



Generalized Derivation, SVEP, Finite Ascent, Range Closure

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Abstract. Let X be an infinite complex Banach space and consider two bounded linear operators $A, B \in L(X)$. Let $L_A \in L(L(X))$ and $R_B \in L(L(X))$ be the left and the right multiplication operators, respectively. The generalized derivation $\delta_{A,B} \in L(L(X))$ is defined by $\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB$. In this paper we give some sufficient conditions for $\delta_{A,B}$ to satisfy SVEP, and we prove that $\delta_{A,B} - \lambda I$ has finite ascent for all complex λ , for general choices of the operators A and B , without using the range kernel orthogonality. This information is applied to prove some necessary and sufficient conditions for the range of $\delta_{A,B} - \lambda I$ to be closed. In [18, Proposition 2.9] Duggal et al. proved that, if $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , and if either (i) A^* and B have SVEP or (ii) $\delta_{A,B}^*$ has SVEP, then $\delta_{A,B} - \lambda$ has closed range for all complex λ if and only if A and B are algebraic operators, we prove using the spectral theory that, if $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , then $\delta_{A,B} - \lambda$ has closed range, for all complex λ if and only if A and B are algebraic operators, without the additional conditions (i) or (ii).

1. Introduction and basic definitions

The single valued extension property (SVEP) dates back to the early days of local spectral theory, appeared first in the work of Dunford [20], [21]. As a witness by the more recent accounts in [1] and [30], SVEP has now developed into one of the major tools in the local spectral theory and Fredholm theory for operators on Banach spaces. In what follows, let X (resp., \mathcal{H}) shall denote an infinite dimensional complex Banach space (resp., Hilbert space) and $L(X)$ will denote the algebra of all bounded linear maps defined on and with values in X . Given $T \in L(X)$, $\ker(T)$, $\mathcal{R}(T)$ and $\sigma_p(T)$ will stand for the null space, the range and the point spectrum of T respectively. Thanks to the work of Finch [25] we have actually the following equivalent localized version of SVEP which is much easier to work with.

Definition 1.1. An operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc \mathbb{D} centered at λ_0 , the only analytic function $f : \mathbb{D} \rightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

Evidently, every operator T , as well as its dual T^* , has SVEP at every point in $\partial\sigma(T)$, where $\partial\sigma(T)$ is the boundary of the spectrum $\sigma(T)$, in particular at every isolated point of $\sigma(T)$. Recall that $T \in L(X)$ is said to be bounded below, if $\ker(T) = \{0\}$ and $\mathcal{R}(T)$ is closed. Denote the approximate point spectrum of T by

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

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Let

$$\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\},$$

denote the surjectivity spectrum of T . In addition, \mathcal{X}^* will denote the dual space of \mathcal{X} , and if $T \in L(\mathcal{X})$, then $T^* \in L(\mathcal{X}^*)$ will stand for the adjoint map of T . Clearly, $\sigma_a(T^*) = \sigma_s(T)$ and $\sigma_a(T) \cup \sigma_s(T) = \sigma(T)$, the spectrum of T . Recall that the ascent $asc(T)$ of an operator T , is defined by $asc(T) = \inf\{n \in \mathbb{N} : \ker(T^n) = \ker(T^{n+1})\}$ and the descent $dsc(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. It is well known that if $asc(T)$ and $dsc(T)$ are both finite, then they are equal.

Definition 1.2. $T \in L(\mathcal{X})$ is said to be

1. Left Drazin invertible if and only if $asc(T) < \infty$ and $\mathcal{R}(T)^{asc(T)+1}$ is closed.
2. Drazin invertible if and only if it has finite ascent and descent.

The left Drazin and the Drazin spectrum are defined respectively by

$$\sigma_{ld}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}.$$

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

We denote by $\Pi^a(T) = \{\lambda \in \sigma_a(T) : T - \lambda I \text{ is left Drazin invertible}\}$ the set of left poles of T , and by $\Pi(T) = \{\lambda \in \mathbb{C} : asc(T - \lambda I) = dsc(T - \lambda I) < \infty\}$ the set of poles of the resolvent. In the sequel we shall denote by $accS$ and $isoS$, the set of accumulation points and the set of isolated points of $S \subset \mathbb{C}$, respectively.

Definition 1.3. An operator $T \in L(\mathcal{X})$ is said to be

1. Polaroid if $iso\sigma(T) \subseteq \Pi(T)$.
2. Left polaroid if $iso\sigma_a(T) \subseteq \Pi^a(T)$.

It is easily seen that, if $T \in L(\mathcal{X})$ is polaroid, then $\Pi(T) = E(T)$, where $E(T)$ is the set of eigenvalues of T which are isolated in the spectrum of T .

It is well known that SVEP is not stable under sums and products of commuting operators. Thus in general SVEP for A and B does not guarantee SVEP for the operators $L_A + R_B$ and $L_A R_B$. A natural question arises under which conditions on A and B , the single-valued extension property holds for the generalized derivation operator $\delta_{A,B}$?

In [30] Laursen and Neumann proved that if $A, B \in L(\mathcal{X})$ and A, B^* satisfy property (β) , then $\delta_{A,B}$ has SVEP (we refer the reader to [30], for the definitions of property (β)). In section 2, we prove that if $A \in L(\mathcal{X})$ has the SVEP (resp., A is quasinilpotent) and B is algebraic (resp., B^* has SVEP), then SVEP holds for $\delta_{A,B}$, and we prove also that if $A, B \in L(\mathcal{X})$, such that the operators $\exp(A)$ and $\exp(-B)$ have spectrum without interior points, then SVEP holds for $\delta_{A,B}$.

