & \lambda^6 I_n & \lambda^6 P_2^T & \lambda^6 P_3^T & \lambda^5 I_n & \lambda^4 I_n & \lambda^4 P_6^T & \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n & P_{11}^T \end{bmatrix}^{\mathcal{B}} \\ (C(\lambda E - A)^{-1})^T \end{bmatrix} y,$$

is a left eigenvector of $\mathbb{L}(\lambda)$, where $y \in \mathcal{N}_l(G(\lambda))$. ■

5.2. Symmetric Linearizations

Let

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A-\lambda E \end{array}\right]$$

be the system matrix and $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ be the associated transfer function of $\mathcal{S}(\lambda)$. We define adjoint of $\mathcal{S}(\lambda)$ by

$$\mathcal{S}^*(\lambda) = \left[\begin{array}{c|c} P^*(\lambda) & B^* \\ \hline C^* & A^*-\lambda E^* \end{array}\right], \tag{20}$$

where $P^*(\lambda) = \sum_{i=0}^{m} \lambda^i A_i^*$ is the adjoint of the matrix polynomial $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$. The adjoint of the associated transfer function $G(\lambda)$ is given by $G^*(\lambda) = P^*(\lambda) + B^*(\lambda E^* - A^*)^{-1}C^*$. A transfer function $G(\lambda)$ is said to be self-adjoint if $G^*(\lambda) = G(\lambda)$. Note that $\mathcal{S}^*(\lambda) = \mathcal{S}(\lambda) \Leftrightarrow G^*(\lambda) = G(\lambda)$. Similarly, if $\mathcal{S}(\lambda)$ is symmetric, then $A_i^T = A_i, C = B^T, A = A^T$ and $E = E^T$. One simple observation is that if $\mathcal{S}(\lambda)$ is symmetric then $\mathbb{M}_i^T = \mathbb{M}_i$ and $\mathbb{M}_i^* = \mathbb{M}_i$ for $i = 0 : m$.

Remark 5.21. *Suppose that $\mathcal{S}(\lambda)$ is symmetric (Hermitian) of degree $m > 1$. Then the GF/PGF pencil $\mathbb{L}(\lambda) = \lambda X + Y$ is symmetric iff $X^T = X$ and $Y^T = Y$. Thus from the definition of GF and PGF pencils it is clear that $X^T = X$ and $Y^T = Y$ only when each Fiedler matrices are commutes with each other. That is, the distance between any two indices is greater than one. So, the symmetric (Hermitian) structure preserving PGF/GF pencil must be of this form $\mathbb{L}(\lambda) = \lambda \mathbb{M}_m \mathbb{M}_{m-2}^{-1} \cdots \mathbb{M}_3^{-1} \mathbb{M}_1^{-1} - \mathbb{M}_0 \mathbb{M}_2 \cdots \mathbb{M}_{m-3} \mathbb{M}_{m-1}$ if m is odd, see [3], and unlike matrix polynomial $\mathbb{L}(\lambda) = \lambda \mathbb{M}_{m-1}^{-1} \mathbb{M}_{m-3}^{-1} \cdots \mathbb{M}_3^{-1} \mathbb{M}_1^{-1} - \mathbb{M}_0 \mathbb{M}_2 \cdots \mathbb{M}_{m-2} \mathbb{M}_m^{-1}$ if m is even, is not symmetric, since $\mathbb{M}_0$ and $\mathbb{M}_m$ never commute. Hence for rational matrix function $G(\lambda)$ only one class of symmetric/ Hermitian PGF pencils exist.*

Suppose that $\mathcal{S}(\lambda)$ is symmetric. Let $\mathbb{L}(\lambda)$ be a GFPR of $\mathcal{S}(\lambda)$ and

$$\mathbb{L}(\lambda) := \lambda X + Y = \lambda \mathbb{M}_{i_1} \mathbb{M}_{i_2} \cdots \mathbb{M}_{i_r} - \mathbb{M}_{j_1} \mathbb{M}_{j_2} \cdots \mathbb{M}_{j_k}.$$

