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Graph Theoretic Representation of Rings of Continuous Functions

Bedanta Bose[a], Angsuman Das[b]

[a]*Swami Niswambalananda Girls' College, Hooghly, India*
[b]*Presidency University, Kolkata, India*

Abstract. In this paper, we introduce a graph structure, called zero-set intersection graph $\Gamma(C(X))$, on the ring of real valued continuous functions, $C(X)$, on a Tychonoff space X. We show that the graph is connected and triangulated. We also study the inter-relationship of cliques of $\Gamma(C(X))$ and ideals in $C(X)$ which helps to characterize the structure of maximal cliques of $\Gamma(C(X))$ by different kind of maximal ideals of $C(X)$. We show that there are at least 2^c many different maximal cliques which are never graph isomorphic to each other. Furthermore, we study the neighbourhood properties of a vertex and show its connection with the topology of X and algebraic properties of $C(X)$. Finally, it is shown that two graphs are isomorphic if and only if the corresponding rings are isomorphic if and only if the corresponding topologies are homeomorphic either for first countable topological spaces or for realcompact topological spaces.

1. Introduction

The study of graph theory, apart from its combinatorial implications, also lends to characterization of various algebraic structures. The benefit of studying these graphs is that one may find some results about the algebraic structures and vice versa. There are three major problems in this area: (1) characterization of the resulting graphs, (2) characterization of the algebraic structures with isomorphic graphs, and (3) realization of the connections between the structures and the corresponding graphs.

The first instance of such work is due to Beck [12] who introduced the idea of zero divisor graph of a commutative ring with unity. Though his key goal was to address the issue of colouring, this initiated the formal study of exposing the relationship between algebra and graph theory and at advancing applications of one to the other. Later on, a different method of associating a zero-divisor graph to a commutative ring R was proposed by Anderson and Livingston [6]. Till then, a lot of research, e.g., [1, 3–5, 7, 10, 11, 13–17, 21], has done in this area. Following those footsteps, Azarpanah et. al. [8] studied zero divisor graph of $C(X)$, the ring of real valued continuous functions on a topological space X. They studied the conditions on X, when the associated graph will be triangulated, connected etc. However, as their work was a follow up of zero divisor graph of a ring, the main topological flavor of characterizing the graph was missing. Apart from that, Amini et al. [2] and Badie [22] also studied another graph structure on $C(X)$ from a co-maximal ideal point of view.

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Email addresses: ana_bedanta@yahoo.com (Bedanta Bose), angsuman.maths@presiuniv.ac.in (Angsuman Das)

In 1967 J.de Groot introduced a new graph structure on a topological space by using 'linked system' on subbase of closed sets. His idea was as follows: A family of sets is 'centered' if every finite subcollection has non-empty intersection and a family of sets is 'linked' if the intersection of every pair of its members is non-empty. Alexander's lemma states that a space X is compact when X possesses a closed subbase such that every centered subcollection has non-empty intersection. Paralleling this lemma, De Groot introduced the following definition in [18]. A space X is supercompact if X possesses a closed subbase such that every linked subcollection has non-empty intersection. Such a subbase is called a binary subbase. By Alexander's lemma, every supercompact space is compact. Let X be a T_1-space and $\mathcal{S}$ a closed T_1 subbase for X. The superextension $\lambda(X, \mathcal{S})$ of X relative to the subbase $\mathcal{S}$ is the set of all maximal linked systems $\mathcal{M} \subseteq \mathcal{S}$ (a subsystem of $\mathcal{S}$ is called linked if every two of its members intersect; a maximal linked system is a linked system not properly contained in any other linked system) topologized by taking $\{\{\mathcal{M} \in \lambda(X, \mathcal{S}) \mid S \in \mathcal{M}\} \mid S \in \mathcal{S}\}$ as a closed subbase. Clearly this subbase is binary, hence $\lambda(X, \mathcal{S})$ is supercompact, while X can be embedded in $\lambda(X, \mathcal{S})$ by the natural embedding $i : X \to \lambda(X, \mathcal{S})$ defined by $i(x) = \{S \in \mathcal{S} \mid x \in S\}$. Verbeek's monograph [23] is a good reference for results on superextensions.

In this paper we initiate the study of a special type of graph structure on $C(X)$, called 'zero-set intersection graph' $\Gamma(C(X))$, depending on the above notion of 'linked system', with a goal to characterize some topological properties of X and ring properties of $C(X)$. We have shown that the graph is connected and triangulated and girth is 3. We have studied the cliques and maximal cliques of this graph and have shown that all ideals are cliques and all maximal ideals are maximal cliques [Theorem 3.2]. Apart from that we have investigated different types of maximal cliques in this graph which are not of the form of maximal ideals [Example 3.3]. We have also characterized all maximal cliques through the maximal ideals which explicitly shows the connection between algebraic structure of $C(X)$ and graphical structure of $\Gamma(C(X))$ [Theorem 3.11]. We define prime clique in a graph to bring the concept of a clique to be contained in an unique maximal clique. The definition [Definition 3.12] is like this: a clique I in a graph G is prime if any two vertex outside of I are adjacent with all vertices of I then they are adjacent to each other. We have shown that all prime ideals are prime cliques [Theorem 3.13]. Furthermore, we have shown that for each maximal clique M there always exists a clique in M of special type O_M which is a prime clique [Theorem 3.16]. We give examples of spaces for which $M \setminus O_M$ contains and does not contain [Example 3.17] prime cliques and incarnating this idea we define GP-space [Definition 3.18] : A Tychonoff space X for which each maximal clique M of $\Gamma(C(X))$, $O_M \setminus M$ contains no prime clique. We show that P-spaces are GP-space [Theorem 3.19]. We point out that non-realcompact first countable spaces can not be a GP-space and hence pose an open question : what is the topological characterization of GP-spaces. Next we move to the study of neighbourhood properties of a vertex of this graph and have shown that it is intimately related with the topology of X [Theorem 4.1]. We have also investigated when a vertex of this graph will be simplicial [Theorem 4.3] and conclude that if X is first countable supercompact space then all the maximal cliques are simplicial with respect to some vertex [Theorem 4.4]. Using neighbourhood property we have shown that how to recognize the unique maximal clique in which a prime clique is contained [Theorem 4.5]. In the last section we have studied the inter-relationship among the graph structure of $\Gamma(C(X))$, ring structure of $C(X)$ and topological structure of X and get a conclusion that all the above three structures are equivalent in nature for realcompact topological spaces [Corollary 5.11, Theorem 5.10, Theorem 5.8, Theorem 5.7] and assuming first countability we have generalized this result for non-realcompact spaces [Theorem 5.9]. In this connection we have shown that there are 2^c-many maximal cliques in $\Gamma(C(X))$ which are never graph isomorphic to each other if the corresponding topological space is first countable and contained a C-embedded copy of $\mathbb{N}$.

