



Several Identities of Classical, Complementary and Generalized Euler Numbers

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Abstract. In this paper, we establish several interesting identities and congruences involving classical, complementary and generalized Euler numbers, by using generating functions.

1. Introduction

For a real or complex parameter x , the *generalized Euler numbers* $E_{2n}^{(x)}$ ([6, 7]) are defined by the generating functions:

$$\left(\frac{2}{e^t + e^{-t}}\right)^x = \sum_{n=0}^{\infty} E_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}\right) \quad (1)$$

or

$$(\sec t)^x = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}\right). \quad (2)$$

When $x = 1$, the numbers $E_{2n}^{(1)} = E_{2n}$ are the classical *Euler numbers*. By (1) or (2), we can get

$$E_{2n}^{(k)} = (2n)! \sum_{\substack{v_1 + \dots + v_k = n \\ v_1, \dots, v_k \geq 0}} \frac{E_{2v_1} \dots E_{2v_k}}{(2v_1)! \dots (2v_k)!} \quad (3)$$

when k is positive. For a real or complex parameter x , the *generalized complementary Euler numbers* are defined by the generating function

$$\left(\frac{2t}{e^t - e^{-t}}\right)^x = \sum_{n=0}^{\infty} \widetilde{E}_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}\right) \quad (4)$$

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or

$$\left(\frac{t}{\sin t}\right)^x = \sum_{n=0}^{\infty} (-1)^n \widehat{E}_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}\right) \tag{5}$$

When $x = 1$, the numbers $\widehat{E}_{2n}^{(1)} = \widehat{E}_{2n}$ are the original complementary Euler numbers or Euler numbers of the second kind ([3, 4]). Notice that $(-1)^n E_{2n} > 0$ and $(-1)^n \widehat{E}_{2n} > 0$ for all $n \geq 0$.

When x is a positive integer, $E_{2n}^{(x)}$ and $\widehat{E}_{2n}^{(x)}$ are called higher-order Euler numbers and higher-order complementary Euler numbers, respectively. In [6, 7], several identities were established, involving the Euler numbers and the Euler numbers of order 2. There are many interesting or useful properties and relations involving classical or generalized Euler numbers, in particular, of order 2. The main purpose of this paper is to show several new identities and congruences involving classical, generalized or complementary Euler numbers, in particular, of order 2.

2. Some basic identities of Euler numbers

In this section, we shall show some new identities of the classical Euler numbers and the generalized Euler numbers of order 2.

Lemma 2.1.

$$\begin{aligned} \sum_{k=0}^{\infty} t^k \cos(2k+1)x &= \frac{(1-t)\cos x}{(1+t)^2 - 4t \cos^2 x}, & \sum_{k=0}^{\infty} t^k \cos 2kx &= \frac{1 - 2t \cos^2 x + t}{(1+t)^2 - 4t \cos^2 x}, \\ \sum_{k=0}^{\infty} t^k \sin(2k+1)x &= \frac{(1+t)\sin x}{(1+t)^2 - 4t \cos^2 x}, & \sum_{k=0}^{\infty} t^k \sin 2kx &= \frac{2t \sin x \cos x}{(1+t)^2 - 4t \cos^2 x}. \end{aligned}$$

Proof. The first and the second identities can be seen in [5]. Put

$$A(t) = \sum_{k=0}^{\infty} t^k \cos kx \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} t^k \sin kx.$$

Then, by De Moivre formula, we have

$$A(t) + \sqrt{-1}B(t) = \sum_{k=0}^{\infty} t^k (\cos x + \sqrt{-1} \sin x)^k = \frac{1 - t \cos x + it \sin x}{1 - 2t \cos x + t^2}.$$

