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Generalized Hypergeometric Identities with Extra Parameters

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Abstract. A new class of hypergeometric identities with extra parameters is introduced in order to generate various kinds of summation theorems for generalized hypergeometric series. Some interesting examples are also given in this direction.

1. Introduction

If z is a complex variable, the generalized hypergeometric function is defined by the series

$$ {}_pF_q\left(\begin{matrix} a_1, & \ldots & ,a_p \\ b_1, & \ldots & ,b_q \end{matrix} \,\middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \cdot \frac{z^k}{k!}, \tag{1} $$

where $b_1, ..., b_q \neq 0, -1, -2, ...$ and

$$ (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & (b = 0,\ a \in \mathbb{C}\backslash\{0\}), \\ a(a+1)\cdots(a+b-1) & (b \in \mathbb{N},\ a \in \mathbb{C}), \end{cases} $$

denotes the Pochhammer symbol in which

$$ \Gamma(z) = \int_0^{\infty} x^{z-1} \mathrm{e}^{-x}\, \mathrm{d}x \qquad (\mathrm{Re}(z) > 0), $$

is the well-known gamma function.

According to the ratio test [9, 17], the series (1) is convergent for all $|z| < \infty$ if $p < q+1$ and for $|z| < 1$ if $p = q+1$, while it is divergent for all $z \neq 0$ if $p > q+1$. Moreover, when $|z| = 1$ with $p = q+1$, the series (1) converges absolutely if

$$ \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \right) > 0, \tag{2} $$

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conditionally convergent if

$$-1 < \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) \leq 0, \quad z \neq 1 \tag{3}$$

and divergent if

$$\mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) \leq -1, \quad z \neq 1. \tag{4}$$

Hypergeometric functions appear in a wide variety of applied mathematics and physics [8, 9]. The main reason for introducing and developing such series is that many special functions can be represented in terms of them and their initial properties can be directly found via the initial properties of hypergeometric functions [5].

It is worth mentioning whenever a generalized hypergeometric series can be summed in terms of gamma functions, the result will be valuable as only a few summation theorems are available in the literature, see e.g. [3, 4, 6, 7, 10, 11].

In this paper, by presenting a new approach, we introduce some hypergeometric identities with extra parameters in order to generate various kinds of summation theorems for generalized hypergeometric series. In section 2, we introduce two main hypergeometric identities having 2 and 3 extra parameters and then study their finite cases containing new kinds of three term relations for generalized hypergeometric functions in detail. In section 3, by using the Karlssonâ–Minton identity [2, 12, 14], we extend our approach to obtain a general hypergeometric identity with n extra parameters. We point out that all results of this paper are valid only if the conditions (2)–(4) are satisfied.

2. Hypergeometric Identities with 2 and 3 Extra Parameters

First, the well-known Gauss formula [1, 15]

$${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{5}$$

directly gives

$$\frac{(c)_k}{(d)_k}\,{}_2F_1\left(\begin{array}{c} -k,\ c-d \\ c \end{array} \middle| 1\right) = 1, \tag{6}$$

provided that $c, d \neq 0, -1, -2, \ldots$.

To establish a hypergeometric identity with two extra parameters, replace (6) in (1) to get

$$\begin{aligned} {}_pF_q\left(\begin{array}{c} a_1,\ \ldots\ ,a_p \\ b_1,\ \ldots\ ,b_q \end{array} \middle| z\right) &= \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} \frac{(c)_k}{(d)_k}\,{}_2F_1\left(\begin{array}{c} -k,\ c-d \\ c \end{array} \middle| 1\right) \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{(1)_k} \frac{(c)_k}{(d)_k}\left(\sum_{j=0}^{k} \frac{(-k)_j (c-d)_j}{(c)_j\, j!}\right). \end{aligned} \tag{7}$$

Since

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} A(j,k) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} A(j,k+j),$$

relation (7) changes to

$$ {}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a_1)_{k+j}\dots(a_p)_{k+j}(c)_{k+j}}{(b_1)_{k+j}\dots(b_q)_{k+j}(d)_{k+j}} \frac{z^{k+j}}{(1)_{k+j}} \frac{(-k-j)_j(c-d)_j}{(c)_j j!}. \tag{8} $$

On the other hand, the following relations hold true

$$ (a)_{k+j} = (a)_k(a+k)_j \quad \text{and} \quad (-k-j)_j = (-1)^j(1+k)_j. \tag{9} $$

Hence, relation (8) takes the final form

$$ {}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{k!} \frac{(c)_k}{(d)_k} {}_{p+2}F_{q+2}\left(\begin{matrix} a_1+k, & \dots & ,a_p+k, & c+k, & c-d \\ b_1+k, & \dots & ,b_q+k, & d+k, & c \end{matrix}\,\middle|\, -z\right), \tag{10} $$

in which $c, d \neq 0, -1, -2, \dots$ are two extra and independent parameters.

As far as we searched in the literature, this approach is being used for the first time for generalized hypergeometric functions. However, the importance of the identity (10) is more for its various subcases, which are as follows. For some generalizations of the classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ see [16] and [13].

Case 1.1. If $d = a_p$ in (10), then

$$ {}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_{p-1})_k(c)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{k!} {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+k, & \dots & ,a_{p-1}+k, & c+k, & c-a_p \\ b_1+k, & \dots & ,b_q+k, & c & \end{matrix}\,\middle|\, -z\right), $$

where $c \neq 0, -1, -2, \dots$.