2. Preliminaries

In this section, for convenience of the reader and also for later use, we recall some definitions and notations concerning elementary graph theory and $C(X)$, the ring of real valued continuous functions over a topological space X. For undefined terms and concepts in graph theory and rings of continuous functions the reader is referred to [24] and [20] respectively.

By a graph $G = (V, E)$, we mean a non-empty set V and a symmetric binary relation (possibly empty) E on V. The set V is called the set of vertices and E is called the set of edges of G. Two element u and v in V are said to be adjacent if $(u, v) \in E$. $H = (W, F)$ is called a *subgraph* of G if H itself is a graph and $\phi \neq W \subseteq V$ and $F \subseteq E$. If V is finite, the graph G is said to be finite, otherwise it is infinite. If all the vertices of G are pairwise adjacent, then G is said to be *complete*. A complete subgraph of a graph G is called a *clique*. A *maximal clique* is a clique which is maximal with respect to inclusion. Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if $\exists$ a bijection $\phi : V \to V'$ such that $(u, v) \in E \Leftrightarrow (\phi(u), \phi(v)) \in E'$. A *path* of length k in a graph is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$, where v_i's are distinct and e_i is the edge joining v_i and v_{i+1}. We call this a path joining v_0 and v_k. A *cycle* is a path with $v_0 = v_k$. A cycle of length 3 is called a *triangle*. A graph is *connected* if for any pair of vertices $u, v \in V, \exists$ a path joining u and v. A graph is said to be *triangulated* if for any vertex $u \in V, \exists v, w \in V$, such that (u, v, w) is a triangle. The *distance* between two vertices $u, v \in V$, $d(u, v)$ is defined as the length of the shortest path joining u and v, if it exists. Otherwise, $d(u, v)$ is defined as ∞. The *diameter* of a graph is defined as $diam(G) = \max_{u,v\in V} d(u, v)$, the largest distance between pairs of vertices of the graph, if it exists. Otherwise, $diam(G)$ is defined as ∞. The *girth* of a graph is the length of its shortest cycle. The neighbourhood of a vertex v of a graph G is the induced subgraph of G consisting of all vertices adjacent to v. Notice that the way neighbourhood has been defined above, it does not include the vertex v itself. For this reason sometimes it is called an open neighbourhood. If one include the vertex v in the induced subgraph then it is called closed neighbourhood of v. Through out the rest of this paper, by neighbourhood of a vertex, we mean the closed neighbourhood unless otherwise mentioned and denote it by $N[v]$. A vertex v is called simplicial if $N[v]$ is a clique.

Coming to the rings of continuous functions, let X be a completely regular Hausdorff topological space (unless otherwise mentioned) and $C(X)$ be the ring of all real valued continuous functions defined on X. A subset Z of X is called a *zero set* of X if $\exists f \in C(X)$ such that $Z = f^{-1}(0)$. The collection of all zero sets in X is denoted by $Z[X]$. A subfamily $\mathcal{F}$ of $Z[X]$ is called a *z-filter* if it satisfies the following: (i) $\phi \notin \mathcal{F}$ (ii) if $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$, and (iii) if $Z \subseteq Z'$ for some $Z' \in Z[X]$ and $Z \in \mathcal{F}$ then $Z' \in \mathcal{F}$.

There is a nice correspondence between ideals and z-filters. For every ideal I of $C(X)$, $Z[I] = \{Z(f) : f \in I\}$ is a z-filter and for every z-filter $\mathcal{F}$, $Z^{-1}[\mathcal{F}] = \{f \in C(X) : Z(f) \in \mathcal{F}\}$ is an ideal of $C(X)$. An ideal I of $C(X)$ is called a *z-ideal* if $Z(f) \in Z[I] \Rightarrow f \in I$. If a z-filter is maximal w.r.t set inclusion, then it is called *z-ultrafilter*. Every z-ultrafilter has the property that if a zero set intersects every member of $\mathcal{F}$, then it belongs to $\mathcal{F}$. The collection of z-ultrafilter with a hull-kernel topology gives the Stone-Čech compactification of X, called βX, in which the space X is densely embedded. A maximal ideal M is called a *real* maximal ideal of $C(X)$ if $C(X)/M$ is isomorphic to the field of real numbers. By vX, we mean the *Hewitt real compactification* of X which can also be seen as the collection of all real maximal ideals of $C(X)$ with the subspace topology.

We initiate the study of a special type of graph structure on $C(X)$, with a goal to characterize some topological properties of X and ring properties of $C(X)$ with the help of this newly defined graph on $C(X)$. The idea is to treat $C(X)$ itself as the set of vertices such that there is an edge between vertices f and g if $Z(f) \cap Z(g) \neq \phi$, where $Z(f) = \{x \in X : f(x) = 0\}$. However, defining the graph in this fashion renders all units in the ring $C(X)$ (i.e., all functions that does not attain zero at any point in X) as isolated vertices. Thus, we consider the set of all non-units in $C(X)$ as the vertex set and define edges between them in the above mentioned way.

3. Zero-Set Intersection Graph of $C(X)$

Definition 3.1. **[Zero-Set Intersection Graph]** Let $\mathcal{N}(X)$ be the set of all non-units in the ring $(C(X), +, \cdot)$. By zero-set intersection graph $\Gamma(C(X))$, we mean the graph whose set of vertices is $\mathcal{N}(X)$ and there is an edge between distinct vertices f and g if $Z(f) \cap Z(g) \neq \phi$.