Hence, we get

$$A(t) = \frac{1 - t \cos x}{1 - 2t \cos x + t^2} \quad \text{and} \quad B(t) = \frac{t \sin x}{1 - 2t \cos x + t^2} \quad (|t| < 1),$$

yielding

$$\begin{aligned} \sum_{k=0}^{\infty} t^k \sin(2k+1)x &= \frac{B(\sqrt{t}) - B(-\sqrt{t})}{2\sqrt{t}} = \frac{(1+t)\sin x}{(1+t)^2 - 4t \cos^2 x}, \\ \sum_{k=0}^{\infty} t^k \sin 2kx &= \frac{B(\sqrt{t}) + B(-\sqrt{t})}{2} = \frac{t \sin 2x}{(1+t)^2 - 4t \cos^2 x}. \end{aligned}$$

□

In [5], it is shown that

$$\sum_{j=0}^n \binom{2n}{2j} (2k+1)^{2n-2j} E_{2j} = 2 \sum_{s=0}^k (-1)^{k-s} (2s)^{2n}.$$

We have a similar formula given by the following.

Theorem 2.2. For positive integers n and k , we have

$$\sum_{j=0}^n \binom{2n+1}{2j} (2k+2)^{2n-2j+1} E_{2j} = 2 \sum_{s=0}^k (-1)^{k-s} (2s+1)^{2n+1}.$$

Proof. By using Lemma 2.1 and the identity

$$\frac{1}{1+t} \frac{(1+t) \sin x}{(1+t)^2 - 4t \cos^2 x} = \frac{2t \sin x \cos x}{(1+t)^2 - 4t \cos^2 x} \frac{1}{2t \cos x},$$

we get

$$\left(\sum_{k=0}^{\infty} (-t)^k \right) \left(\sum_{k=0}^{\infty} t^k \sin(2k+1)x \right) = \sum_{k=0}^{\infty} t^k \sin(2k+2)x \frac{\sec x}{2}$$

or

$$2 \sum_{k=0}^{\infty} t^k \sum_{s=0}^k (-1)^{k-s} \sin(2s+1)x = \sum_{k=0}^{\infty} t^k \sin(2k+2)x \sec x.$$

Comparing the coefficients on both sides, we can get

$$2 \sum_{s=0}^k (-1)^{k-s} \sin(2s+1)x = \sin(2k+2)x \sec x.$$

By using Taylor series, we can obtain

$$\begin{aligned} & 2 \sum_{s=0}^k (-1)^{k-s} \sum_{n=0}^{\infty} (-1)^n (2s+1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n (2k+2)^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. \square

We show two relations between Euler numbers of order 2 and the classical Euler numbers.

Proposition 2.3. For positive integers n and k , we have

$$\sum_{j=0}^n \binom{2n}{2j} (2k+1)^{2n-2j} E_{2j} - (2k)^{2n} = \sum_{j=0}^{n-1} \binom{2n}{2j+1} E_{2j}^{(2)} (2k)^{2n-2j-1}.$$

Proof. Integrating (2) in the case of order 2, we have

$$\tan x = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(2)} \frac{x^{2n+1}}{(2n+1)!}. \tag{6}$$

By using Taylor expansions of $\cos 2kx = \sin 2kx \tan x + \cos(2k+1)x \sec x$ and comparing the coefficients, we get the desired result. \square

Proposition 2.4. For positive integers n and k , we have

$$\sum_{j=0}^n \binom{2n+1}{2j+1} (2k+1)^{2n-2j} E_{2j}^{(2)} = - \sum_{j=0}^n \binom{2n+1}{2j} (2k)^{2n-2j+1} E_{2j} + (2k+1)^{2n+1}.$$

Proof. Since $\cos(2k+1)x \tan x = -\sin 2kx \sec x + \sin(2k+1)x$, by using Taylor series with (2) and (6), and comparing the coefficients, we can get the desired result. \square

Theorem 2.5. For positive integers n and k , we have

$$\sum_{j=0}^n \frac{(2k)^{2j+1} E_{n-j}}{(2j+1)!(2n-2j)!} = \frac{(2k+1)^{2n+1}}{(2n+1)!} - \sum_{j=0}^n \frac{1}{(2j+1)!} \sum_{s=0}^{2k} \frac{(2k-2s)^{2n-2j}}{(2n-2j)!}.$$