Then $\mathbb{L}(\lambda)$ should be symmetric if $X^T = X$ and $Y^T = Y$. That is $(\mathbb{M}_{i_1} \mathbb{M}_{i_2} \cdots \mathbb{M}_{i_r})^T = \mathbb{M}_{i_1} \mathbb{M}_{i_2} \cdots \mathbb{M}_{i_r}$ and $(\mathbb{M}_{j_1} \mathbb{M}_{j_2} \cdots \mathbb{M}_{j_k})^T = \mathbb{M}_{j_1} \mathbb{M}_{j_2} \cdots \mathbb{M}_{j_k}$. Since $\mathbb{M}_i^T = \mathbb{M}_i$ for symmetric $\mathcal{S}(\lambda)$ we have $(\mathbb{M}_{i_1} \mathbb{M}_{i_2} \cdots \mathbb{M}_{i_r})^T = \mathbb{M}_{i_r}^T \mathbb{M}_{i_{r-1}}^T \cdots \mathbb{M}_{i_1}^T = \mathbb{M}_{i_r} \mathbb{M}_{i_{r-1}} \cdots \mathbb{M}_{i_1}$. So all we need for $\mathbb{L}(\lambda)$ is symmetric for $\mathcal{S}(\lambda)$ symmetric is that

$$\mathbb{M}_{i_r} \mathbb{M}_{i_{r-1}} \cdots \mathbb{M}_{i_1} = \mathbb{M}_{i_1} \mathbb{M}_{i_2} \cdots \mathbb{M}_{i_r} \text{ and } \mathbb{M}_{j_k} \mathbb{M}_{j_{k-1}} \cdots \mathbb{M}_{j_1} = \mathbb{M}_{j_1} \mathbb{M}_{j_2} \cdots \mathbb{M}_{j_k}$$

i.e., $rev(i_1, i_2, \ldots, i_r) \sim (i_1, i_2, \ldots, i_r)$ and $rev(j_1, j_2, \ldots, j_k) \sim (j_1, j_2, \ldots, j_k)$.

Lemma 5.22. *Let $\boldsymbol{q}$ be a tuple satisfying SIP with indices from either $\{0, 1, \ldots, m-1\}$ or $\{-m, \ldots, -1\}$. Then $\mathbb{M}_{\boldsymbol{q}}$ is symmetric for any symmetric $\mathcal{S}(\lambda)$ of degree m if and only if $\mathbb{M}_{\boldsymbol{q}}$ is block-symmetric for any $\mathcal{S}(\lambda)$ of degree m.*

Definition 5.23. *Let $\boldsymbol{q}$ be an index tuple with indices from either $\{0, 1, \ldots, m-1\}$ or $\{-m, \ldots, -1\}$. Then we say that i, j from $\boldsymbol{q}$ commute if $||i| - |j|| > 1$.*

Definition 5.24. *An index tuple $\boldsymbol{q}$ of nonnegative (resp. negative) indices is symmetric if $\boldsymbol{q} \sim rev(\boldsymbol{q})$.*

Lemma 5.25. *Let $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ be two tuples with the same indices from either $\{0, 1, \ldots, m-1\}$ or $\{-m, \ldots, -1\}$. Assume that $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ satisfy the SIP. Then,*

(a) If $\boldsymbol{t}_1$ is equivalent to $\boldsymbol{t}_2$ then $\mathbb{M}_{\boldsymbol{t}_1} = \mathbb{M}_{\boldsymbol{t}_2}$ for any system matrix $\mathcal{S}(\lambda)$

(b) If $\mathbb{M}_{\boldsymbol{t}_1} = \mathbb{M}_{\boldsymbol{t}_2}$ for some system matrix $\mathcal{S}(\lambda)$ with A_0 nonsingular and $A_i \neq -I_n$ $\boldsymbol{t}_1$ for $i = 0 : m$, then $\boldsymbol{t}_1$ is equivalent to $\boldsymbol{t}_2$.