For any two $f, g \in \mathcal{N}(X)$, $fg \in \mathcal{N}(X)$ and both f and g are adjacent with fg, means that $\Gamma(C(X))$ is connected and $diam(\Gamma(C(X))) = 2$. Also for non trivial X and $f \in \mathcal{N}(X)$, one can easily produce two different non-zero functions $2f$ and $3f \in \mathcal{N}(X)$ so that all three functions are adjacent to each other which makes the graph $\Gamma(C(X))$ triangulated and as a simple consequence, girth of $\Gamma(C(X))$ is 3.

The way we have defined the graph $\Gamma(C(X))$, one may identify any two vertex $f, g \in \mathcal{N}(X)$ if and only if $Z(f) = Z(g)$ and gives a induced graph structure on the equivalence class which is identical with the graph defined on the subbase $Z[X]$ using De Groot's 'linked system'. Viewing the graph in this fashion it may seem to one that the graph of $\mathcal{N}(X)$ is similar to the graph that studied by De Groot on topological spaces. Later on we will see that these two graphs are completely different with respect to their behaviour of maximal clique, though it is quite clear by the above identification that the space of maximal linked system of the graph $\Gamma(C(X))$ is homeomorphic to $\lambda(X, Z[X])$.

3.1. *Cliques in* $\Gamma(C(X))$

In this subsection we study the cliques and maximal cliques in $\Gamma(C(X))$ and their relations with ideals, prime ideals and maximal ideals of $C(X)$. In this connection observe that for any ideal I of $C(X)$ and any two function $f, g \in I$, $Z(f) \cap Z(g) = Z(f^2 + g^2)$, showing that every ideal in $C(X)$ is a clique in $\mathcal{N}(X)$ and hence every maximal ideal of $C(X)$ becomes a clique. The next theorem shows that maximal ideals of $C(X)$ are also maximal cliques.

Theorem 3.2. *Maximal ideals* $M^p = \{f \in C(X) \mid p \in cl_{\beta X} Z(f)\}$, $p \in \beta X$ *are maximal cliques in* $\Gamma(C(X))$.

Proof. If not and if possible, let M^p be not a maximal clique in $\Gamma(C(X))$. Then there exists $\varphi \notin M^p$ such that $M^p \cup \{\varphi\}$ is again a clique. Again the definition of clique ensures that $Z(\varphi)$ intersects every member of the z-ultrafilter $Z[M^p]$ and then the property of z-ultrafilter forces $Z(\varphi)$ to be included in $Z[M^p]$ and hence $\varphi \in M^p$, since M^p is a z-ideal, which is a contradiction and this concludes the proof. □

Note that the converse of this theorem is not true in general, i.e., $\exists$ compact topological space X, such that $\Gamma(C(X))$ contains maximal cliques which are not maximal ideals.

Example 3.3. Let $X = \beta\mathbb{N}$ with its usual topology. Let $M_{1,2}, M_{2,3}, M_{1,3}$ be the collection of all (real valued continuous) functions which vanish at $\{1, 2\}, \{2, 3\}, \{1, 3\}$ respectively. Let $M = M_{1,2} \cup M_{2,3} \cup M_{1,3}$. Then M is a maximal clique which is not a maximal ideal.

Remark 3.4. We can classify the maximal cliques into three different categories. The first category consists of all fixed maximal ideals and we call it 'fixed maximal clique'. In the second category we include all free maximal ideals and we call it 'free maximal clique of ideal type' and the third category consists of rest of all the maximal cliques, i.e., all maximal cliques which are not of ideals. We call 'maximal clique of non-ideal type'.

It is known that a prime ideal in $C(X)$ is contained in a unique maximal ideal. However, as seen in the above example, there are maximal cliques which are not maximal ideals. Thus, it is natural to ask whether prime ideals in $C(X)$ are contained in a unique maximal clique or not. We answer this question assertively in the next theorem.

Theorem 3.5. *Every prime ideal in* $C(X)$ *is contained in a unique maximal clique.*

Proof. Let P be a prime ideal in $C(X)$. Suppose P is contained in at least two distinct maximal cliques. Since every prime ideal in $C(X)$ is contained in some unique maximal ideal, therefore, one of the maximal cliques should be a maximal ideal, say, M^p for some $p \in \beta X$. Let N be another maximal clique containing P. Obviously for every $f \in N$, $Z(f)$ intersects every member of $Z[P]$. Hence $Z[P] \cup \{Z(f)\}$ is contained in a unique z-ultrafilter and it should be equal to $Z[M^p]$, where M^p is the unique maximal ideal containing P. Then $Z(f)$ intersects every member of the z-ultrafilter $Z[M^p]$ and consequently $Z(f) \in Z[M^p]$ i.e., $f \in M^p$. This holds for every $f \in N$ and it follows that $N = M^p$. □

3.2. **Characterizing Maximal Cliques in** $\Gamma(C(X))$

In this section, we characterize the structure of maximal cliques in $\Gamma(C(X))$ by showing that every maximal clique can be expressed as union of intersection of a suitable collection of maximal ideals of $C(X)$.

Theorem 3.6. *Let M be a maximal clique in $\Gamma(C(X))$, then M always contains an ideal of $C(X)$.*

Proof. Let $f \in M$. Construct $I = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. Clearly, I is an ideal of $C(X)$. Again, f is adjacent with all element $h \in M$, i.e., $Z(f) \cap Z(h) \neq \emptyset$ for all $h \in M$, showing that $Z(g) \cap Z(h) \neq \emptyset$ for each $g \in I$ and for all $h \in M$, means that each $g \in I$ is adjacent with all elements of M and as a consequence of maximality of M, $I \subseteq M$. $\square$

Remark 3.7. Consider the set of all ideals properly contained in a maximal clique M with a partial ordering (set inclusion). Every chain has an upper bound (union). By Zorn's lemma, it has at least one maximal element. Let $\mathcal{A}$ be the collection of all such maximal elements. Let $N \in \mathcal{A}$ and $\mathcal{N}'$ be the collection of all maximal ideals of $C(X)$ in which N can be extended.