Proof. Since

$$\sum_{s=0}^{2k} (-1)^s \cos(2k-2s)x = \frac{\cos(2k+1)x}{\cos x} \tag{7}$$

(e.g., [5, (5)]), we get

$$\sin x \sum_{s=0}^{2k} (-1)^s \cos(2k-2s)x = \sin(2k+1)x - \sin 2kx \sec x.$$

By using Taylor series with (2), and comparing the coefficients, we get the desired result. \square

We consider some special cases of the above identities. Let p be a prime number with $p \equiv 1 \pmod{4}$, we may denote by

$$n = \frac{p-1}{4} \quad \text{and} \quad k = \frac{p-1}{2} \quad (\text{even}).$$

By Theorem 2.2, we have

$$\begin{aligned} 4 \sum_{s=0}^{\frac{p-1}{2}} (-1)^{\frac{p-1}{2}-s} (2s+1)^{\frac{p+1}{2}} &= 4 \sum_{s=0}^{\frac{p-1}{2}} (-1)^s (2s+1)^{\frac{p+1}{2}} \\ &= (p+1)^2 E_{\frac{p-1}{2}} + 2 \sum_{j=0}^{\frac{p-5}{4}} E_{2j} \binom{\frac{p+1}{2}}{2j} (p+1)^{\frac{p+1}{2}-2j}. \end{aligned}$$

Thus, we can get the congruence relations

$$4 \sum_{s=0}^{\frac{p-1}{2}} (-1)^s (2s+1)^{\frac{p+1}{2}} \equiv (p+1)^2 E_{\frac{p-1}{2}} \pmod{(p+1)^3}$$

and

$$4 \sum_{s=0}^{\frac{p-3}{2}} (-1)^s (2s+1) \binom{2s+1}{p} \equiv E_{\frac{p-1}{2}} + 2 \sum_{j=0}^{\frac{p-5}{4}} E_{2j} \binom{\frac{p+1}{2}}{2j} \pmod{p},$$

where $\binom{a}{p}$ denotes the Legendre symbol.

Similarly, by Proposition 2.3 and Proposition 2.4, we get the alternative congruence relations

$$\left(\frac{-1}{p}\right) - E_{\frac{p-1}{2}} \equiv \sum_{j=0}^{\frac{p-5}{4}} E_{2j}^{(2)} \binom{\frac{p-1}{2}}{2j+1} \pmod{p},$$

and

$$E_{\frac{p-1}{2}}^{(2)} \equiv \sum_{j=0}^{\frac{p-1}{4}} \binom{\frac{p+1}{2}}{2j} E_{2j} \pmod{p}.$$

3. Some identities of complementary Euler numbers

In this section, we shall study the complementary Euler numbers and higher-order complementary Euler numbers.

From the definitions (1) and (4),

$$E_{2n+1} = \widehat{E}_{2n+1} = 0 \quad (n \geq 0).$$

Euler numbers E_{2n} are integers, but complementary Euler numbers \widehat{E}_{2n} are rational numbers. We can know the denominator of \widehat{E}_{2n} completely.

Theorem 3.1. For an integer $n \geq 1$, the denominator of complementary Euler numbers \widehat{E}_{2n} is given by

$$\prod_{(p-1)|2n} p,$$

where p runs over all odd primes with $(p-1)|2n$. In other word,

$$\left(\prod_{(p-1)|2n} p\right) \widehat{E}_{2n}$$

is an integer, where p runs over all odd primes with $(p-1)|2n$.

Remark 3.2. For any integer $n \geq 0$,

$$(2n+1)(2n-1)\cdots 3\widehat{E}_{2n} = \frac{(2n+1)!}{2^n n!} \widehat{E}_{2n}$$

is an integer.