Case 1.2. For $c = b_q$ in (10) we have

$$ {}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_{q-1})_k(d)_k} \frac{z^k}{k!} {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+k, & \dots & ,a_p+k, & b_q-d & \\ b_1+k, & \dots & ,b_{q-1}+k, & d+k, & b_q \end{matrix}\,\middle|\, -z\right), $$

where $d \neq 0, -1, -2, \dots$.

Case 1.3. If simultaneously $d = a_p$ and $c = b_q$, (10) reduces to

$$ {}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_{p-1})_k}{(b_1)_k\dots(b_{q-1})_k} \frac{z^k}{k!} {}_pF_q\left(\begin{matrix} a_1+k, & \dots & ,a_{p-1}+k, & b_q-a_p \\ b_1+k, & \dots & ,b_{q-1}+k, & b_q \end{matrix}\,\middle|\, -z\right). $$

There is also an important finite subcase for the identity (10) which must be studied separately. If $c - d = -m$ $(m \in \mathbb{Z}^+)$ in (10), then the relation

$$ \begin{aligned} {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\,\middle|\, z\right) &= \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k(c)_k}{(b_1)_k\dots(b_q)_k(c+m)_k} \frac{z^k}{k!} {}_{p+2}F_{q+2}\left(\begin{matrix} -m, & a_1+k, \dots, a_p+k, & c+k \\ b_1+k, & \dots, b_q+k, c+m+k, & c \end{matrix}\,\middle|\, -z\right) \\ &= \sum_{j=0}^{m} \frac{(-m)_j(-z)^j}{(c)_j j!} \sum_{k=0}^{\infty} \frac{(a_1)_k(a_1+k)_j\dots(a_p)_k(a_p+k)_j(c)_k(c+k)_j}{(b_1)_k(b_1+k)_j\dots(b_q)_k(b_q+k)_j(c+m)_k(c+m+k)_j} \frac{z^k}{k!}, \end{aligned} $$

is finally simplified, by employing the relations (9), as

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\,\middle|\, z\right) = \sum_{j=0}^{m} \frac{(-m)_j(a_1)_j\dots(a_p)_j}{(b_1)_j\dots(b_q)_j(c+m)_j} \frac{(-z)^j}{j!} {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+j, & \dots & ,a_p+j, & c+j \\ b_1+j, & \dots & ,b_q+j, & c+m+j \end{matrix}\,\middle|\, z\right). \tag{11} $$

The finite identity (11) contains two remarkable subcases as follows:
Case 2.1. If $c + m = a_p$ in (11), we eventually obtain

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-1}, a_p + m \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \dots (a_{p-1})_j}{(b_1)_j \dots (b_q)_j} \frac{(-z)^j}{j!} {}_pF_q\left(\begin{matrix} a_1 + j, \dots, a_p + j \\ b_1 + j, \dots, b_q + j \end{matrix} \,\middle|\, z \right). \tag{12} $$

For example, by noting the second kind of Gauss formula [1, 15]

$$ {}_2F_1\left(\begin{matrix} a, \ b \\ (a+b+1)/2 \end{matrix} \,\middle|\, \frac{1}{2} \right) = \frac{\sqrt{\pi}\,\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}, \tag{13} $$

if $(p, q) = (2, 1)$, $z = \frac{1}{2}$ and $b_1 = (a_1 + a_2 + 1)/2$ in (12), then

$$ {}_2F_1\left(\begin{matrix} a_1, \ a_2 + m \\ (a_1 + a_2 + 1)/2 \end{matrix} \,\middle|\, \frac{1}{2} \right) = \sqrt{\pi}\,\Gamma\left(\frac{a_1 + a_2 + 1}{2}\right) \sum_{j=0}^{m} \frac{(-1)^j}{j! 2^j} \frac{(-m)_j (a_1)_j}{\Gamma((j + a_1 + 1)/2)\Gamma((j + a_2 + 1)/2)}, $$

is derived as a generalization of the Gauss formula (13) for $m = 0$.

Case 2.2. For $c = b_q$, the identity (11) reads, after a minor change, as

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, b_q - m \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \dots (a_p)_j}{(b_q - m)_j (b_1)_j \dots (b_q)_j} \frac{(-z)^j}{j!} {}_pF_q\left(\begin{matrix} a_1 + j, \dots, a_p + j \\ b_1 + j, \dots, b_q + j \end{matrix} \,\middle|\, z \right). \tag{14} $$

For example, if $(p, q) = (2, 1)$, $z = 1/2$ and $b_1 = (a_1 + a_2 + 1)/2$ are replaced in (14), then

$$ {}_2F_1\left(\begin{matrix} a_1, \ a_2 \\ \frac{a_1+a_2+1}{2} - m \end{matrix} \,\middle|\, \frac{1}{2} \right) = \sqrt{\pi}\,\Gamma\left(\frac{a_1 + a_2 + 1}{2}\right) \sum_{j=0}^{m} \frac{(-1)^j}{j! 2^j} \frac{(-m)_j (a_1)_j (a_2)_j}{((a_1 + a_2 + 1)/2 - m)_j \Gamma((j + a_1 + 1)/2)\Gamma((j + a_2 + 1)/2)}, $$

is another generalization of the formula (13) for $m = 0$.