Lemma 3.8. $\bigcup_{N\in\mathcal{A}} N = M$.

Proof. Clearly, $\bigcup_{N\in\mathcal{A}} N \subset M$. For the other direction, let $f \in M$. Consider the ideal generated by f in $C(X)$, $\langle f\rangle$. As for any $h \in \langle f\rangle, Z(f) \subset Z(h)$ and M is a maximal clique, therefore $\langle f\rangle \subset M$. Hence, $\langle f\rangle$ is contained in some maximal element $N \in \mathcal{A}$ and in particular $f \in N$. Thus, $\langle f\rangle \subseteq \bigcup_{N\in\mathcal{A}} N$ and hence the lemma. $\square$

Lemma 3.9. *If $N, P \in \mathcal{A}$, then $\mathcal{N}' \cap \mathcal{P}' \neq \phi$, where the symbols have their meaning as in Remark 3.7.*

Proof. We first show that $Z[N] \cup Z[P]$ has finite intersection property. Let $\{Z(f_i)\}_i \cup \{Z(g_j)\}_j$ be a finite collection of elements in $Z[N] \cup Z[P]$, where $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. Now, $\bigcap_{i,j}(Z(f_i) \cap Z(g_j)) = Z(f) \cap Z(g)$, where $f = \sum_i f_i^2$ and $g = \sum_j g_j^2$ and being to be an ideal, $f \in N$ and $g \in P$. Again, M is a maximal clique and $N, P \subset M$, therefore $Z(f) \cap Z(g) \neq \phi, \forall f \in N$ & $\forall g \in P$. Thus, $Z[N] \cup Z[P]$ has finite intersection property and as a result, it can be extended to a z-ultrafilter $Z[\mathbf{M}]$ for some maximal ideal $\mathbf{M}$ of $C(X)$. Since, $Z[N] \subset Z[\mathbf{M}]$, therefore $N \subset \mathbf{M}$. Similarly, $P \subset \mathbf{M}$. Hence, $\mathbf{M} \in \mathcal{N}' \cap \mathcal{P}'$, thereby proving the lemma. $\square$

Lemma 3.10. $\bigcap_{P\in\mathcal{N}'} P = N$, *where the symbols have their meaning as in Remark 3.7.*

Proof. Clearly, $N \subset \bigcap_{P\in\mathcal{N}'} P$. For the other direction, let $f \in \bigcap_{P\in\mathcal{N}'} P$. Now, for any $P \in \mathcal{A}$, by Lemma 3.9, $\mathcal{N}' \cap \mathcal{P}' \neq \phi$. Thus there exists a maximal ideal in $C(X)$ containing P and N and also containing f, i.e., f is adjacent to all members of P and hence f is adjacent to all members of M (since, P is arbitrary in $\mathcal{A}$). Thus, due to maximality of M, $f \in M$.

Now consider the ideal $\langle N, f\rangle = N_1$. Since, f belongs to all maximal ideals in $C(X)$ which contains N, therefore N_1 extends to exactly all maximal ideals belonging to $\mathcal{N}'$.

Let $g \in N_1$. Now Lemma 3.9 shows that there exists a maximal ideal in $C(X)$ containing N and P, for each $P \in \mathcal{A}$ and also we have seen that N_1 & N extends to the same maximal ideals in $C(X)$, therefore there exists a maximal ideal in $C(X)$ containing N_1 and P, for all $P \in \mathcal{A}$. Since $g \in N_1$, therefore g is adjacent to every element of P, for all $P \in \mathcal{A}$. Thus g is adjacent to every element of $\bigcup_{P\in\mathcal{A}} P$ which is eventually equal to M (by Lemma 3.8). Thus, $g \in M$ and hence $N_1 \subset M$.

Now, N is an ideal which is maximal in M and N_1 is an ideal containing N and contained in M. Thus, $N = N_1$ and hence $f \in N$. Hence the lemma follows. $\square$

Theorem 3.11. *Every maximal clique in $\Gamma(C(X))$ can be expressed as union of intersection of some maximal ideals in $C(X)$.*

Proof. Let M be a maximal clique. Then, by Lemma 3.8, $\bigcup_{N\in\mathcal{A}} N = M$. Now, by Lemma 3.10, every $N \in \mathcal{A}$ can be expressed as $\bigcap_{P\in\mathcal{N}'} P$. Thus,

$$M = \bigcup_{N\in\mathcal{A}}\left(\bigcap_{P\in\mathcal{N}'} P\right)$$

$\square$

Definition 3.12. A clique I in a graph G is defined to be a prime clique if for any two vertex $f, g \in G \setminus I$ which are adjacent with all elements of I, then f and g are adjacent to each other.

It is quite clear from the above definition that the central idea behind it is to incorporate the concept of a clique being contained in an unique maximal clique. This concept is not new in literature, rather it is a very familiar concept in algebra, specially, ring theory and ideal theory, *viz.* Gelfand ring. Next we will investigate the structure of prime cliques.

Theorem 3.13. *Every prime ideal in $C(X)$ is a prime clique.*

Proof. Let P be a prime ideal of $C(X)$ and $f, g \in \mathcal{N}(X) \setminus P$ such that f and g are both adjacent to each element of P individually. We have to show that both f and g are adjacent. If not and if possible let f and g are not adjacent to each other. Then consider the ideals $I =< P, f >$ and $I' =< P, g >$. Since $Z(f) \cap Z(g) = \emptyset$, both I and I' are not same. Since $C(X)$ is a gelfand ring therefore every prime ideal in $C(X)$ is contained in an unique maximal ideal and therefore I and I' is contained in a unique maximal ideal in $C(X)$. Hence f and g are contained in a same ideal which shows that $Z(f) \cap Z(g) \neq \emptyset$ which is a contradiction. $\square$

Theorem 3.11 gives us a specific structure of maximal clique in terms of maximal ideals. Using this structure we now construct prime cliques for each maximal ideal. Recall, from Theorem 3.11, that for any maximal clique M of $\mathcal{N}(X)$

$$O_M = \bigcup_{N \in \mathcal{A}} \left(\bigcap_{P \in \mathcal{N}'} P \right),$$

where $\mathcal{A}$ is the collection of maximal ideals contained in M and for each $N \in \mathcal{A}$, $\mathcal{N}'$ is the collection of all maximal ideals of $C(X)$ in which N can be extended. Keeping this structure in mind, we define O_M as follows

$$O_M = \bigcup_{N \in \mathcal{A}} \left(\bigcap_{M^p \in \mathcal{N}'} O^p \right)$$

where O^p is the ideal consisting of all $f \in C(X)$ such that $\mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p \in \beta X$.