Proof. Notice that for $n \geq 1$, we have

$$\widehat{E}_n = 2^n B_n\left(\frac{1}{2}\right) = (2-2^n)B_n,$$

where $B_n(x)$ is the Bernoulli polynomial, defined by

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

When $x = 0$, $B_n = B_n(0)$ is the classical Bernoulli number with $B_1 = -1/2$. By Von Staud-Clausen theorem ([2, 8]), for $n \geq 1$

$$B_{2n} + \sum_{(p-1)|2n} \frac{1}{p}$$

is an integer, where the sum extends over all primes p with $(p - 1)|2n$. By Fermat’s Little Theorem, if $(p - 1)|2n$, then $m^{2n} \equiv 1 \pmod{p}$ for $m = 1, 2, \dots, p - 1$. Thus, $2^{2n} \equiv 1 \not\equiv 2 \pmod{p}$ for any odd prime p . Therefore, the denominator of Euler numbers of the second kind is given by

$$\prod_{(p-1)|2n} p$$

where the product extends over all odd primes p with $(p - 1)|2n$. \square

Examples. The odd primes p satisfying $(p - 1)|24$ are 3, 5, 7, 13, and

$$\widehat{E}_{24} = \frac{1982765468311237}{1365} = \frac{47 \cdot 103 \cdot 178481 \cdot 2294797}{3 \cdot 5 \cdot 7 \cdot 13}.$$

The odd prime p satisfying $(p - 1)|26$ is 3, and

$$\widehat{E}_{26} = -\frac{286994504449393}{3} = -\frac{13 \cdot 31 \cdot 601 \cdot 1801 \cdot 657931}{3}.$$

It is well-known that Euler numbers satisfy the recurrence relation

$$\sum_{j=0}^n \binom{2n}{2j} E_{2j} = 0 \quad (n \geq 1)$$

with $E_0 = 1$. Similarly, complementary Euler numbers satisfy the following recurrence relation.

Theorem 3.3. For $n \geq 1$,

$$\sum_{j=0}^n \binom{2n+1}{2j} \widehat{E}_{2j} = 0$$

and $\widehat{E}_0 = 1$.

Proof. From the definition (4), we have

$$\begin{aligned} t &= \frac{t}{\sinh t} \sinh t \\ &= \left(\sum_{j=0}^{\infty} \widehat{E}_{2j} \frac{t^{2j}}{(2j)!} \right) \left(\sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{2n+1}{2j} \widehat{E}_{2j} \frac{t^{2n+1}}{(2n+1)!} \quad (n = j + l). \end{aligned}$$

Comparing the coefficients on both sides, we get the result. \square

Similarly to Theorem 2.2 and the result in [5], complementary Euler numbers satisfy the following recurrence relation, complementary Euler numbers satisfy the following relation.

Theorem 3.4. For a positive integer n and a nonnegative integer k ,

$$\sum_{j=0}^n \binom{2n+1}{2j} (2k+1)^{2n-2j+1} \widehat{E}_{2j} = 2(2n+1) \sum_{l=1}^k (2l)^{2n}.$$

Proof. From Lemma 2.1, for $|t| < 1$ we have

$$\begin{aligned} \sum_{k=0}^{\infty} t^k \sum_{j=0}^{2k} \cos(2k - 2j)x &= \sum_{k=0}^{\infty} t^k \left(2 \sum_{j=0}^k \cos(2k - 2j)x - 1 \right) \\ &= 2 \left(\sum_{k=0}^{\infty} t^k \right) \left(\sum_{l=0}^{\infty} t^l \cos 2lx \right) - \sum_{k=0}^{\infty} t^k \\ &= \frac{2}{1-t} \frac{1 - 2t \cos^2 x + t}{(1+t)^2 - 4t \cos^2 x} - \frac{1}{1-t} \\ &= \frac{1+t}{(1+t)^2 - 4t \cos^2 x} = \frac{1}{\sin x} \sum_{k=0}^{\infty} t^k \sin(2k+1)x. \end{aligned}$$