2.1. A Hypergeometric identity with three extra parameters

Similarly, in order to find an identity with three extra parameters, we can e.g. refer to the Pfaff-Saalschutz formula [1, 15]

$$ {}_3F_2\left(\begin{matrix} -k, \ c - a, \ c - b \\ c, \ 1 + c - a - b - k \end{matrix} \,\middle|\, 1 \right) = \frac{(a)_k (b)_k}{(c)_k (a + b - c)_k}, \tag{15} $$

which is equivalent to

$$ \frac{(c)_k (a + b - c)_k}{(a)_k (b)_k} {}_3F_2\left(\begin{matrix} -k, \ c - a, \ c - b \\ c, \ 1 + c - a - b - k \end{matrix} \,\middle|\, 1 \right) = 1, \tag{16} $$

provided that $a, b, c \neq 0, -1, -2, \dots$.

Now, replace (16) in (1) to get

$$ \begin{aligned} {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z \right) &= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} \frac{(c)_k (a + b - c)_k}{(a)_k (b)_k} {}_3F_2\left(\begin{matrix} -k, \ c - a, \ c - b \\ c, \ 1 + c - a - b - k \end{matrix} \,\middle|\, 1 \right) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1)_{k+j} \dots (a_p)_{k+j}}{(b_1)_{k+j} \dots (b_q)_{k+j}} \frac{z^{k+j}}{(1)_{k+j}} \frac{(c)_{k+j} (a + b - c)_{k+j}}{(a)_{k+j} (b)_{k+j}} \frac{(-k - j)_j (c - a)_j (c - b)_j}{(c)_j (1 + c - a - b - k - j)_j j!}. \end{aligned} $$

By employing the relations (9) we therefore obtain

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} \frac{(c)_k (a+b-c)_k}{(a)_k (b)_k} {}_{p+3}F_{q+3}\left(\begin{matrix} a_1+k, & \dots & , a_p+k, \; c+k, \; c-a, \; c-b \\ b_1+k, & \dots & , b_q+k, \; a+k, \; b+k, \; c \end{matrix}\,\middle|\, z\right), \quad (17)
$$

which is valid for $a, b, c \neq 0, -1, -2, \dots$.

Once again, the importance of the identity (17) is due to having diverse subcases. In this sense, there are four infinite subcases of the identity (17) as follows.

Case 3.1. For $c = b_q$, the identity (17) reads as

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & a_p \\ b_1, & \dots & b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_{q-1})_k} \frac{(a+b-b_q)_k}{(a)_k (b)_k} \frac{z^k}{k!} {}_{p+2}F_{q+2}\left(\begin{matrix} a_1+k, \dots, a_p+k, \; b_q-a, \; b_q-b \\ b_1+k, \dots, b_{q-1}+k, \; a+k, \; b+k, \; b_q \end{matrix}\,\middle|\, z\right),
$$

where $a, b \neq 0, -1, -2, \dots$.

Case 3.2. If $a = a_p$ and $b = a_{p-1}$ in (17), then

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_{p-2})_k (a_p+a_{p-1}-c)_k (c)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+k, \dots, a_{p-2}+k, \; c+k, \; c-a_{p-1}, \; c-a_p \\ b_1+k, \; \dots \; , b_q+k, \quad c \end{matrix}\,\middle|\, z\right),
$$

where $c \neq 0, -1, -2, \dots$.

Case 3.3 If $a = a_p$ and $b = a_{p-1}$ in (17), then

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_{p-2})_k (a_p+a_{p-1}-c)_k (c)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+k, \dots, a_{p-2}+k, \; c+k, \; c-a_{p-1}, \; c-a_p \\ b_1+k, \; \dots \; , b_q+k, \quad c \end{matrix}\,\middle|\, z\right),
$$

where $c \neq 0, -1, -2, \dots$.

Case 3.4 Finally for $a = a_p$, $b = a_{p-1}$ and $c = b_q$ in (17), we have

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_{p-2})_k (a_p+a_{p-1}-b_q)_k}{(b_1)_k \dots (b_{q-1})_k} \frac{z^k}{k!} {}_pF_q\left(\begin{matrix} a_1+k, \dots, a_{p-2}+k, \; b_q-a_{p-1}, \; b_q-a_p \\ b_1+k, \dots, b_{q-1}+k, \quad b_q \end{matrix}\,\middle|\, z\right).
$$

There are also two important finite subcases for the identity (17) which must be studied separately. The first case is when $a+b-c = -m$ for any $m \in \mathbb{Z}^+$. In this case, relation (17) changes to

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{m} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{(-m)_k (a+b+m)_k}{(a)_k (b)_k} \frac{z^k}{k!} {}_{p+3}F_{q+3}\left(\begin{matrix} a_1+k, \dots, a_p+k, \; a+b+m+k, \; a+m, \; b+m \\ b_1+k, \dots, b_q+k, \; a+k, \; b+k, \; a+b+m \end{matrix}\,\middle|\, z\right). \quad (18)
$$

Now assume in (18) that $a = a_{p-1}$, $b = a_p$ and $b_q = a_{p-1} + a_p + m$. Then we have

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_{p-1}, \ a_p \\ b_1, & \dots & ,b_{q-1}, \ a_{p-1}+a_p+m \end{matrix}\ \middle|\ z\right) = \sum_{k=0}^{m} \frac{(-m)_k (a_1)_k \dots (a_{p-2})_k}{(b_1)_k \dots (b_{q-1})_k} \frac{z^k}{k!}\, {}_pF_q\left(\begin{matrix} a_1+k, \dots, a_{p-2}+k, \ a_{p-1}+m, \ a_p+m \\ b_1+k, \dots, b_{q-1}+k, \ a_{p-1}+a_p+m \end{matrix}\ \middle|\ z\right). \quad (19)
$$