Theorem 3.14. *For each maximal clique M of $\Gamma(X)$, $O_M \subseteq M$ and hence O_M is a clique.*

Proof. For each $N \in \mathcal{A}$, $\bigcap_{P \in \mathcal{N}'} P = N$. Again $O^p \subseteq M^p$ for all $p \in \beta X$ and hence $\bigcap_{M^p \in \mathcal{N}'} O^p \subseteq N$ which shows that $O_M \subseteq M$. $\square$

Lemma 3.15. *For each $N \in \mathcal{A}$, $\bigcap_{M^p \in \mathcal{N}'} O^p$ is contained in an unique element of $\mathcal{A}$, i.e., N.*

Proof. Observe that Lemma 3.10 shows that each N is a closed set in $C(X)$ with m-topology, i.e., $\bar{N} = \{p \in \beta X \mid N \subseteq M^P\}$ is closed in βX. If $\bigcap_{M^p \in \mathcal{N}'} O^p$ is contained in another element $N_1 \in \mathcal{A}$ then consider $\bar{N}_1$ and $q \in \bar{N}_1 \setminus \bar{N}$. Using complete regularity we can find out a function $f \in C^*(X)$ such that $\bar{N} \subseteq Z(f^\beta)$ and $f^\beta(q) = 1$. Therefore, $f \notin O^q$ though $f \in \bigcap_{M^p \in \mathcal{N}'} O^p$ which is a contradiction. $\square$

Theorem 3.16. *For every maximal clique M of $\mathcal{N}(X)$, O_M is a prime clique.*

Proof. Let $f, g \notin O_M$ such that f and g are both adjacent with every element of O_M. Suppose f and g are not adjacent to each other. Then $Z(f) \cap Z(g) = \emptyset$. Now Lemma 3.15 and the property that for each $N \in \mathcal{A}$, f is adjacent to all elements of $\bigcap_{M^p \in \mathcal{N}'} O^p$ shows that $f \in N$ and similarly, $g \in N$. This happens for each and every element $N \in \mathcal{A}$. Hence f and g are both contained in the same maximal clique M which contradicts the fact that f and g are not adjacent. $\square$

There may be a prime clique contained in a maximal clique M of $\Gamma(C(X))$ which is not of the form O_M, in fact, disjoint from O_M e.g., consider $\Gamma(C(\mathbb{R}))$ and the maximal clique M^p for some $p \in \mathbb{R}$. Now construct the clique $J^p = \{f \in \mathcal{N}(\mathbb{R}) \mid Z(f) = \{p\}\}$. Obviously, $J^p \subseteq M^p$ and J^p is a prime clique. But $J^p \cap O_{M^p} = \emptyset$, i.e., J^p is a prime clique which is contained in the complement of O_{M^p} in M^p. Also one can construct a maximal clique M of $\Gamma(C(X))$ such that the complement $M \setminus O_M$ contains no prime clique. Following is an example of such maximal clique.

Example 3.17. Consider the topological space $\lambda(\mathbb{N}^*)$, i.e., superextension of the remainder space $\mathbb{N}^*$. Assuming continuum hypothesis, $\mathbb{N}^*$ has both P-points as well as non P-points also. Let $p, q \in \mathbb{N}^*$ be two P-points of $\mathbb{N}^*$ and $r \in \mathbb{N}^*$ be a non P-point. Consider the maximal clique of $\Gamma(C(\mathbb{N}^*))$ corresponding to these three points $M_{pqr} = (M^p \cap M^q) \cup (M^q \cap M^r) \cup (M^r \cap M^p)$. Since p and q are P-points so $M^p = O^p$ and $M^q = O^q$. Therefore the complement of $O_{M_{pqr}}$ in M_{pqr} consists of those functions which are zero either on $\{p, r\}$ or on $\{q, r\}$ which shows that $M_{pqr} \setminus O_{M_{pqr}} \subseteq M^r$ and hence $M_{pqr} \setminus O_{M_{pqr}}$ can not be prime.

Observing this fact and keeping it in mind we define GP-space as following.

Definition 3.18. A Tychonoff space X is called a GP-space if for each maximal clique M of $\Gamma(C(X))$, the complement of O_M in M contains no prime clique.

For an example of GP-space one can observe that every finite space is a GP-space. Following theorem gives examples of GP-spaces other than finite spaces.

Theorem 3.19. *Every realcompact P-space is a GP-space.*

Proof. Let X be a realcompact P-space. Then for each $p \in \beta X$, O^p is maximal [[20], Theorem 14.29], i.e., $M^p = O^p$ and therefore for each maximal clique M of $\Gamma(C(X))$, $O_M = M$. Hence the theorem follows. □

One can observe that if X is first countable and not a realcompact P-space then there will be a fixed maximal clique M^q, $q \in X$ such that $M^q \setminus O^q$ is prime, because, in that case $M^q \setminus O^q$ contains a prime clique $\{f \in M^q \mid Z(f) = \{q\}\}$. Therefore first countable non-realcompact P-spaces can not be a GP-space. This observation shows that topological characterization of GP-spaces has some importance and we leave this as an open question:

Problem 3.20. *Find a topological characterization of GP-spaces.*

In connection of the above question, one may think that *almost P-spaces* may be one of the characterization of GP-spaces. We note that in [25], S.Watson constructed an example of a compact Hausdorff space without P-points which is an *almost P-space* but without lacking first countability.

4. Neighbourhood Properties

In this section we investigate some properties of the graph related to the neighbourhood of a vertex and show their connection with the ring properties of $C(X)$ and topological properties of X.