The following implications hold for a general bounded linear operator T on a Banach space \mathcal{X} , in particular

$$\ker(T) \perp \mathcal{R}(T) \Rightarrow \ker(T) \cap \mathcal{R}(T) = \{0\} \Rightarrow asc(T) \leq 1,$$

where $\ker(T) \perp \mathcal{R}(T)$ denote that the kernel of T is orthogonal to the range of T in the sense of G. Birkhoff. So the range kernel orthogonality of an operator is related to its ascent. The range-kernel orthogonality of $\delta_{A,B}$ in the sense of G. Birkhoff, was studied by numerous mathematicians, see [10, 22] and the references therein. Anderson [8], Anderson and Foias [9] considered the generalized derivation $\delta_{A,B}$ to prove that if A and B are normal Hilbert space operators, then $\ker(\delta_{A,B}) \perp \mathcal{R}(\delta_{A,B})$, this implies that $asc(\delta_{A,B}) \leq 1$. Duggal, Djordjevic and Kubrusly [17, Proposition 2.3] proved that if $A, B \in L(\mathcal{X})$ such that A is a contraction and B is right invertible by a contraction, then $\ker(\delta_{A,B}) \perp \mathcal{R}(\delta_{A,B})$, this implies that $asc(\delta_{A,B}) \leq 1$, and in [17, Corollary 2.7] the authors proved that if $A, B \in L(\mathcal{H})$, such that A and B^* are w -hyponormal with $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then $asc(\delta_{A,B}) \leq 1$. In section 3, we prove that if $A, B^* \in L(\mathcal{H})$ are reduced by each of its eigenspaces and have property $H(1)$, then

$$asc(\delta_{A,B} - \lambda I) \leq 1, \forall \lambda \in \mathbb{C}.$$

Recall that an operator $T \in L(\mathcal{X})$, is said to have property $H(1)$, if

$$\forall \lambda \in \mathbb{C}, H_0(T - \lambda I) = \ker(T - \lambda I),$$

where $H_0(T - \lambda I)$ is the quasi-nilpotent part of $T - \lambda I$ defined by

$$H_0(T - \lambda I) = \{x \in \mathcal{X} : \lim_{n \rightarrow \infty} \|(T - \lambda I)^n(x)\|^{\frac{1}{n}} = 0\}.$$

Several authors have studied the problem of characterizing when the the range of $\delta_{A,B}$ is norm closed in $L(\mathcal{X})$. Moreover Anderson and Foias [9, Theorem 4.2] proved that if $A, B \in L(\mathcal{H})$ are scalar Hilbert space operators, then $\delta_{A,B} - \lambda$ has closed range for every complex λ if and only if $\sigma(A) \cup \sigma(B)$ is finite. Scalar Hilbert space operators are similar to normal operators, and normal operators are polaroid. Hence [1, Theorem 3.83], if $A, B \in L(\mathcal{H})$ are scalar operators, then $\delta_{A,B} - \lambda$ has closed range, for all complex λ if and only if A and B are algebraic operators. In [18, Propostion 2.9], the authors proved that, if $asc(\delta_{A,B} - \lambda) \leq 1$, for all complex λ and if either (i) A^* and B have SVEP or (ii) $\delta_{A,B}^*$ has SVEP, then $\delta_{A,B} - \lambda$ has closed range for all complex λ if and only if A and B are algebraic operators. In section 4, we prove that, if $asc(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , then $\delta_{A,B} - \lambda$ has closed range, for all complex λ if and only if A and B are algebraic operators, without the additional conditions (i) or (ii).

In [17, Remark 3.3] the authors noted that if $A, B \in L(\mathcal{X})$ such that A, B are polaroid, $asc(\delta_{A,B}) \leq 1$ and $\delta_{A,B}^*$ has SVEP at 0, the following conditions are mutually equivalent

1. $0 \in iso\sigma_a(\delta_{A,B})$,
2. $\delta_{A,B}$ is left polaroid,
3. $\delta_{A,B}$ has closed range,
4. There exist finite sequences $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$, where $\alpha_i \in iso\sigma_a(A)$ and $\beta_i \in iso\sigma_s(B)$ such that $\alpha_i - \beta_i = 0$, for all $1 \leq i \leq n$,
5. $L(\mathcal{X}) = \ker \delta_{A,B} \oplus \mathcal{R}(\delta_{A,B})$,
6. $0 \in iso\sigma(\delta_{A,B})$.

In section 4, we prove that if $A, B^* \in L(\mathcal{H})$ are reduced by each of its eigenspaces and have property $H(1)$, then the conditions (1) to (6) are mutually equivalent.

2. SVEP property for $\delta_{A,B}$

In this section we give some sufficient conditions for $\delta_{A,B}$ to satisfy property SVEP. Before giving our results, we need the following definitions.

Definition 2.1. An operator $T \in L(\mathcal{X})$ is said to be a Fredholm if $\alpha(T) = \dim \ker(T)$ and $\beta(T) = \text{codim} \mathcal{R}(T)$ are finite dimensional. Let $\sigma_e(T)$ denote the essential spectrum of T .

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not Fredholm}\},$$

Definition 2.2. An operator $T \in L(\mathcal{X})$ is said to be algebraic if there exists a non trivial complex polynomial h such that $h(T) = 0$.

Definition 2.3. An operator $T \in L(\mathcal{X})$ is said to be Riesz if for all $\lambda \in \mathbb{C}, \lambda \neq 0, T - \lambda I$ is a Fredholm operator, equivalently T is Riesz operator if $\sigma_e(T) = \{0\}$.

Theorem 2.4. [16, Lemma 2.8] For an operator $A \in L(\mathcal{X})$, A (resp., A^*) has SVEP at μ if and only if L_A (resp., R_A) has SVEP at μ .