For the second finite case, if $c - b = -m$ $(m \in \mathbb{Z}^+)$ in (17), then the relation

$$
\begin{aligned}
&{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\ \middle|\ z\right) \\
&= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_{p-1})_k}{(b_1)_k \dots (b_q)_k} \frac{(b-m)_k (a+m)_k}{(a)_k (b)_k} \frac{z^k}{k!}\, {}_{p+3}F_{q+3}\left(\begin{matrix} -m, \ a_1+k, \ \dots, a_p+k, \ b-m+k, b-m-a \\ b_1+k, \ \dots, \ b_q+k, \ a+k, \ b+k, \ b-m \end{matrix}\ \middle|\ z\right) \\
&= \sum_{j=0}^{m} \frac{(-m)_j (b-m-a)_j}{(b-m)_j} \frac{z^j}{j!} \sum_{k=0}^{\infty} \frac{(a_1)_k (a_1+k)_j \dots (a_p)_k (a_p+k)_j (b-m)_k (b-m+k)_j (a+m)_k}{(b_1)_k (b_1+k)_j \dots (b_q)_k (b_q+k)_j (a)_k (a+k)_j (b)_k (b+k)_j} \frac{z^k}{k!},
\end{aligned}
$$

is simplified, by employing the relations (9), as

$$
\begin{aligned}
&{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\ \middle|\ z\right) \\
&= \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \dots (a_p)_j}{(b_1)_j \dots (b_q)_j} \frac{(b-a-m)_j}{(a)_j (b)_j} \frac{z^j}{j!}\, {}_{p+2}F_{q+2}\left(\begin{matrix} a_1+j, & \dots & ,a_p+j, b-m+j, a+m \\ b_1+j, & \dots & ,b_q+j, \ \ a+j, \ b+j \end{matrix}\ \middle|\ z\right), \quad (20)
\end{aligned}
$$

in which $a, b \neq 0, -1, -2, \dots$ are two extra and independent parameters.
The finite identity (20) contains five notable subcases as follows:

Case 4.1. If $a = a_p$ in (20), then

$$
\begin{aligned}
&{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\ \middle|\ z\right) \\
&= \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \dots (a_{p-1})_j (b-m-a_p)_j}{(b_1)_j \dots (b_q)_j (b)_j} \frac{z^j}{j!}\, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+j, & \dots & ,a_{p-1}+j, b-m+j, a_p+m \\ b_1+j, & \dots & ,b_q+j, \ b+j \end{matrix}\ \middle|\ z\right),
\end{aligned}
$$

where $b \neq 0, -1, -2, \dots$.

Case 4.2. For $b = a_p$ in (20) we have

$$
\begin{aligned}
&{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\ \middle|\ z\right) \\
&= \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \dots (a_{p-1})_j (a_p-m-a)_j}{(b_1)_j \dots (b_q)_j (a)_j} \frac{z^j}{j!}\, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+j, & \dots & ,a_{p-1}+j, a_p-m+j, a+m \\ b_1+j, & \dots & ,b_q+j, \ a+j \end{matrix}\ \middle|\ z\right),
\end{aligned}
$$

where $a \neq 0, -1, -2, \dots$.

Case 4.3. Let $b = b_q + m$ in (20). Hence

$$
{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\ \middle|\ z\right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \dots (a_p)_j (b_q-a)_j}{(b_1)_j \dots (b_q)_j (b_q+m)_j (a)_j} \frac{z^j}{j!}\, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+j, \ \dots \ ,a_p+j, \quad a+m \\ b_1+j, \dots, b_{q-1}+j, a+j, b_q+m+j \end{matrix}\ \middle|\ z\right),
$$

where $a \neq 0, -1, -2, \ldots$.

Case 4.4. For $a = a_{p-1}$ and $b = a_p$ in (20), we get

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \ldots (a_{p-2})_j (a_p - a_{p-1} - m)_j}{(b_1)_j \ldots (b_q)_j} \frac{z^j}{j!} {}_pF_q\left(\begin{matrix} a_1 + j, \ldots, a_{p-2} + j, a_p - m + j, a_{p-1} + m \\ b_1 + j, \ldots, b_q + j \end{matrix} \,\middle|\, z \right).
$$

Also for $a = a_p$ and $b = b_q + m$ in (20), we have

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \ldots (a_{p-1})_j (b_q - a_p)_j}{(b_1)_j \ldots (b_q)_j (b_q + m)_j} \frac{z^j}{j!} {}_pF_q\left(\begin{matrix} a_1 + j, \ldots, a_{p-1} + j, a_p + m \\ b_1 + j, \ldots, b_{q-1} + j, b_q + m + j \end{matrix} \,\middle|\, z \right).
$$

Case 4.5. Finally, if $a = a_{p-1}$, $b = a_p$ and $b_q = a_p - m$ in (20), then

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_{p-1}, b_q + m \\ b_1, \ldots, b_{q-1}, b_q \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j \ldots (a_{p-2})_j (b_q - a_{p-1})_j}{(b_1)_j \ldots (b_q)_j} \frac{z^j}{j!} {}_{p-1}F_{q-1}\left(\begin{matrix} a_1 + j, \ldots, a_{p-2} + j, a_{p-1} + m \\ b_1 + j, \ldots, b_{q-1} + j \end{matrix} \,\middle|\, z \right). \quad (21)
$$