Lemma 4.1. *For any two $f, g \in \mathcal{N}(X)$, $N[f] \subseteq N[g]$ if and only if $Z(f) \subseteq Z(g)$, where $N[f]$ represents the closed neighbourhood of f in $\Gamma(C(X))$.*

Proof. If possible let $Z(f) \nsubseteq Z(g)$. Then there exists $p \in Z(f)$ such that $p \notin Z(g)$. Since X is complete regular, there exists $h \in C(X)$ such that $h(p) = 0$ and $h(Z(g)) = 1$, meaning that h is adjacent to f but not to g, i.e., $h \in N[f]$ but $h \notin N[g]$ which contradicts the fact that $N[f] \subseteq N[g]$. The other way implication is trivial. □

The above lemma shows an inter-relation between graph structure of $\Gamma(C(X))$ and topological structure of X. As an application, we get the following result which is an easy consequence of the above lemma.

Corollary 4.2. *If $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ is a graph isomorphism then for any two $f, g \in C(X)$, $Z(f) \subseteq Z(g)$ if and only if $Z(\varphi(f)) \subseteq Z(\varphi(g))$.*

For any $f \in \mathcal{N}(X)$, in general, $N[f]$ may not be a clique. In fact, if $Z(f)$ can be expressed as a union of two disjoint closed sets, then complete regularity ensures that $N[f]$ contains at least two vertices which are never adjacent to each other means that f is not a simplicial vertex. Simplicial property depends on the topology of X also. On the next theorem we characterize simplicial vertex for first countable topological spaces.

Theorem 4.3. *If X is first countable, then a vertex $f \in \mathcal{N}(X)$ is simplicial if and only if $Z(f)$ is singleton.*

Proof. Suppose $f \in \mathcal{N}(X)$ is simplicial, i.e., $N[f]$ is a clique. If not and if possible suppose $Z(f)$ is not a singleton set. Then there exists at least two distinct points $p, q \in Z(f)$ and hence due to first countability and complete regularity of X, there exists $g, h \in \mathcal{N}(X)$ such that $Z(g) = \{p\}$, $g(q) = 1$ and $Z(h) = \{q\}$, $h(p) = 1$. Therefore, $g, h \in N[f]$ though g and h are not adjacent, which contradicts our initial assumption. The converse part follows trivially. □

One can observe from the proof of the above theorem that for first countable topological space X and $f \in \mathcal{N}(X)$ such that $Z(f)$ is a singleton set, $N[f]$ is a maximal clique and that maximal clique is of 'fixed maximal clique' corresponding to the singleton point of $Z(f)$. In other way, clique corresponding to a simplicial vertex becomes fixed maximal clique. Keeping this in mind, we investigate whether the partial converse of this statement is true: that when every maximal clique becomes simplicial corresponding to some vertex of $\mathcal{N}(X)$. We know that if X is supercompact then all the maximal cliques are centered, i.e., they are fixed. Hence we can conclude the following

Theorem 4.4. *If X is first countable supercompact space then every maximal clique is a neighbourhood of some simplicial vertex.*

It is clear that every prime clique is contained in an unique maximal clique. But how to find that maximal clique is not clear. In the next theorem we give a way to find out that unique maximal clique containing a given prime clique using the neighbourhood property.

Theorem 4.5. *For a given prime clique P of $\Gamma(C(X))$, $M = \cap\{N[f] \mid f \in P\}$ is a maximal clique containing P.*

Proof. $P \subseteq M$ is trivial. First we prove that M is a clique. Let $h, k \in M$, i.e., h, k are adjacent with all the elements of P. Since P is prime therefore there is an edge between h and k. For maximality let $M \subset N$ where N is a maximal clique containing P. Let, $g \in N \setminus M$. Then $g \notin N[f]$ for some $f \in P$. This contradicts that N is a clique containing P. Hence the theorem follows. □

5. Γ(C(X)) and Graph Isomorphisms

In this section, we study the inter-relationships between graph isomorphisms of $\Gamma(C(X))$, ring isomorphisms of $C(X)$ and homeomorphisms of X. Finally, we show that graph isomorphisms in $\Gamma(C(X))$ is equivalent to ring isomorphisms in $C(X)$ as well as homeomorphism in X.

For $f \in \mathcal{N}(X)$, let us denote $I_f = \{g \in C(X) \mid Z(f) \subseteq Z(g)\}$. One can easily check that I_f is an ideal of $C(X)$. Corollary 4.2 shows that if $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ is a graph isomorphism then $\varphi(I_f) = I_{\varphi(f)}$.

Lemma 5.1. *Let $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ be a graph isomorphism and $N \subseteq \Gamma(C(X))$, then $\varphi(\cup_N I_f) = \cup_N \varphi(I_f)$.*

Proof. Let $\varphi(g) \in \varphi(\cup_N I_f)$ then $g \in \cup_N I_f$, i.e., $g \in I_f$ for some $f \in N$ and therefore $\varphi(g) \in \varphi(I_f)$, $f \in N$ which proves that $\varphi(\cup_N I_f) \subseteq \cup_N \varphi(I_f)$. For the other part, let $\varphi(g) \in \varphi(\cup_N I_f)$ for some $g \in \cup_N I_f$, i.e., $g \in I_{f'}$ for some $f' \in N$. Then $\varphi(g) \in \varphi(I_{f'})$ and hence $\varphi(g) \in \cup_N \varphi(I_f)$. □

Lemma 5.2. *If $N \subseteq \Gamma(C(X))$, then $\cup_N I_f \subseteq \{h \in \Gamma(C(X)) \mid \cap_N cl_{\beta X} Z(f) \subseteq cl_{\beta X} Z(h)\}$.*

Proof. Let $g \in \cup_N I_f$ then $g \in I_f$ for some $f \in N$. Therefore, $Z(f) \subseteq Z(g)$ and hence $\cap_N \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} Z(g)$ which imply that $g \in \{h \in \Gamma(C(X)) \mid \cap_N \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} Z(h)\}$. Hence the lemma follows. □

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Theorem 5.3. *If $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ is a graph isomorphism then for each $p \in \beta X$ there exists $q \in \beta Y$ such that $\varphi(M_p) = M_q$.*