Remark 2.5. SVEP for A and B does not guarantee SVEP for the operators $A + B$ and AB , even when A and B commute. Thus in general SVEP for A and B does not guarantee SVEP for the operators $L_A + R_B$ and $L_A R_B$.

As a consequence from the above theorem, we give some sufficient conditions for $\delta_{A,B}$ to satisfy SVEP property

Proposition 2.6. Suppose that $A \in L(X)$ has the SVEP (resp., A is algebraic) and B is algebraic (resp., B^* has SVEP), then SVEP holds for $\delta_{A,B}$.

Proof. Since $\sigma(R_B) = \sigma(B)$ and $\sigma(L_A) = \sigma(A)$, we deduce that R_B is also algebraic. It follows from Theorem 2.4 and [4, Theorem 2.3] that SVEP holds for $\delta_{A,B}$. \square

Proposition 2.7. Let $A, B \in L(X)$, if A is quasinilpotent (resp., A has SVEP) and B^* has SVEP (resp., B is quasinilpotent), then $\delta_{A,B}$ has SVEP.

Proof. Suppose that A is quasinilpotent, according to [24, Corollary 3.4], we have $\sigma_e(L_A) = \sigma(A) = \{0\}$, it follows that L_A is Riesz operator, since B^* has SVEP it follows from [16, Lemma 2.8] that R_B has SVEP, apply [2, Theorem 0.3], we get $\delta_{A,B}$ has SVEP. Similarly if we suppose that A has SVEP and B is quasinilpotent we get the result. \square

Example 2.8. Let $1 \leq p < \infty$ arbitrarily given and let A be a weighted right shift operator on the Banach space $l^p(\mathbb{N})$ with weight sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}}$,

$$Ax = \sum_{n=1}^{\infty} \alpha_n x_n e_{n+1},$$

where (e_n) is the canonical basis of $l^p(\mathbb{N})$, suppose that $(\alpha_k \dots \alpha_{k+n-1})^{\frac{1}{n}} \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $k \in \mathbb{N}$, it follows from [30, Proposition 1.6.14] that A is quasinilpotent. Let B be a weighted right shift operator on the Banach space $l^p(\mathbb{N})$ with weight sequence $\beta = (\beta_n)_{n \in \mathbb{N}}$, such that $\liminf_{n \rightarrow \infty} (\beta_1 \dots \beta_n)^{\frac{1}{n}} = 0$, it follows from [1, Theorem 2.88] that B^* has SVEP. Applying Proposition 2.7, we get SVEP holds for $\delta_{A,B}$.

Theorem 2.9. Let $A, B \in L(X)$, suppose that $\exp(A)$ and $\exp(-B)$ have spectrum without interior points, then SVEP holds for $\delta_{A,B}$.

Proof. Since $\sigma(R_{\exp(-B)}) = \sigma(\exp(-B))$ and $\sigma(L_{\exp(A)}) = \sigma(\exp(A))$, we deduce that the operators $L_{\exp(A)}$ and $R_{\exp(-B)}$ have spectrum without interior points and they are two commuting operators. So by [22, Theorem 1.3] $L_{\exp(A)} \cdot R_{\exp(-B)}$ has the SVEP. On the other hand we have

$$\begin{aligned} \exp(L_A - R_B) &= \exp(L_A) \exp(R_{(-B)}) \\ &= L_{\exp(A)} R_{\exp(-B)}, \end{aligned}$$

consequently SVEP holds for $\exp(L_A - R_B)$. By [1, Theorem 2.39] this implies that $\delta_{A,B}$ has SVEP. \square

The following theorem was proved by B. P. Duggal, S. V. Djordjevic and C. S. Kubrusly in [18, Propostion 2.6], by using the spectral theory, we give another proof

Theorem 2.10. Let $A, B \in L(X)$, then A and B are algebraic if and only if $\delta_{A,B}$ is algebraic.

Proof. We prove that $\sigma(\delta_{A,B})$ is a finite set of poles of the resolvent. Since A and B are algebraic, it follows that $\sigma(\delta_{A,B})$ is finitely countable and since algebraic operators are polaroid, it follows from [14, Theorem 3.6] that $\delta_{A,B}$ is polaroid, this implies that $\sigma(\delta_{A,B}) = iso\sigma(\delta_{A,B}) = \Pi(\delta_{A,B})$, consequently $\delta_{A,B}$ is algebraic. Suppose now that $\delta_{A,B}$ is algebraic, then $acc\sigma(\delta_{A,B}) = \emptyset$ and $\sigma(\delta_{A,B}) = iso\sigma(\delta_{A,B}) = \Pi(\delta_{A,B})$, it follows from [14, Remark 3.1] and the proof of [14, Theorem 3.6] that

$$\begin{aligned} iso\sigma(\delta_{A,B}) &= (iso\sigma(A) - iso\sigma(B)) \setminus acc\sigma(\delta_{A,B}) \\ &= (\Pi(A) - \Pi(B)) \setminus acc\sigma(\delta_{A,B}). \end{aligned}$$

Hence $\sigma(A) = iso\sigma(A) = \Pi(A)$ and $\sigma(B) = iso\sigma(B) = \Pi(B)$, consequently A and B are algebraic. \square

Corollary 2.11. *Suppose that A and B are algebraic, then $\delta_{A,B}$ and $\delta_{A,B}^*$ have SVEP.*

Remark 2.12. *In the following example we show that if A is Riesz operator, then L_A is not in general Riesz operators.*

Example 2.13. *Let A be compact non quasinilpotent operator, then A is Riesz operator. According to [24, Corollary 3.4], we have $\sigma_e(L_A) = \sigma(A) \neq \{0\}$, then L_A is not Riesz operator.*

In the following theorem we give necessary and sufficient condition for $\delta_{A,B}$ to be Riesz operator.