For example, by noting the kummer formula [1, 15]

$$
{}_2F_1\left(\begin{matrix} a, b \\ a + 1 - b \end{matrix} \,\middle|\, -1 \right) = \frac{\Gamma(a + 1 - b)\Gamma(a/2 + 1)}{\Gamma(a/2 + 1 - b)\Gamma(a + 1)}, \quad (22)
$$

if $(p, q) = (3, 2)$, $z = -1$ and $b_1 = a_1 + 1 - a_2 - m$ in (21), then

$$
{}_3F_2\left(\begin{matrix} a_1, & a_2, & b_2 + m \\ a_1 + 1 - a_2 - m, & b_2 \end{matrix} \,\middle|\, -1 \right) = \frac{\Gamma(a_1 + 1 - a_2 - m)}{\Gamma(a_1 + 1)} \times \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(-m)_j (a_1)_j (b_2 - a_2)_j}{(a_1 + 1)_j (b_2)_j} \frac{\Gamma((a_1 + 2 + j)/2)}{\Gamma((a_1 + 2 - 2a_2 - 2m + j)/2)},
$$

is derived as a generalization of the kummer formula (22) for $m = 0$.

Another interesting example of relation (21) is when $(p, q) = (4, 3)$, i.e.

$$
{}_4F_3\left(\begin{matrix} a_1, a_2, a_3, b_3 + m \\ b_1, b_2, b_3 \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{m} \frac{(-m)_j (a_1)_j (a_2)_j (b_3 - a_3)_j}{(b_1)_j (b_2)_j (b_3)_j} \frac{z^j}{j!} {}_3F_2\left(\begin{matrix} a_1 + j, a_2 + j, a_3 + m \\ b_1 + j, b_2 + j \end{matrix} \,\middle|\, z \right). \quad (23)
$$

Now, by noting the formula (15), replacing $z = 1$, $b_2 = a_1 + a_2 + 1 - b_1 - n$ and $a_3 = -m - n$ for $m, n \in \mathbb{Z}^+$ in (23) finally yield

$$
{}_4F_3\left(\begin{matrix} -(m + n), a_1, a_2, b_3 + m \\ b_1, a_1 + a_2 + 1 - b_1 - n, b_3 \end{matrix} \,\middle|\, 1 \right) = \frac{(b_1 - a_1)_n (b_1 - a_2)_n}{(b_1 - a_1 - a_2)_n (b_1)_n} {}_4F_3\left(\begin{matrix} -m, a_1, a_2, b_3 + m + n \\ b_1 + n, a_1 + a_2 + 1 - b_1, b_3 \end{matrix} \,\middle|\, 1 \right). \quad (24)
$$

Although for $m = 0$, (24) leads to the same as Pfaff-Saalschutz formula, for e.g. $m = 1$ in this relation we obtain

$$
{}_4F_3\left(\begin{matrix} -(n + 1), a_1, a_2, b_3 + 1 \\ b_1, a_1 + a_2 + 1 - b_1 - n, b_3 \end{matrix} \,\middle|\, 1 \right) = \left(1 - \frac{a_1 a_2 (b_3 + n + 1)}{b_3 (b_1 + n)(a_1 + a_2 + 1 - b_1)} \right) \frac{(b_1 - a_1)_n (b_1 - a_2)_n}{(b_1 - a_1 - a_2)_n (b_1)_n}.
$$

2.2. New kinds of three term relations for generalized hypergeometric series

Due to having an extra parameter, the finite identity (11) is notable especially when $m = 1$, i.e.

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) = {}_{p+1}F_{q+1}\left(\begin{matrix} a_1, \dots, a_p, c \\ b_1, \dots, b_q, c+1 \end{matrix} \,\middle|\, z\right) + \frac{a_1 \dots a_p}{b_1 \dots b_q (c+1)} z \, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+1, \dots, a_p+1, c+1 \\ b_1+1, \dots, b_q+1, c+2 \end{matrix} \,\middle|\, z\right). \quad (25) $$

For $c + 1 = a_p$, the three term relation (25) reduces to

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p - 1 \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) + \frac{a_1 \dots a_{p-1}}{b_1 \dots b_q} z \, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_{p-1}+1, a_p \\ b_1+1, \dots, b_q+1 \end{matrix} \,\middle|\, z\right), $$

and for $c = b_q$, it reads as

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, b_q+1 \end{matrix} \,\middle|\, z\right) + \frac{a_1 \dots a_p}{b_1 \dots b_q (b_q+1)} z \, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_p+1 \\ b_1+1, \dots, b_{q-1}+1, b_q+2 \end{matrix} \,\middle|\, z\right). $$

Similarly, due to having two extra parameters, for $m = 1$ the finite identity (18) reads as

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) = {}_{p+2}F_{q+2}\left(\begin{matrix} a_1, \dots, a_p, a+1, b+1 \\ b_1, \dots, b_q, a, b \end{matrix} \,\middle|\, z\right) - \frac{a_1 \dots a_p}{b_1 \dots b_q} \frac{a+b+1}{ab} z \, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+1, \dots, a_p+1, a+b+2 \\ b_1+1, \dots, b_q+1, a+b+1 \end{matrix} \,\middle|\, z\right). \quad (26) $$

For example, if $a = a_{p-1}$, $b = a_p$ and $b_q = a_{p-1} + a_p + 1$, the formula (26) reduces to