Proof. Since $M^p = \cup_{f \in M^p} I_f$, therefore lemma 5.1 shows that $\varphi(M^p) = \cup_{f \in M^p} I_{\varphi(f)}$ and due to lemma 5.2, $\varphi(M^p) \subseteq \{h \in \Gamma(C(Y)) \mid \cap_{M^p} \mathrm{cl}_{\beta Y} Z(\varphi(f)) \subseteq \mathrm{cl}_{\beta Y} Z(h)\} = M(\text{say})$. Since βY is compact, $\cap_{M^p} \mathrm{cl}_{\beta Y} Z(\varphi(f)) \neq \emptyset$ and let it be Z. As a consequence of this fact M becomes a clique in $\Gamma(C(Y))$. Since M^p is a maximal clique in $\Gamma(C(X))$ and ϕ is a graph isomorphism, therefore, $\varphi(M^p) = M$ and also $Z = \{q\}$, a singleton set for some $q \in \beta Y$ which is an easy consequence of the fact that βY is a completely regular space. Hence the proof is complete. □

The importance of Theorem 5.3 is that any graph isomorphism preserves the algebraic structure. More precisely, any graph isomorphism takes a maximal clique of $\Gamma(C(X))$ to some maximal clique of $\Gamma(C(Y))$ and here, we have already observed that $\Gamma(C(X))$ has two types of maximal cliques, first type being the maximal ideals of $C(X)$ and other type being those which are not maximal ideals of $C(X)$. Therefore it may happen that for some graph isomorphism, a maximal clique of $\Gamma(C(X))$ of the form maximal ideal is mapped to a maximal clique which is not a maximal ideal. But incidentally, Theorem 5.3 assures that it is not the case.

Theorem 5.4. *If $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ is a graph isomorphism then φ induces a homeomorphism $\overline{\varphi} : \beta X \to \beta Y$ given by $\overline{\varphi}(M^p) = \varphi(M^p)$.*

Proof. It is clear that bijection of $\overline{\varphi}$ is a simple consequence of graph isomorphism property of φ. For continuity of $\overline{\varphi}$, we consider a basic closed set $M(f) = \{M^p : f \in M^p, p \in \beta X\}$ for some $f \in C(X)$. Then $\overline{\varphi}(M(f)) = M(\varphi(f))$, as well as $\overline{\varphi}^{-1}(M(g)) = M(\varphi^{-1}(g))$, for some $g \in C(Y)$. □

Theorem 5.4 and Theorem 5.9 together show that if we consider $Y = X$ and the space X to be first countable, then $\overline{\varphi}$ is a homeomorphism from βX to itself which one can interpret as the maximal clique M^p, $p \in \beta X$, of $\Gamma(C(X))$ will be mapped to another maximal clique M^q, $q \in \beta X$, of $\Gamma(C(X))$ by any graph isomorphism φ if and only if there exists a homeomorphism $\overline{\varphi}$ from βX to βX that map p to q. Therefore, mapping of one maximal clique of $\Gamma(C(X))$ into any other maximal clique of $\Gamma(C(X))$ under graph isomorphism is merely a direct consequence of homogeneity of βX. But βX is not homogeneous, meaning that, we can get at least two maximal cliques M^p and M^q ($p, q \in \beta X$ and $p \neq q$) which will never be mapped to each other by any graph isomorphism. It is now clear that the number of different kind of maximal cliques under this graph isomorphism sense is equivalent with the number of different equivalent classes under the relation homeomorphism on βX. Now using the Frolik's work [[26], 4.11] on "Type of points" on βX we can conclude the following

Theorem 5.5. *There are at least 2^c many different maximal cliques in $\Gamma(C(X))$, for first countable Tychonoff space X (in which $\mathbb{N}$ is C-embedded) of the form M^p, $p \in \beta X$ which are never graph isomorphic to each other.*

Since maximal cliques of the form M^p are also maximal ideals, Theorem 5.5, also holds for maximal ideals. Here, as a graph entity M^p and as an algebraic entity M^p, we have got the same consequence. If one try to understand through algebraic sense that, why it is happening to maximal ideals or more precisely which algebraic property of maximal ideals causes it not to isomorphic to other particular maximal ideals, one get an unclear idea. Rather, the same fact has an explanation through graph theoretic point of view. It happens due to the closed neighbourhood $N[M^p] = \cup\{N[f] \mid f \in M^p\}$ has different graph structure than the closed neighbourhood $N[M^q]$, where $p, q \in \beta X$ are two non-homogeneous points.

We have seen that a graph isomorphism preserves the maximal ideal structure in Theorem 5.3. In the next theorem we prove that not only the maximal ideal structure, the graph isomorphism also preserves more deeper algebraic structure of maximal ideals which is called real maximal ideals. Apparently, it shows that deeper algebraic structure is also preserved by a graph isomorphism. This happens due to "closed under countable intersection" property of real z-ultrafilters.

Theorem 5.6. *If $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ is a graph isomorphism, then φ induces a bijective map $\overline{\varphi} : \upsilon X \to \upsilon Y$ given by $\overline{\varphi}(M^p) = \varphi(M^p)$, where υX stands for the Hewitt real compactification of X.*

Proof. To prove this, it is sufficient to show that the image of a real maximal ideal under φ is again a real maximal ideal. Since, real maximal ideals are maximal cliques and φ is a graph isomorphism, image of a real maximal ideal under φ is a maximal clique. Let $M^p,\ p \in \upsilon X$ be a real maximal ideal. Therefore, $\varphi(M^p)$ is a maximal clique and it is also a maximal ideal (by Theorem 5.3). Now, to show $\varphi(M^p)$ is a real maximal ideal it is sufficient to show that $Z[\varphi(M^p)]$ has a countable intersection property. Let $\{Z(\varphi(f_i))\}$ be a countable collection of zero sets in $Z[\varphi(M^p)]$. Then $\{Z(f_i)\}$ is a countable collection of zero sets of $Z[M^p]$. Now, since M^p is a real maximal ideal, $Z[M^p]$ is closed under countable intersection, i.e., $\bigcap_{i=1}^{\infty} Z(f_i) = Z(f),$ for some $f \in M^p$. Then $Z(f) \subseteq Z(f_i), \forall i = 1, 2, \dots$ and due to Corollary 4.2, $Z(\varphi(f)) \subseteq Z(\varphi(f_i)), \forall i = 1, 2, \dots$ Hence $\phi \neq Z(\varphi(f)) \subseteq \bigcap_{i=1}^{\infty} Z(\varphi(f_i))$.Thus, $Z[\varphi(M^p)]$ has a countable intersection property and hence, $\varphi(M^p)$ is a real maximal ideal. It is also to be noted that $\overline{\varphi}$ is a bijection. □