Proposition 2.14. *Let $A, B \in L(X)$. Then $\delta_{A,B}$ is Riesz operator, if and only if $\sigma_e(A - \lambda) \cap \sigma(B) = \emptyset$ and $\sigma(A - \lambda) \cap \sigma_e(B) = \emptyset$, for all $\lambda \in \mathbb{C}, \lambda \neq 0$.*

Proof. Suppose that $\delta_{A,B}$ is Riesz operator, then for all $\lambda \in \mathbb{C}, \lambda \neq 0$, we have $\delta_{A,B} - \lambda$ is Fredholm, this is equivalent to $\lambda \notin \sigma_e(\delta_{A,B})$, since the spectral mapping theorem holds for the essential spectrum see [1, Corollary 3.61], it follows that $0 \notin \sigma_e(\delta_{A-\lambda,B})$ and according to [24, Corollary 3.4], we have

$$\sigma_e(\delta_{A-\lambda,B}) = (\sigma(A - \lambda) - \sigma_e(B)) \cup (\sigma_e(A - \lambda) - \sigma(B)),$$

hence we obtain $0 \notin (\sigma(A - \lambda) - \sigma_e(B))$ and $0 \notin (\sigma_e(A - \lambda) - \sigma(B))$, this implies that $\sigma_e(A - \lambda) \cap \sigma(B) = \emptyset$ and $\sigma(A - \lambda) \cap \sigma_e(B) = \emptyset$, for all $\lambda \in \mathbb{C}, \lambda \neq 0$. \square

Before giving an example which illustrate the precedent proposition, we recall that if $T \in L(X)$, the analytic core $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$, see [1] for information on $K(T)$.

Example 2.15. *Let A be quasinilpotent operator such that $K(A) = \{0\}$, it follows from [1, Theorem 6.42] that A is Riesz operator, consequently $\lambda \notin \sigma_e(A)$, for all $\lambda \in \mathbb{C}, \lambda \neq 0$. Let $B : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be defined by $B(x_1, x_2, x_3, \dots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$, $\sigma(B) = \sigma_e(B) = \{0\}$, then we have $\sigma_e(A - \lambda) \cap \sigma(B) = \emptyset$ and $\sigma(A - \lambda) \cap \sigma_e(B) = \emptyset$, for all $\lambda \in \mathbb{C}, \lambda \neq 0$. Hence $\delta_{A,B}$ is Riesz operator and $\sigma(\delta_{A,B}) = \sigma_e(\delta_{A,B}) = \{0\}$.*

Remark 2.16. *The class of operators satisfying the condition $K(T) = \{0\}$ was introduced by Mbekhta in [33] in the case of Hilbert space and studied in more general setting of Banach spaces, see [1]. Such condition is verified by every weighted unilateral right shift T on $l^p(\mathbb{N})$ ($1 \leq p < \infty$) defined by*

$$Te_n = \omega_n e_{n+1},$$

where the weight $(\omega_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive numbers, and $(e_n)_{n \in \mathbb{N}}$ stands for the canonical basis of $l^p(\mathbb{N})$.

Proposition 2.17. *Let $A, B \in L(X)$. Suppose that L_A and R_B are Riesz operators, then $\delta_{A,B}$ is Riesz operator.*

Proof. Since L_A and R_B are commuting operators, according to [1, Theorem 3.112], we get the result. \square

Proposition 2.18. *Let $A \in L(X)$ and $B \in L(X)$ be Riesz operators, then $\delta_{A,B}$ has SVEP.*

Proof. If A and B are Riesz operators, then $\sigma(A)$ and $\sigma(B)$ are finite or countable. Therefore by [30, Proposition 1.4.5], the operators A and B are super-decomposable, and by [30, Theorem 3.6.17], L_A and R_B are super-decomposable and $\delta_{A,B}$ is decomposable. Hence $\delta_{A,B}$ has SVEP. \square

Now, we give some results on the stability of the single-valued extension property under perturbations by commuting algebraic, Riesz and quasi-nilpotent operators.

Observe that for operators A, B, C and $D \in L(X)$, we have

$$\delta_{A+C,B+D} = \delta_{A,B} + \delta_{C,D}.$$

Hence perturbation of the operators A and B in $\delta_{A,B}$ by operators C and D , such that C commute with A and D commute with B results in a perturbation of $\delta_{A,B}$ by an operator $\delta_{C,D}$ which commutes with it.

Proposition 2.19. Let $C, D \in L(X)$ be algebraic operators and let $A, B \in L(X)$, such that C commute with A , D commute with B and A, B^* have property (β) , then

$$\text{SVEP holds for } \delta_{A+C, B+D}.$$

Proof. According to Theorem 2.10, $\delta_{C,D}$ is algebraic and from [30, Theorem 3.6.3] $\delta_{A,B}$ has SVEP, applying [4, Theorem 2.3], we get $\delta_{A+C, B+D} = \delta_{A,B} + \delta_{C,D}$ has SVEP. \square

Proposition 2.20. Let $C, D \in L(X)$ be quasinilpotent operators and let $A, B \in L(X)$, such that A, B^* have property (β) , then

$$\text{SVEP holds for } \delta_{A+C, B+D}.$$

Proposition 2.21. Let $C, D \in L(X)$ such that $\sigma_e(C - \lambda) \cap \sigma(D) = \emptyset$ and $\sigma(C - \lambda) \cap \sigma_e(D) = \emptyset$, for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and let $A, B \in L(X)$, such that C commute with A , D commute with B , then

$$\delta_{A,B} \text{ has SVEP if and only if } \delta_{A+C, B+D} \text{ has SVEP.}$$