Lemma 5.26. *Let $\boldsymbol{t}$ be a tuple satisfying the SIP with indices from either $\{0, 1, \ldots, m-1\}$ or $\{-m, \ldots, -1\}$. Then for any symmetric $\mathcal{S}(\lambda)$, $\mathbb{M}_{\boldsymbol{t}}(\mathcal{S})$ is symmetric if and only if $\boldsymbol{t}$ is symmetric.*

Proof. Assume that $\mathbf{t}$ is symmetric and $\mathcal{S}(\lambda)$ is symmetric. Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$. Then $\mathbb{M}_{\mathbf{t}}(\mathcal{S}) = \mathbb{M}_{i_1}\mathbb{M}_{i_2}\ldots\mathbb{M}_{i_r}$ Now $\mathbb{M}_{\mathbf{t}}^T(\mathcal{S}) = \mathbb{M}_{i_r}^T\mathbb{M}_{i_{r-1}}^T\ldots\mathbb{M}_{i_1}^T = \mathbb{M}_{i_r}\mathbb{M}_{i_{r-1}}\ldots\mathbb{M}_{i_1} = \mathbb{M}_{rev(\mathbf{t})}(\mathcal{S}) = \mathbb{M}_{\mathbf{t}}(\mathcal{S})$. So $\mathbb{M}_{\mathbf{t}}(\mathcal{S})$ is symmetric. Conversely, assume that $\mathbb{M}_{\mathbf{t}}(\mathcal{S})$ is symmetric. Then $\mathbb{M}_{\mathbf{t}}(\mathcal{S}) = \mathbb{M}_{\mathbf{t}}^T(\mathcal{S}) = \mathbb{M}_{rev(\mathbf{t})}(\mathcal{S})$, since $\mathcal{S}(\lambda)$ is symmetric. Thus $\mathbf{t} \sim rev(\mathbf{t})$. Hence proved. □

Theorem 5.27. *Let $\mathcal{S}(\lambda)$ be a symmetric/self-adjoint system matrix. If m is odd and $\sigma = (1,2), \tau = (-4,-6,\ldots,-(m-3),-(m-1),-3,-5,\ldots,-(m-2)), \sigma_2 = (1), \tau_2 = (-4,-6,\ldots,-(m-3),-(m-1)), \sigma_1 = \phi, \tau_1 = \phi$ then*

$$\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{-m}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{0}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$$

is a symmetric/self-adjoint operation free GFPR pencil of $\mathcal{S}(\lambda)$. Hence $\mathbb{L}(\lambda)$ is symmetric/self-adjoint linearization of $\mathcal{S}(\lambda)$. In particular, if $G(\lambda)$ is minimal then $\mathbb{L}(\lambda)$ is a symmetric/self-adjoint linearization of $G(\lambda)$.

Proof. By commutativity relations we have $\mathrm{rev}(\tau_1,\sigma_1,\tau,-m,\sigma_2,\tau_2) \sim (\tau_1,\sigma_1,\tau,-m,\sigma_2,\tau_2)$ and $\mathrm{rev}(\tau_1,\sigma_1,\sigma,0,\sigma_2,\tau_2) \sim (\tau_1,\sigma_1,\sigma,0,\sigma_2,\tau_2)$, since

$$\begin{aligned}
&rev(-4,-6,\ldots,-(m-3),-(m-1),-3,-5,\ldots,-(m-2),-m,1,-4,-6,\ldots,-(m-3),-(m-1))\\
&\sim(-(m-1),-(m-3),\ldots,-6,-4,1,-m,-(m-2),\ldots,-5,-3,-(m-1),-(m-3),\ldots,-6,-4)\\
&\sim(-6,-4,\ldots,-(m-3),-(m-1),-m,-(m-2),\ldots,-5,-3,1,-(m-1),-(m-3),\ldots,-6,-4)\\
&\sim(-4,-6,\ldots,-(m-3),-(m-1),-3,-5,\ldots,-(m-2),-m,1,-4,-6,\ldots,-(m-3),-(m-1)).
\end{aligned}$$

and

$$\begin{aligned}
&rev(1,2,0,1,-4,-6,\ldots,-(m-3),-(m-1)) \sim (-(m-1),-(m-3),\ldots,-6,-4,1,0,2,1)\\
&\sim(-4,-6,\ldots,-(m-3),-(m-1),1,2,0,1) \sim (1,2,0,1,-4,-6,\ldots,-(m-3),-(m-1)).
\end{aligned}$$