Comparing the coefficients of t^k , we get

$$\sum_{j=0}^{2k} \cos(2k - 2j)x = \frac{\sin(2k+1)x}{\sin x}. \tag{8}$$

The right-hand side of (8) is equal to

$$\begin{aligned} \frac{x}{\sin x} \frac{\sin(2k+1)x}{x} &= \left(\sum_{j=0}^{\infty} (-1)^j \widehat{E}_{2j} \frac{x^{2j}}{(2j)!} \right) \left(\sum_{m=0}^{\infty} (-1)^m (2k+1)^{2m+1} \frac{x^{2m}}{(2m+1)!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^n \binom{2n+1}{2j} (2k+1)^{2n-2j+1} \widehat{E}_{2j} \frac{x^{2n}}{(2n+1)!} \quad (j+m=n). \end{aligned}$$

The left-hand side of (8) is equal to

$$\sum_{j=0}^{2k} \sum_{n=0}^{\infty} (-1)^n (2k-2j)^{2n} \frac{x^{2n}}{(2n)!}.$$

Comparing the coefficients on both sides, we have

$$\begin{aligned} \sum_{j=0}^n \binom{2n+1}{2j} (2k+1)^{2n-2j+1} \frac{\widehat{E}_{2j}}{(2n+1)!} &= \sum_{l=0}^{2k} \frac{(2k-2l)^{2n}}{(2n)!} \\ &= 2 \sum_{l=1}^k \frac{(2l)^{2n}}{(2n)!}. \end{aligned}$$

Therefore, we get the desired result. \square

Now, we show some relations with complementary Euler numbers of order 2.

Lemma 3.5.

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n \widehat{E}_{2n}^{(2)}}{(2n)!} \frac{t^{2n-1}}{2n-1}.$$

Proof. By (4), we have

$$-t^2 \frac{d}{dt} \cot t = \left(\frac{t}{\sin t} \right)^2 = \sum_{n=0}^{\infty} (-1)^n \widehat{E}_{2n}^{(2)} \frac{t^{2n}}{(2n)!}.$$

Dividing $-t^2$ and integrating both sides, we get the relation. \square

The first relation is about the classical Bernoulli numbers B_n , defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Theorem 3.6. For $n \geq 1$,

$$\frac{\widehat{E}_{2n}}{2^{4n-1}(2n-1)} = -1 - \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} B_k 2^k.$$

Proof. Since

$$\frac{1}{2} \frac{2it}{e^{2it} - 1} + \frac{1}{2} \frac{2it}{e^{2it} - 1} e^{it} + \frac{it}{2} = \frac{it e^{it} + 1}{2 e^{it} - 1} = \frac{t}{2} \cot \frac{t}{2}$$

where $t^2 = -1$, we obtain

$$\frac{1}{2} \sum_{n=0}^{\infty} B_n \frac{(2it)^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} B_n \frac{(2it)^n}{n!} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} + \frac{it}{2} = \frac{t}{2} \left(\frac{2}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n \widehat{E}_{2n}^{(2)}}{2^{2n-1}(2n)!} \frac{t^{2n-1}}{2n-1} \right).$$

Comparing the coefficient of t^{2n} ($n \geq 1$), we have

$$2^{2n} + \sum_{k=0}^{2n} \binom{2n}{k} B_k 2^k = -\frac{\widehat{E}_{2n}}{2^{2n-1}(2n-1)}.$$

□

We give a relation with the original complementary Euler numbers.

Theorem 3.7. For $n \geq 1$,

$$\sum_{k=1}^n \binom{2n}{2k} \frac{\widehat{E}_{2k}^{(2)}}{2k-1} = -\widehat{E}_{2n} - 2n + 1.$$

Proof. By $1/\sin t - \sin t = \cos t \cdot \cot t$ and using Lemma 3.5,

$$\frac{1}{t} \sum_{n=0}^{\infty} (-1)^n \widehat{E}_{2n} \frac{t^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \right) \left(\frac{1}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n \widehat{E}_{2n}^{(2)}}{(2n)!} \frac{t^{2n-1}}{2n-1} \right).$$