Proof. According to Proposition 2.14 $\delta_{C,D}$ is Riesz operator. Since the operators $\delta_{C,D}$ and $\delta_{A,B}$ commute, it follows from [4] that $\delta_{A,B}$ has SVEP if and only if $\delta_{A+C, B+D}$ has SVEP. \square

3. Finite ascent for $\delta_{A,B}$

Let $T \in L(\mathcal{H})$ be reduced by each of its eigenspaces. If we let $M = \bigvee \{\ker(T - \mu I), \mu \in \sigma_p(T)\}$ (where $\bigvee(\cdot)$ denotes the closed linear span), it follows that M reduces T . Let $T_1 = T|_M$ and $T_2 = T|_{M^\perp}$. By [13, Proposition 4.1] we have

- T_1 is normal with pure point spectrum,
- $\sigma_p(T_1) = \sigma_p(T)$,
- $\sigma(T_1) = cl\sigma_p(T_1)$ (here cl denotes the closure),
- $\sigma_p(T_2) = \emptyset$.

Before giving our main result, we recall the definition of normal operators on Banach spaces

Definition 3.1. An operator $A \in L(X)$ is said to be hermitian if $\|\exp(itA)\| = 1$, for all real t . An operator T on X is said to be normal if $T = A + iB$, where A and B are commuting hermitian operators.

Theorem 3.2. Suppose that $A, B^* \in L(\mathcal{H})$ are reduced by each of its eigenspaces and have property $H(1)$, then

$$asc(\delta_{A,B} - \lambda I) \leq 1, \quad \forall \lambda \in \mathbb{C}.$$

Proof. Since A and B^* are reduced by each of its eigenspaces, then there exists

$$M_1 = \bigvee \{\ker(A - \beta I), \beta \in \sigma_p(A)\} \text{ and } M_2 = H \ominus M_1$$

on the one hand and

$$N_1 = \bigvee \{\ker(B^* - \bar{\alpha} I), \bar{\alpha} \in \sigma_p(B^*)\} \text{ and } N_2 = H \ominus N_1$$

on the other hand such that A and B have the representations

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } \mathcal{H} = M_1 \oplus M_2,$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ on } H = N_1 \oplus N_2.$$

Recall from [23] that $\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)$. We consider the following cases:

Case 1: If $\lambda \in \mathbb{C} \setminus \sigma(\delta_{A,B})$, then $\ker(\delta_{A,B} - \lambda I) = \{0\}$ and hence

$$\text{asc}(\delta_{A,B} - \lambda I) \leq 1.$$

Case 2: If $\lambda \in \text{iso}\sigma(\delta_{A,B})$, then there exists finite sequences $\{\mu_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$, where $\mu_i \in \text{iso}\sigma(A)$ and $v_i \in \text{iso}\sigma(B)$ such that

$$\lambda = \mu_i - v_i, \text{ for all } 1 \leq i \leq n.$$

Since the spectrum of A_2 and the spectrum of B_2 does not contains isolated points, then $\lambda \notin \sigma(\delta_{A_i,B_j})$ for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Consider $X \in \ker(\delta_{A,B} - \lambda I)$ such that $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$ have the representation $X = [X_{kl}]_{k,l=1}^2$. Hence

$$(\delta_{A,B} - \lambda I)(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)(X_{11}) & (\delta_{A_1,B_2} - \lambda I)(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)(X_{21}) & (\delta_{A_2,B_2} - \lambda I)(X_{22}) \end{pmatrix} = 0.$$

Observe that $\delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Hence $X_{22} = X_{21} = X_{12} = 0$. Since $A_1 - \mu_i$ and $B_1 - v_i$ are normal, it follows from [9, Theorem 5.4] that $(\delta_{A_1,B_1} - \lambda I) \in L(L(N_1, M_1))$ is normal Banach space operator, and from [19, Theorem 3.4] that $\text{asc}(\delta_{A_1,B_1} - \lambda I) \leq 1$. Hence $\text{asc}(\delta_{A,B} - \lambda I) \leq 1$.

Case 3: If $\lambda \in \text{acc}\sigma(\delta_{A,B})$, it follows from [31, Lemma 3.1] that $\lambda \in (\sigma(A) - \text{acc}\sigma(B)) \cup (\text{acc}\sigma(A) - \sigma(B))$, then there exists $\mu \in \sigma(A)$ and $v \in \sigma(B)$ such that $\lambda = \mu - v \in (\sigma(A) - \text{acc}\sigma(B))$ or $\lambda = \mu - v \in (\text{acc}\sigma(A) - \sigma(B))$. Since A and B^* have property $H(1)$, then they are polaroid, hence $\text{acc}\sigma(A) = \sigma_D(A)$ and $\text{acc}\sigma(B) = \sigma_D(B)$, it is easy to see that

$$\sigma_D(A) = \sigma_D(A_1) \cup \sigma_D(A_2) \text{ and } \sigma_D(B) = \sigma_D(B_1) \cup \sigma_D(B_2),$$

since $\sigma_p(A_2) = \sigma_p(B_2^*) = \emptyset$, then $\sigma_D(A_2) = \sigma(A_2)$ and $\sigma_D(B_2) = \sigma(B_2)$. Hence we have

$$\mu \in \sigma_D(A_1) \cup \sigma(A_2) \text{ and } v \in \sigma(B_1) \cup \sigma(B_2).$$

Or

$$\mu \in \sigma(A_1) \cup \sigma(A_2) \text{ and } v \in \sigma_D(B_1) \cup \sigma(B_2).$$