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-1}, a_p \\ b_1, \dots, b_{q-1}, a_{p-1}+a_p+1 \end{matrix} \,\middle|\, z\right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-2}, a_{p-1}+1, a_p+1 \\ b_1, \dots, b_{q-1}, a_{p-1}+a_p+1 \end{matrix} \,\middle|\, z\right) - \frac{a_1 \dots a_{p-2}}{b_1 \dots b_{q-1}} z \, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_{p-2}+1, a_{p-1}+1, a_p+1 \\ b_1+1, \dots, b_{q-1}+1, a_{p-1}+a_p+1 \end{matrix} \,\middle|\, z\right). \quad (27) $$

Finally, due to having two extra parameters in the identity (20), the following three term relation is derived for $m = 1$:

$$ {}_{p+2}F_{q+2}\left(\begin{matrix} a_1, \dots, a_p, a+1, b-1 \\ b_1, \dots, b_q, a, b \end{matrix} \,\middle|\, z\right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) + \frac{b-a-1}{ab} \frac{a_1 \dots a_p}{b_1 \dots b_q} z \, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+1, \dots, a_p+1, b \\ b_1+1, \dots, b_q+1, b+1 \end{matrix} \,\middle|\, z\right). \quad (28) $$

There are five particular cases for relation (28) respectively as follows.

For $a = a_p$, (28) reduces to

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) = {}_{p+1}F_{q+1}\left(\begin{matrix} a_1, \dots, a_{p-1}, a_p+1, b-1 \\ b_1, \dots, b_q, b \end{matrix} \,\middle|\, z\right) - \frac{a_1 \dots a_{p-1}}{b_1 \dots b_q} \frac{b-1-a_p}{b} z \, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1+1, \dots, a_p+1, b \\ b_1+1, \dots, b_q+1, b+1 \end{matrix} \,\middle|\, z\right), $$

while for $b = a_p$ we have

$$ {}_{p+1}F_{q+1}\left(\begin{matrix} a_1, \dots, a_p, a+1 \\ b_1, \dots, b_q, a \end{matrix} \,\middle|\, z\right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p+1 \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) + \frac{a_1 \dots a_{p-1}}{b_1 \dots b_q} \frac{a_p - a}{a} z \, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_p+1 \\ b_1+1, \dots, b_q+1 \end{matrix} \,\middle|\, z\right). $$

Also if $b = b_q + 1$ in (28), then

$$ {}_{p+1}F_{q+1}\left(\begin{matrix} a_1, \dots, a_p, a+1 \\ b_1, \dots, b_q, \quad a \end{matrix} \middle| z \right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q - 1 \end{matrix} \middle| z \right) + \frac{a_1 \dots a_p}{b_1 \dots b_q} \frac{b_q - 1 - a}{a(b_q - 1)} z\, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_p+1 \\ b_1+1, \dots, b_q+1 \end{matrix} \middle| z \right). $$

Now if $a = a_{p-1}$ and $b = a_p$ simultaneously, (28) reads as

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-2}, a_{p-1}+1, a_p - 1 \\ b_1, \dots, b_q \end{matrix} \middle| z \right) - \frac{a_1 \dots a_{p-2}(a_p - a_{p-1} - 1)}{b_1 \dots b_q} z\, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_{p-1}+1, a_p \\ b_1+1, \dots, b_q+1 \end{matrix} \middle| z \right). $$

Also for $a = a_p$ and $b = b_q + 1$ in (28), we have

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-1}, a_p+1 \\ b_1, \dots, b_{q-1}, b_q+1 \end{matrix} \middle| z \right) - \frac{a_1 \dots a_{p-1}}{b_1 \dots b_q} \frac{b_q - a_p}{b_q + 1} z\, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_p+1 \\ b_1+1, \dots, b_{q-1}+1, b_q+2 \end{matrix} \middle| z \right). $$

Finally if $a = a_{p-1}$, $b = a_p$ and $b_q = a_p - 1$, (28) reduces to

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-1}, b_q+1 \\ b_1, \dots, b_{q-1}, b_q \end{matrix} \middle| z \right) = {}_{p-1}F_{q-1}\left(\begin{matrix} a_1, \dots, a_{p-2}, a_{p-1}+1 \\ b_1, \dots, b_{q-1} \end{matrix} \middle| z \right) - \frac{a_1 \dots a_{p-2}(b_q - a_{p-1})}{b_1 \dots b_q} z\, {}_{p-1}F_{q-1}\left(\begin{matrix} a_1+1, \dots, a_{p-1}+1 \\ b_1+1, \dots, b_{q-1}+1 \end{matrix} \middle| z \right). $$

Many new hypergeometric identities can be derived from the above-mentioned three term relations. For example, replacing $(p, q) = (2, 1)$ in (25) eventually gives

$$ {}_3F_2\left(\begin{matrix} a_1+1, a_2+1, c+1 \\ b_1+1, c+2 \end{matrix} \middle| z \right) = \frac{b_1(c+1)}{a_1 a_2 z} \left({}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| z \right) - {}_3F_2\left(\begin{matrix} a_1, a_2, c \\ b_1, c+1 \end{matrix} \middle| z \right) \right). \tag{29} $$

Now by referring to the Watson formula [1, 15]