Hence,

$$\sum_{n=0}^{\infty} (-1)^n \widehat{E}_{2n} \frac{t^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} - \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^n \binom{2n}{2k} \frac{\widehat{E}_{2k}^{(2)}}{2k-1} \frac{t^{2n}}{(2n)!}.$$

Comparing the coefficients on both sides, we get the result. □

4. Some congruence relations of complementary Euler numbers

In this section, we use another definition of complementary Euler numbers. Put

$$F(x) = \frac{1}{\sin x}.$$

Then complementary Euler numbers are defined by

$$xF(x) = \sum_{n=0}^{\infty} \widehat{\mathcal{E}}_n \frac{x^n}{n!}.$$

From (4), we know that $\widehat{\mathcal{E}}_{2n} = (-1)^n \widehat{E}_{2n} > 0$. Define the constants $c_i(m)$ ($i = 0, 1, \dots, m$) by the coefficients of the polynomial

$$(x + 1^2)(x + 3^2)(x + 5^2) \cdots (x + (2m - 1)^2) = \sum_{i=0}^m c_i(m)x^{m-i}.$$

Then, as in [9, (6)] there is the recurrence relation $c_k(n + 1) = c_k(n) + (2n + 1)^2 c_{k-1}(n)$ with initial conditions $c_0(n) = 1$ and $c_n(n) = ((2n - 1)!!)^2$ ($n \geq 1$).

In [9], the calculating problem of the summation involving the Euler numbers

$$\sum_{a_1 + \cdots + a_k = n} \frac{E_{2a_1} \cdots E_{2a_k}}{(2a_1)! \cdots (2a_k)!}$$

were studied. We consider the similar summation involving the complementary Euler numbers $\widehat{\mathcal{E}}_n$.

Theorem 4.1. For $n = 0, 1, \dots, m$ and $k = 2m + 1 > 1$, we have

$$(2m)! \sum_{a_1 + \cdots + a_k = n} \frac{\widehat{\mathcal{E}}_{2a_1} \cdots \widehat{\mathcal{E}}_{2a_k}}{(2a_1)! \cdots (2a_k)!} = c_n(m)(2m - 2n)!.$$

For any nonnegative integer n , we have

$$\begin{aligned} (2m)! \sum_{a_1 + \cdots + a_k = n+m+1} \frac{(2n)!}{(2a_1)! \cdots (2a_k)!} \widehat{\mathcal{E}}_{2a_1} \cdots \widehat{\mathcal{E}}_{2a_k} \\ = \sum_{i=0}^m \frac{c_i(m) \widehat{\mathcal{E}}_{2n+2m-2i+2}}{(2n + 2m - 2i + 1)(2n + 2m - 2i) \cdots (2n + 2)(2n + 2m - 2i + 2)!}. \end{aligned}$$

Proof. First, we shall prove

$$(2m)! F^{2m+1}(x) = \sum_{i=0}^m c_i(m) F^{(2m-2i)}(x). \tag{9}$$

From the definition of $F(x)$,

$$F'(x) = -\frac{\cos x}{\sin^2 x} \quad \text{and} \quad F''(x) = \frac{1 + \cos^2 x}{\sin^3 x}.$$

So, we have $2F^3(x) = F''(x) + F(x)$. This is the case $m = 1$ in (9). By induction, using the recurrence relation of $c_i(m)$, we can prove (9). The procedure is similar to the case $F(x) = 1/\cos x$ ([9, Lemma]). Next, since $\widehat{\mathcal{E}}_{2n-1} = 0$ and

$$F(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \widehat{\mathcal{E}}_{2n} \frac{x^{2n-1}}{(2n)!},$$

we have for $i = 0, 1, \dots$

$$F^{(2i)}(x) = \frac{(2i)!}{x^{2i+1}} + \sum_{n=i+1}^{\infty} \frac{\widehat{\mathcal{E}}_{2n}}{(2n-1)(2n-2) \cdots (2n-2i)} \frac{x^{2n-2i-1}}{(2n)!}$$