Let $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$ have the representation $X = [X_{kl}]_{k,l=1}^2$. Hence

$$(\delta_{A,B} - \lambda I)(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)(X_{11}) & (\delta_{A_1,B_2} - \lambda I)(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)(X_{21}) & (\delta_{A_2,B_2} - \lambda I)(X_{22}) \end{pmatrix}.$$

We consider the following cases

- $\mu \in \sigma(A_1)$ and $v \in \sigma_D(B_1)$, or
- $\mu \in \sigma(A_1)$ and $v \in \sigma(B_2)$, or
- $\mu \in \sigma(A_2)$ and $v \in \sigma_D(B_1)$, or
- $\mu \in \sigma(A_2)$ and $v \in \sigma(B_2)$.

or

- $\mu \in \sigma_D(A_1)$ and $v \in \sigma(B_1)$, or
- $\mu \in \sigma_D(A_1)$ and $v \in \sigma(B_2)$, or
- $\mu \in \sigma(A_2)$ and $v \in \sigma(B_1)$.

We start by studying these cases

- If $\mu \in \sigma(A_1)$ and $\nu \in \sigma_D(B_1)$. Since $\mu \notin \sigma(A_2)$ and $\nu \notin \sigma(B_2)$, then $\delta_{A_i, B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Hence $asc(\delta_{A_i, B_j} - \lambda I) = 0$, for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Since $A_1 - \mu$ and $B_1 - \nu$ are normal, it follows from [9, Theorem 5.4] that $(\delta_{A_1, B_1} - \lambda I) \in L(L(N_1, M_1))$ is normal Banach space operator, and from [19, Theorem 3.4] that $asc(\delta_{A_1, B_1} - \lambda I) \leq 1$. Hence $asc(\delta_{A, B} - \lambda I) \leq 1$.
- If $\mu \in \sigma(A_1)$ and $\nu \in \sigma(B_2)$, let $X \in \ker(\delta_{A, B} - \lambda I)$, then $X_{22} = X_{21} = X_{11} = 0$ and

$$A_1 X_{12} = X_{12} (B_2 - \lambda). \tag{1}$$

We argue as in the proof of [32, Theorem 2.2], we get $X_{12} = 0$, consequently $X = 0$. Hence $asc(\delta_{A, B} - \lambda I) \leq 1$.

- If $\mu \in \sigma(A_2)$ and $\nu \in \sigma_D(B_1)$, let $X \in \ker(\delta_{A, B} - \lambda I)$, then $X_{11} = X_{22} = X_{12} = 0$ and $(A_2 - \lambda)X_{21} = X_{21}B_1$. We argue as in the proof of [32, Theorem 2.2], we obtain $X_{21} = 0$, hence $X = 0$ and $asc(\delta_{A, B} - \lambda I) \leq 1$.
- If $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_2)$. Since A has property $H(1)$ it follows from [1, Theorem 3.99] that A_2 has property $H(1)$. Hence $H_0(A_2 - \mu) = \ker(A_2 - \mu) = \{0\}$. Let $X \in \ker(\delta_{A, B} - \lambda I)$, then $X_{21} = X_{12} = X_{11} = 0$ and $(A_2 - \mu)X_{22} = X_{22}(B_2 - \nu)$, this implies that, if $t \in H_0(B_2 - \nu)$, then $X_{22}t \in H_0(A_2 - \mu) = \{0\}$. Hence $X_{22}t = 0$. Since $t \in H_0(B_2 - \nu)$, using properties of quasinilpotent part, we get $(B_2 - \nu)(t) \in H_0(B_2 - \nu)$, consequently $N_2 = clH_0(B_2 - \nu)$. So $X_{22} = 0$, hence $X = 0$ and $asc(\delta_{A, B} - \lambda I) \leq 1$.

The cases

- $\mu \in \sigma_D(A_1)$ and $\nu \in \sigma(B_1)$, or
- $\mu \in \sigma_D(A_1)$ and $\nu \in \sigma(B_2)$, or
- $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_1)$,

can be proved similarly. \square

The class of operators having property $H(1)$ and reduced by each of its eigenspaces is considerably large, it contains the following class of operators. An operator $T \in L(\mathcal{H})$ is said to be p -hyponormal, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. An invertible operator $T \in L(H)$ is log-hyponormal if $\log |T^*|^2 \leq \log |T|^2$. An operator $T \in L(\mathcal{H})$ is said to be w -hyponormal if $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$, see [28]. It is shown in [5, 6] that the class of w -hyponormal properly contains the class of p -hyponormal ($0 < p \leq 1$), and log-hyponormal. T. Furuta, M. Ito and T. Yamazaki [26] introduced a very interesting class \mathcal{A} operators defined by $|T^2| - |T|^2 \geq 0$, and they showed that class \mathcal{A} is a subclass of paranormal operators (i.e., $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$) and contains w -hyponormal operators. An operator $T \in L(\mathcal{H})$ is said to be class $\mathcal{A}(s, t)$, where s and t are strictly positive integers, if $|T^*|^{2t} \leq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{t+s}}$. Then $T \in \mathcal{A}(\frac{1}{2}, \frac{1}{2})$ if and only if T is w -hyponormal and $T \in \mathcal{A}(1, 1)$ if and only if T is class \mathcal{A} .

I. H. Jeon and I. H. Kim [29] introduced quasi-class \mathcal{A} operators defined by $T^*(|T^2| - |T|^2)T \geq 0$, as an extension of the notion of class \mathcal{A} operators. K. Tanahash, I. H. Jeon, I. H. Kim and A. Uchiyama [34] introduced k -quasi-class \mathcal{A} operators defined by $T^{*k}(|T^2| - |T|^2)T^k \geq 0$, for a positive integer k as an extension of the notion of quasi-class \mathcal{A} operators, for interesting properties of k -quasi-class \mathcal{A} operators, called also quasi-class (\mathcal{A}, k) , see [27, 34].