Now, since $\mathcal{S}(\lambda)$ is symmetric, so $\mathbb{L}(\lambda)$ is symmetric. Note that $\mathbb{L}(\lambda)$ is strictly equivalent to a PGF pencil, since $\mathbb{L}(\lambda)$ can be written as $\mathbb{L}(\lambda) = \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}(\lambda\mathbb{M}_{\tau}\mathbb{M}_{-m} - \mathbb{M}_{\sigma}\mathbb{M}_0)\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$. Hence linearizations of $\mathcal{S}(\lambda)$. It is also easy to show that $\mathbb{L}(\lambda)$ is operation free. □

Corollary 5.28. *Let $\mathcal{S}(\lambda)$ be the system matrix of degree m is odd. Consider the pencil $\mathbb{L}(\lambda)$ defined in Theorem 5.27. Then $\mathbb{L}(\lambda)$ is block-symmetric for $\mathcal{S}(\lambda)$.*

Example 5.29. *Suppose that $\mathcal{S}(\lambda)$ is symmetric system matrix with $m = 5$ and*

$$\mathbb{L}(\lambda) = \lambda\mathbb{M}_{-4}\mathbb{M}_{-3}\mathbb{M}_{-5}\mathbb{M}_1\mathbb{M}_{-4} - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_{-4}$$

$$= \lambda\left[\begin{array}{ccccc|c} 0 & 0 & I_n & & & \\ 0 & A_5 & A_4 & & & \\ I_n & A_4 & A_3 & & & \\ & & & -A_1 & I_n & \\ & & & I_n & 0 & \\ \hline & & & & & -E \end{array}\right] - \left[\begin{array}{ccccc|c} 0 & I_n & & & & \\ I_n & A_4 & & & & \\ & & -A_2 & -A_1 & I_n & 0 \\ & & -A_1 & -A_0 & 0 & -C \\ & & I_n & 0 & 0 & 0 \\ \hline & & 0 & -B & 0 & -A \end{array}\right].$$

Then by the Theorem 5.27 we have $\mathbb{L}(\lambda)$ is symmetric and operation free. Note that $\mathbb{L}(\lambda)$ is block-symmetric.

Theorem 5.30. *Let $\mathcal{S}(\lambda)$ be symmetric/self-adjoint system matrix. If m is odd and $\sigma = (2,\ldots,m-1,3,5,,m-2), \tau = (-1), \sigma_2 = \phi,\ \tau_2 = \phi,\ \sigma_1 = (3,5,\ldots,(m-2)),\ \tau_1 = \phi$ then*

$$\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{-m}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{0}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$$

is symmetric/self-adjoint operation free GFPR pencil of $\mathcal{S}(\lambda)$. Hence $\mathbb{L}(\lambda)$ is symmetric/self-adjoint linearization of $\mathcal{S}(\lambda)$. In particular, if $G(\lambda)$ is minimal then $\mathbb{L}(\lambda)$ is a symmetric/self-adjoint linearization of $G(\lambda)$.

Proof. By commutativity relations we have $\mathrm{rev}(\tau_1, \sigma_1, \tau, -m, \sigma_2, \tau_2) \sim (\tau_1, \sigma_1, \tau, -m, \sigma_2, \tau_2)$ and $\mathrm{rev}(\tau_1, \sigma_1, \sigma, 0, \sigma_2, \tau_2) \sim (\tau_1, \sigma_1, \sigma, 0, \sigma_2, \tau_2)$, since $rev(3, 5, \ldots, (m-2), -1, -m) \sim (-m, -1, (m-2), \ldots, 5, 3) \sim (3, 5, \ldots, (m-2), -1, -m)$, and

$$\begin{aligned} &rev(3,5,\ldots,(m-2),2,\ldots,(m-1),3,5,\ldots,(m-2),0) \sim (0,(m-2),\ldots,5,3,(m-1),\ldots,2,(m-2),\ldots,5,3) \\ &\sim (3,5,\ldots,(m-2),2,\ldots,(m-1),3,5,\ldots,(m-2),0). \end{aligned}$$