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ (a+b+1)/2, 2c \end{matrix} \middle| 1 \right) = \frac{\sqrt{\pi}\,\Gamma(1+c/2)\Gamma((a+b+1)/2)\Gamma(c-(a+b-1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)\Gamma(c-(a-1)/2)\Gamma(c-(b-1)/2)}, \tag{30} $$

assume in (29) that $c = 1$ and $b_1 = (a_1 + a_2 + 1)/2$. Then, by noting (30) we can finally obtain

$$ {}_3F_2\left(\begin{matrix} a, b, 2 \\ (a+b+1)/2, 3 \end{matrix} \middle| 1 \right) = \frac{a+b-1}{(a-1)(b-1)} \Gamma\left(\frac{a+b-1}{2}\right) \times \left(\frac{\Gamma((3-a-b)/2)}{\Gamma((1-a+b)/2)\Gamma((1+a-b)/2)} - \frac{\pi}{2} \frac{\Gamma((4-a-b)/2)}{\Gamma(a/2)\Gamma(b/2)\Gamma(2-a/2)\Gamma(2-b/2)} \right). $$

Another example is when $(p, q) = (3, 2)$ in (27), i.e.

$$ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, a_2+a_3+1 \end{matrix} \middle| z \right) = {}_3F_2\left(\begin{matrix} a_1, a_2+1, a_3+1 \\ b_1, a_2+a_3+1 \end{matrix} \middle| z \right) - \frac{a_1}{b_1} z\, {}_3F_2\left(\begin{matrix} a_1+1, a_2+1, a_3+1 \\ b_1+1, a_2+a_3+1 \end{matrix} \middle| z \right). \tag{31} $$

By noting the Gauss formula (5), a remarkable case in (31) is when $z = 1$ and $a_1 = a_2 + a_3 + 1$, i.e.

$$ {}_3F_2\left(\begin{matrix} a_2+a_3+2, a_2+1, a_3+1 \\ a_2+a_3+1, \quad b_1+1 \end{matrix} \middle| 1 \right) = \frac{b_1}{a_2+a_3+1} \left({}_2F_1\left(\begin{matrix} a_2+1, a_3+1 \\ b_1 \end{matrix} \middle| 1 \right) - {}_2F_1\left(\begin{matrix} a_2, a_3 \\ b_1 \end{matrix} \middle| 1 \right) \right). $$

3. A Hypergeometric Identity with n Extra Parameters

Now, the approach used in the previous section can be extended in a more general case. Let $\{m_k\}_{k=1}^{n}$ denote a set of positive integers. If $b = k$, the Karlsson-Minton identity [2, 12, 14]:

$$ {}_{n+2}F_{n+1}\left(\begin{matrix} a,\ b,\ c_1+m_1,\ \dots,\ c_n+m_n \\ b+1,\ c_1,\ \dots,\ c_n \end{matrix} \,\middle|\, 1 \right) = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(c_1-b)_{m_1}\dots(c_n-b)_{m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}}, $$

is transformed to

$$ {}_{n+2}F_{n+1}\left(\begin{matrix} a,\ k,\ 1-c_1+m_1,\ \dots,\ 1-c_n+m_n \\ k+1,\ 1-c_1,\ \dots,\ 1-c_n \end{matrix} \,\middle|\, 1 \right) = \frac{k!}{(1-a)_k} \frac{(c_1)_k\dots(c_n)_k}{(c_1-m_1)_k\dots(c_n-m_n)_k}, \tag{32} $$

where $\mathrm{Re}(-a) > m_1 + \ldots + m_n - 1$ because in general we have

$$ \frac{(c-k)_m}{(c)_m} = \frac{(1-c)_k}{(1-c-m)_k}. $$

We can now apply our approach for the identity (32) to get

$$ \begin{aligned} {}_pF_q\left(\begin{matrix} a_1,\ \dots\ ,a_p \\ b_1,\ \dots\ ,b_q \end{matrix} \,\middle|\, z \right) &= \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{k!} \frac{(1-a)_k(c_1-m_1)_k\dots(c_n-m_n)_k}{(1)_k(c_1)_k\dots(c_n)_k} \\ &\qquad \times {}_{n+2}F_{n+1}\left(\begin{matrix} a,\ k,\ 1-c_1+m_1,\ \dots,\ 1-c_n+m_n \\ k+1,\ 1-c_1,\ \dots,\ 1-c_n \end{matrix} \,\middle|\, 1 \right) \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{k!} \frac{(1-a)_k(c_1-m_1)_k\dots(c_n-m_n)_k}{(1)_k(c_1)_k\dots(c_n)_k} \left(1 + \sum_{j=1}^{\infty} \frac{(a)_j(1-c_1+m_1)_j\dots(1-c_n+m_n)_j}{(1-c_1)_j\dots(1-c_n)_j\, j!} \frac{(k)_j}{(k+1)_j} \right). \end{aligned} \tag{33} $$

Since

$$ \frac{(k)_j}{(k+1)_j} = \frac{k}{j} \frac{(j)_k}{(j+1)_k}, $$

relation (33) changes to

$$ \begin{aligned} {}_pF_q\left(\begin{matrix} a_1,\ \dots\ ,a_p \\ b_1,\ \dots\ ,b_q \end{matrix} \,\middle|\, z \right) &= {}_{p+n+1}F_{q+n+1}\left(\begin{matrix} a_1,\ \dots,a_p,1-a,c_1-m_1,\dots,c_n-m_n \\ b_1,\ \dots\ ,b_q,1,c_1,\ \dots\ ,c_n \end{matrix} \,\middle|\, z \right) \\ &\quad + \sum_{j=1}^{\infty} \frac{(a)_j(1-c_1+m_1)_j\dots(1-c_n+m_n)_j}{(1-c_1)_j\dots(1-c_n)_j\,(j)j!} \sum_{k=1}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{k!} \frac{(1-a)_k(c_1-m_1)_k\dots(c_n-m_n)_k}{(1)_k(c_1)_k\dots(c_n)_k} \frac{k(j)_k}{(j+1)_k}. \end{aligned} \tag{34} $$