The following result is a straight forward application of the above theorem.

Corollary 3.3. *Let $A, B^* \in L(\mathcal{H})$ be k -quasi-class \mathcal{A} operators. Assume that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then*

$$asc(\delta_{A, B} - \lambda I) \leq 1, \text{ for all complex } \lambda.$$

Corollary 3.4. [17, Corollary 2.7] Let $A, B^* \in L(\mathcal{H})$ be w -hyponormal operators. Assume that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then

$$\text{asc}(\delta_{A,B}) \leq 1.$$

4. Range closure for $\delta_{A,B}$

If we combine [12, Theorem 2.7], [11, Theorem 2.7] and [1, Theorem 3.83], we get the following Lemma

Lemma 4.1. Let $T \in L(X)$. Then The following statements are equivalent

1. $\sigma_{\text{id}}(T) = \emptyset$,
2. $\sigma_{\text{dsc}}(T) = \emptyset$, where $\sigma_{\text{dsc}}(T) = \{\lambda \in \mathbb{C} : \text{dsc}(T - \lambda I) = \infty\}$,
3. T is algebraic,
4. $\sigma(T)$ is a finite set of poles of the resolvent.

In [18, Propostion 2.9], the authors proved that, if $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ and if either (i) A^* and B have SVEP or (ii) $\delta_{A,B}^*$ has SVEP, then $\delta_{A,B} - \lambda$ has closed range for all complex λ if and only if A and B are algebraic operators. In the following result, we prove that, if $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , then $\delta_{A,B} - \lambda$ has closed range, for all complex λ if and only if A and B are algebraic operators, without the additional conditions (i) or (ii).

Proposition 4.2. Let $A, B \in L(X)$. If $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , then $\delta_{A,B} - \lambda$ has closed range, for all complex λ if and only if A and B are algebraic operators.

Proof. Suppose that A and B are algebraic operators, then $\delta_{A,B}$ is algebraic, form Lemma 4.1, we deduce that $\sigma_{\text{id}}(\delta_{A,B}) = \emptyset$, then $\delta_{A,B} - \lambda$ is left Drazin, for all complex λ , since $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , then from [2, Lemma 1.7] $\delta_{A,B} - \lambda$ has closed range, for all complex λ . Conversely if $\text{asc}(\delta_{A,B} - \lambda) \leq 1$ and $\delta_{A,B} - \lambda$ has closed range, for all complex λ , then $\delta_{A,B} - \lambda$ is left Drazin invertible, for all complex λ , consequently $\sigma_{\text{id}}(\delta_{A,B}) = \emptyset$. Hence from Lemma 4.1 $\delta_{A,B}$ is algebraic. \square

Corollary 4.3. Suppose that $A, B^* \in L(\mathcal{H})$ are reduced by each of its eigenspaces and have property $H(1)$, then $\delta_{A,B} - \lambda$ has closed range, for all complex λ if and only if A and B are algebraic operators.

Proof. From Theorem 3.2, we get $\text{asc}(\delta_{A,B} - \lambda) \leq 1$, for all complex λ , consequently the equivalence follows from Proposition 4.2. \square

Theorem 4.4. Suppose that $A, B^* \in L(\mathcal{H})$ are reduced by each of its eigenspaces and have property $H(1)$, then the following conditions are pairwise equivalent

1. $\lambda \in \text{iso}\sigma_a(\delta_{A,B})$,
2. $\delta_{A,B}$ is left polaroid,
3. $\delta_{A,B} - \lambda I$ has closed range,
4. There exist finite sequences $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$, where $\alpha_i \in \text{iso}\sigma_a(A)$ and $\beta_i \in \text{iso}\sigma_s(B)$ such that $\alpha_i - \beta_i = \lambda$, for all $1 \leq i \leq n$,
5. $L(\mathcal{H}) = \ker(\delta_{A,B} - \lambda) \oplus \mathcal{R}(\delta_{A,B} - \lambda)$,
6. $\lambda \in \text{iso}\sigma(\delta_{A,B})$.

Proof. We deduce from Theorem 3.2 that $\text{asc}(\delta_{A,B} - \alpha I) \leq 1$, for all complex α .

(1) \implies (2) Since A and B^* have property $H(1)$, it follows that they are polaroid, hence from [17, Proposition 3.1] $\delta_{A,B}$ is left polaroid, then for $\lambda \in \text{iso}\sigma_a(\delta_{A,B})$, we have $\delta_{A,B} - \lambda$ is left Drazin invertible.

(2) \Leftrightarrow (3) Since $\text{asc}(\delta_{A,B} - \alpha I) \leq 1$, for all complex α , then we have the equivalence.

(3) \implies (4) Since $\delta_{A,B} - \lambda$ has closed range and $\text{asc}(\delta_{A,B} - \lambda I) \leq 1$, it follows that $\delta_{A,B} - \lambda$ is left Drazin invertible, hence from [3, Theorem 2.4] $\lambda \in \text{iso}\sigma_a(\delta_{A,B})$, according to [30, Theorem 3.5.1] we have $\sigma_a(\delta_{A,B}) = \sigma_a(A) - \sigma_s(B)$,

it is not difficult to conclude that $iso\sigma_a(\delta_{A,B}) = iso\sigma_a(A) - iso\sigma_s(B) \setminus acc\sigma_a(\delta_{A,B})$, hence we get (4).

(5) \implies (6) \implies (1) are evident.