Now, since $\mathcal{S}(\lambda)$ is symmetric, so $\mathbb{L}(\lambda)$ is symmetric. Note that $\mathbb{L}(\lambda)$ is strictly equivalent to a PGF pencil. Hence linearizations of $\mathcal{S}(\lambda)$. It is also easy to show that $\mathbb{L}(\lambda)$ is operation free. □

Corollary 5.31. *Let $\mathcal{S}(\lambda)$ be the system matrix of degree m is odd. Consider the pencil $\mathbb{L}(\lambda)$ defined in Theorem 5.30. Then $\mathbb{L}(\lambda)$ is block-symmetric for $\mathcal{S}(\lambda)$.*

Example 5.32. *Suppose that $\mathcal{S}(\lambda)$ is symmetric system matrix of degree $m = 7$ and*

$$\mathbb{L}(\lambda) = \lambda \mathbb{M}_3 \mathbb{M}_5 \mathbb{M}_{-7} \mathbb{M}_{-1} - \mathbb{M}_3 \mathbb{M}_5 \mathbb{M}_0 \mathbb{M}_2 \mathbb{M}_4 \mathbb{M}_6 \mathbb{M}_3 \mathbb{M}_5$$

$$= \lambda \left[\begin{array}{ccccccc|c} A_7 & & & & & & & \\ & -A_5 & I_n & & & & & \\ & I_n & 0 & & & & & \\ & & & -A_3 & I_n & & & \\ & & & I_n & 0 & & & \\ & & & & & 0 & I_n & \\ & & & & & I_n & A_1 & \\ \hline & & & & & & & -E \end{array}\right] - \left[\begin{array}{ccccccc|c} -A_6 & -A_5 & I_n & 0 & 0 & 0 & 0 & \\ -A_5 & -A_4 & 0 & -A_3 & I_n & 0 & 0 & \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & -A_3 & 0 & -A_2 & 0 & I_n & 0 & \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_0 & -C \\ \hline & & & & & & -B & -A \end{array}\right].$$

By the Theorem 5.30 we have $\mathbb{L}(\lambda)$ is symmetric and operation free. Note that $\mathbb{L}(\lambda)$ is block-symmetric.

Remark 5.33. *The eigenvector formula and eigenvector recovery property of symmetric system matrix follows directly from Theorem 6.5.*

6. Conclusions and future work

We have introduced operation-free generalized Fiedler pencil with repetition(GFPR) for rational matrix functions $G(\lambda)$ and shown that GFPR pencils are linearizations of $G(\lambda)$. We have obtained the explicit formulas for the eigenvectors of the GF linearizations in terms of the eigenvectors of $G(\lambda)$. Also, we have discussed that the eigenvectors of $\mathcal{S}(\lambda)$ and $G(\lambda)$ can be recovered from those of the GFPR pencils without performing any arithmetic operations. Finally, we have introduced Symmetric/Hermitian pencils for Symmetric/Hermitian $G(\lambda)$ using GFPR pencils and shown that these pencils are linearizations of $G(\lambda)$. It would be interesting to consider multi-linear algebra, i.e., tensor eigenvalue problems and tensor functions and study its spectral properties, sensitivity analysis and linearizations. In [10, 13] and [7, 17] it has been studied spectral properties and perturbation analysis of generalized tensor eigenvalue problems and tensor functions, respectively. I would like to generalize these ideas to study spectral analysis, sensitivity analysis and linearizations of tensor polynomial eigenvalue problems in details.

Acknowledgement

I express my gratitude to Prof.Rafikul Alam from IIT Guwahati for his valuable discussions and suggestions towards the manuscript. I would also like to thank reviewer for his valuable comments to improve the article.

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