$$= \frac{(2i)!}{x^{2i+1}} + \sum_{n=0}^{\infty} \frac{\widehat{\mathcal{E}}_{2n+2i+2}}{(2n+2i+1)(2n+2i)\cdots(2n+2)} \frac{x^{2n+1}}{(2n+2i+2)!}$$

and

$$\begin{aligned} F^{(2i+1)}(x) &= -\frac{(2i+1)!}{x^{2i+2}} + \sum_{n=i+1}^{\infty} \frac{\widehat{\mathcal{E}}_{2n}}{(2n-1)(2n-2)\cdots(2n-2i-1)} \frac{x^{2n-2i-2}}{(2n)!} \\ &= -\frac{(2i+1)!}{x^{2i+2}} + \sum_{n=0}^{\infty} \frac{\widehat{\mathcal{E}}_{2n+2i+2}}{(2n+2i+1)(2n+2i)\cdots(2n+1)} \frac{x^{2n}}{(2n+2i+2)!}. \end{aligned}$$

On one hand,

$$\begin{aligned} (2m)!x^k F^k(x) &= x^k \sum_{i=0}^m c_i(m) F^{(2m-2i)}(x) \\ &= \sum_{i=0}^m c_i(m) \left(\frac{(2m-2i)!x^{2m+1}}{x^{2m-2i+1}} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{\widehat{\mathcal{E}}_{2n+2m-2i+2}}{(2n+2m-2i+1)(2n+2m-2i)\cdots(2n+2)} \frac{x^{2n+2m+2}}{(2n+2m-2i+2)!} \right) \\ &= \sum_{i=0}^m c_i(m) \cdot (2m-2i)!x^{2i} \\ &\quad + \sum_{n=0}^{\infty} \sum_{i=0}^m \frac{c_i(m)\widehat{\mathcal{E}}_{2n+2m-2i+2}}{(2n+2m-2i+1)(2n+2m-2i)\cdots(2n+2)} \frac{x^{2n+2m+2}}{(2n+2m-2i+2)!}. \end{aligned}$$

On the other hand,

$$x^k F^k(x) = \sum_{n=0}^{\infty} \sum_{a_1+\dots+a_k=n} \frac{(2n)!}{(2a_1)! \cdots (2a_k)!} \widehat{\mathcal{E}}_{2a_1} \cdots \widehat{\mathcal{E}}_{2a_k} \frac{x^{2n}}{(2n)!}.$$

Comparing the coefficient of x^{2n} ($n = 0, 1, \dots, m$) on both sides, we get the first result. Comparing the coefficient of $x^{2n+2m+2}$ on both sides, we get the second result. \square

For any odd prime p , by using the congruences ([9])

$$c_i \left(\frac{p-1}{2} \right) \equiv 0 \pmod{p} \quad \left(i = 1, \dots, \frac{p-3}{2} \right),$$

we can get the following result.

Theorem 4.2. Let $m = \frac{p-1}{2}$ and $k = p$. Then, we obtain the following congruence relations: For $n = 0$, we have the trivial identity

$$(p-1)! = c_0 \left(\frac{p-1}{2} \right) (p-1)!.$$

For $n = \frac{p-1}{2}$, we have

$$(p-1)! \sum_{a_1+\dots+a_p=\frac{p-1}{2}} \frac{\widehat{\mathcal{E}}_{2a_1} \cdots \widehat{\mathcal{E}}_{2a_p}}{(2a_1)! \cdots (2a_p)!} = ((p-2)!!)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.$$

For $n = 1, \dots, \frac{p-3}{2}$, we have

$$(p-1)! \sum_{a_1+\dots+a_p=n} \frac{\widehat{\mathcal{E}}_{2a_1} \cdots \widehat{\mathcal{E}}_{2a_p}}{(2a_1)! \cdots (2a_p)!} = c_n \left(\frac{p-1}{2} \right) (p-2n-1)! \equiv 0 \pmod{p}.$$

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