The implication (4) \implies (1) is evident. Since A and B^* have property $H(1)$, it follows from [1, Theorem 3.96] that they are polaroid and have SVEP, then from the proof of [7, Proposition 3.5] we have $\Pi^a(\delta_{A,B}) = \Pi(\delta_{A,B})$. Hence (2) \implies (6), and (6) \implies (5), then the proof is complete.

□

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References

- [1] P. Aiena, Fredholm and local spectral theory with applications to multipliers, Kluwer Acad, publishers, 2004.
- [2] P. Aiena and T. L. Miller, On generalized a-Browder's theorem, *Studia Math.* 180 (3)(2007), 285–300.
- [3] P. Aiena and J. E. Sanabria, On left and right poles of the resolvent, *Acta Sci. Math. (Szeged)* 74 (2008) 669–687.
- [4] P. Aiena and M. M. Neumann, On the stability of the localized single valued extension property under commuting perturbations, *Proc. Amer. Math. Soc.* 141 (6) (2013) 2039–2050.
- [5] A. Aluthge and D. Wang, On w-hyponormal operators II, *Intergr. Equ. Oper. Theory* 37 (2000) 324–331.
- [6] A. Aluthge and D. Wang, An operator inequality which implies paranormality, *Math. Ineq. Appl* 2 (1999) 113–119.
- [7] M. Amouch and F. Lombarkia, Rakocević's property for generalized derivations, *Rend. Circ. Mat. Palermo* 64 (1) (2015) 57–66.
- [8] J. Anderson, On normal derivations, *Proc. Amer. Math. Soc.* 38 (1973), 136–140
- [9] J. Anderson and C. Foias, Properties which normal operators share with normal derivations, and related operators, *Pacific. J. Math.* 61 (1975), 313–325.
- [10] A. Bachir and F. Lombarkia, Fuglede Putnam's theorem for w-hyponormal operators, *Math. Ineq. Appl* 12 (2012) 777–786.
- [11] O. Bel Hadj Fredj, M. Burgos, M. Oudghiri, Ascent spectrum and essential ascent spectrum, *Studia Math* 187 (2008), 59–73.
- [12] O. Bel Hadj Fredj, Essential descent spectrum and commuting compact perturbations, *Extracta Math* 21(3) (2006), 261–271.
- [13] S. K. Berberian, The Weyl spectrum of an operator, *Indiana Univ. Math. J* 20 (1970) 529–544.
- [14] E. Boasso and M. Amouch, Generalized Browder's and Weyl's theorem for generalized derivations, *Mediterr. J. Math.* 1 (12) (2015) 117–131.
- [15] M. Cho, S. V. Djordjević, B. P. Duggal and T. Yamazaki, On an elementary operator with w-hyponormal operator entries, *Linear Algebra Appl* 433 (2010) 2070–2079.
- [16] D. S. Djordjević, B. P. Duggal and R. E. Harte, Consistent in invertibility operators and SVEP, *Filomat* 27(5)(2013) 767–776.
- [17] B. P. Duggal, S. D. Djordjevic and C.S. Kubrusly Elementary operators, finite ascent, range closure and compactness, *Linear Algebra Appl* 449 (2014) 334–340.
- [18] B. P. Duggal, S. D. Djordjevic and C.S. Kubrusly Algebraic elementary operators, *Functional Analysis, Approximation and Computation* 6(2) (2014) 43–50.
- [19] B. P. Duggal, The closure of the range of an elementary operator, *Linear Algebra Appl.* 392 (2004) 305–319.
- [20] N. Dunford, Spectral theory II. Resolution of the identity, *Pacific. J. Math.* 2 (1952) 559–614.
- [21] N. Dunford, Spectral theory, *Pacific. J. Math.* 4 (1954) 321–354.
- [22] D. Drissi, The Single-Valued Extension Property and Subharmonicity, *Extracta Math.* 22 (3) (2007), 235–244.
- [23] M. R. Embry and M. Rosenblum, Spectra, tensor products and linear operator equations, *Pacific J. Math* 53 (1974) 95–107.
- [24] J. Eschmeier, Tensor products and elementary operators, *J. Reine Angew. Math.* 390 (1988), 47–66.
- [25] J. K. Finch, The single valued extension property on a Banach space, *Pacific. J. Math.* 58 (1975), 61–69.
- [26] T. Furuta, M. Ito and T. Yamazaki, A subclass of operators including class of log-hyponormal and several classes, *Sci. Math* 3 (1998) 389–403.
- [27] F. G. Gao and X. C. Fang, On k-quasiclass A operators, *Math. Inequal. Appl* 2009 (2009) 1–10.
- [28] M. Ito, T. Yamazaki, Relation between two inequalities $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications, *Integral Equations Operator Theory* 44 (2002) 442–450.
- [29] I. H. Jeon and I. H. Kim, On operators satisfying $T^*|T^2|T \geq T^*|T|^2T$, *Linear Algebra Appl* 418 (2006) 854–862.
- [30] K. B. Laursen and M. M. Neumann, An introduction to local spectral theory, *Lon. Math. Soc. Monographs*, Oxford Univ. Press, 2000.
- [31] F. Lombarkia and H. Zguitti, On the Browder's Theorem of an Elementary Operator, *extracta mathematicae* 28 (2013), 213–224.
- [32] F. Lombarkia and M. Amouch, Asymmetric Fuglede Putnam's theorem of operators reduced by their eigenspaces, *Filomat* 20 (2017), 6409–6417.
- [33] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasgow Math. J* 29 (2) (1987) 159–175.
- [34] K. Tanahashi, I. H. Jeon, I. H. Kim and A. Uchiyama, Quasinilotent part of class A or (p,k)-quasihyponormal operators, *Oper. Theory Adv. Appl* 187 (2008) 199–210.