On the other hand, since

$$ \begin{aligned} \sum_{k=1}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{(k-1)!} \frac{(1-a)_k(c_1-m_1)_k\dots(c_n-m_n)_k}{(1)_k(c_1)_k\dots(c_n)_k} \frac{(j)_k}{(j+1)_k} &= \frac{a_1\dots a_p}{b_1\dots b_q} \frac{(1-a)(c_1-m_1)\dots(c_n-m_n)}{c_1\dots c_n} \frac{j}{j+1} z \\ &\quad \times \sum_{k=0}^{\infty} \frac{(a_1+1)_k\dots(a_p+1)_k}{(b_1+1)_k\dots(b_q+1)_k} \frac{z^k}{k!} \frac{(2-a)_k(c_1+1-m_1)_k\dots(c_n+1-m_n)_k}{(2)_k(c_1+1)_k\dots(c_n+1)_k} \frac{(j+1)_k}{(j+2)_k}, \end{aligned} $$

relation (34) becomes

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = {}_{p+n+1}F_{q+n+1}\left(\begin{matrix} a_1,\dots,a_p, 1-a, c_1-m_1,\dots,c_n-m_n \\ b_1,\ \dots\ ,b_q, 1, c_1,\ \dots\ ,c_n \end{matrix}\,\middle|\, z\right)
$$
$$
+\frac{a_1\dots a_p}{b_1\dots b_q}\frac{(1-a)(c_1-m_1)\dots(c_n-m_n)}{c_1\dots c_n}z\sum_{j=1}^{\infty}\frac{(a)_j(1-c_1+m_1)_j\dots(1-c_n+m_n)_j}{(1-c_1)_j\dots(1-c_n)_j\,(2)_j}
$$
$$
\times\, {}_{p+n+2}F_{q+n+2}\left(\begin{matrix} a_1+1,\dots,a_p+1, 2-a, c_1+1-m_1,\dots,c_n+1-m_n, j+1 \\ b_1+1,\ \dots\ ,b_q+1, 2, c_1+1,\ \dots\ ,c_n+1,\ j+2 \end{matrix}\,\middle|\, z\right),
$$

which is finally equivalent to

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = {}_{p+n+1}F_{q+n+1}\left(\begin{matrix} a_1,\dots,a_p, 1-a, c_1-m_1,\dots,c_n-m_n \\ b_1,\ \dots\ ,b_q, 1, c_1,\ \dots\ ,c_n \end{matrix}\,\middle|\, z\right)
$$
$$
+\frac{a_1\dots a_p}{b_1\dots b_q}\frac{(c_1-m_1)(1-(c_1-m_1))\dots(c_n-m_n)(1-(c_n-m_n))}{c_1(1-c_1)\dots c_n(1-c_n)}\frac{a\,(1-a)}{2}\,z
$$
$$
\times\sum_{j=0}^{\infty}\frac{(a+1)_j(2-c_1+m_1)_j\dots(2-c_n+m_n)_j}{(2-c_1)_j\dots(2-c_n)_j\,(3)_j}
$$
$$
\times\, {}_{p+n+2}F_{q+n+2}\left(\begin{matrix} a_1+1,\dots,a_p+1, 2-a, c_1+1-m_1,\dots,c_n+1-m_n, j+2 \\ b_1+1,\dots,b_q+1, 2, c_1+1,\dots,c_n+1,\ j+3 \end{matrix}\,\middle|\, z\right), \quad (35)
$$

where $\mathrm{Re}(-a) > m_1+\cdots+m_n-1$ and $\{c_i-m_i\}_{i=1}^{n} \neq 0, 1$.

The finite case of (35) happens when $a = -(m_1+\dots+m_n) = -M_n$, i.e.

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & ,a_p \\ b_1, & \dots & ,b_q \end{matrix}\,\middle|\, z\right) = {}_{p+n+1}F_{q+n+1}\left(\begin{matrix} a_1,\dots,a_p, 1+m_1+\dots+m_n, c_1-m_1,\dots,c_n-m_n \\ b_1,\ \dots\ ,b_q, 1, c_1,\ \dots\ ,c_n \end{matrix}\,\middle|\, z\right)
$$
$$
-\frac{a_1\dots a_p}{b_1\dots b_q}\frac{(c_1-m_1)(1-(c_1-m_1))\dots(c_n-m_n)(1-(c_n-m_n))}{c_1(1-c_1)\dots c_n(1-c_n)}\frac{M_n\,(M_n+1)}{2}\,z
$$
$$
\times\sum_{j=0}^{M_n-1}\frac{(-(M_n-1))_j(2-c_1+m_1)_j\dots(2-c_n+m_n)_j}{(2-c_1)_j\dots(2-c_n)_j\,(3)_j}
$$
$$
\times\, {}_{p+n+2}F_{q+n+2}\left(\begin{matrix} a_1+1,\dots,a_p+1, 2+m_1+\dots+m_n, c_1+1-m_1,\dots,c_n+1-m_n, j+2 \\ b_1+1,\ \dots\ ,b_q+1, 2, c_1+1,\ \dots\ ,c_n+1,\ j+3 \end{matrix}\,\middle|\, z\right).
$$

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