Theorem 5.7. *If $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$ is a graph isomorphism, then the induced map $\overline{\varphi} : \upsilon X \to \upsilon Y$ given by $\overline{\varphi}(M^p) = \varphi(M^p)$ is a homeomorphism. Hence, $C(X)$ is isomorphic to $C(Y)$ as rings.*

Proof. At first, we show that $\overline{\varphi}$ maps basic closed sets of υX to basic closed sets of υY. Now considering υX as a real maximal ideal space, $M(f) = \{M^p : f \in M^p, p \in \upsilon X\}$ becomes a basic closed set in υX for $f \in C(X)$. Then as a direct application of Theorem 5.6, $\overline{\varphi}(M(f)) = M(\varphi(f))$ as well as $\overline{\varphi}^{-1}(M(g)) = M(\varphi^{-1}(g))$ for some $g \in C(Y)$ which conclude the theorem. □

Since $C(X)$ and $C(Y)$ are isomorphic in Theorem 5.7, one of the consequences of Theorem 5.7 is that βX and βY are homeomorphic. Again if X and Y are both realcompact topological spaces then $\upsilon X = X$ and $\upsilon Y = Y$ and hence we can conclude the following from Theorem 5.7.

Theorem 5.8. *For realcompact topological spaces X and Y, if $\Gamma(C(X))$ and $\Gamma(C(Y))$ are graph isomorphic then X and Y are homeomorphic.*

We can get the above result for non-realcompact space X also if we assume first countability on X. For a first countable topological space X every point is a G_δ-point but on the contrary no point of $\beta X \setminus X$ is G_δ in nature in βX [[20], Corollary 9.6] which leads us to conclude that the induced homeomorphism which we get in Theorem 5.7 is reduced to a homeomorphism from X onto Y if X and Y are both first countable topological spaces. Hence the following is an easy conclusion.

Theorem 5.9. *For first countable topological spaces X and Y, if $\Gamma(C(X))$ and $\Gamma(C(Y))$ are graph isomorphic then X and Y are homeomorphic.*

Theorem 5.10. *Let X and Y be two topological spaces such that $C(X)$ and $C(Y)$ are isomorphic as rings. Then $\Gamma(C(X))$ and $\Gamma(C(Y))$ are graph isomorphic.*

Proof. Let $\varphi : C(X) \to C(Y)$ be a ring isomorphism. Clearly, the restriction of φ on the set of non-units in $C(X)$ is also a bijection from $\Gamma(C(X))$ onto $\Gamma(C(Y))$. Without loss of generality, we denote that restricted bijection also by φ, i.e., we write $\varphi : \Gamma(C(X)) \to \Gamma(C(Y))$. The only thing left to be proved is that φ preserves adjacency. Let f and g be adjacent in $\Gamma(C(X))$, i.e., $Z(f^2 + g^2) = Z(f) \cap Z(g) \neq \phi$. This implies that $f^2 + g^2$ is a non-unit and as φ is a ring isomorphism, $\varphi(f^2 + g^2)$ is also a non-unit, i.e., $Z(\varphi(f^2 + g^2)) \neq \phi$. Now, $Z(\varphi(f)) \cap Z(\varphi(g)) = Z((\varphi(f))^2 + (\varphi(g))^2) = Z(\varphi(f^2 + g^2)) \neq \phi$. Similarly, it can be shown that $\varphi(f)$ and $\varphi(g)$ are adjacent in $\Gamma(C(Y))$ implies that f and g are adjacent in $\Gamma(C(X))$. Hence, $\Gamma(C(X))$ and $\Gamma(C(Y))$ are graph isomorphic. □

Corollary 5.11. *Let X and Y be two topological spaces. If X is homeomorphic to Y, then $\Gamma(C(X))$ is isomorphic to $\Gamma(C(Y))$.*

Proof. If X is homeomorphic to Y, then it is known that $C(X)$ and $C(Y)$ are isomorphic as ring. Thus, the corollary follows from Theorem 5.10. □

$$\begin{tikzcd} & \overset{\text{(as toplogical spaces)}}{X \cong Y}\text{(First Countable/ Realcompact)} \arrow[dl, shift left, Rightarrow] \arrow[dr, shift left, Rightarrow] & \\ \underset{\text{(as rings)}}{C(X) \cong C(Y)} \arrow[ur, shift left, Rightarrow] \arrow[rr, shift left] & & \underset{\text{(as graphs)}}{\Gamma(C(X)) \cong \Gamma(C(Y))} \arrow[ul, shift left, Rightarrow] \arrow[ll, shift left] \end{tikzcd}$$

Figure 1: The Combined Picture

The inter-relationship established in the above theorems can be represented diagrammatically as in Figure 1.

The inter relationship between graph and supercompact space shows that the space of maximal linked system of $\Gamma(C(X))$ is homeomorphic to $\lambda(X, Z[X])$ and the space of maximal linked system of $\Gamma(C(\beta X))$ is homeomorphic to $\lambda(\beta X, Z[\beta X])$. In general, $\lambda(X, Z[X])$ and $\lambda(\beta X, Z[\beta X])$ may not be topologically same, rather, one can at once conclude that $\lambda(X, Z[X])$ can be naturally embedded into $\lambda(\beta X, Z[\beta X])$. In the following theorem we prove a necessary condition on X for $\lambda(X, Z[X])$ to be homeomorphic with $\lambda(\beta X, Z[\beta X])$.

Theorem 5.12. *If X is pseudocompact then $\lambda(X, Z[X]) \cong \lambda(\beta X, Z[\beta X])$.*

Proof. Let X be a pseudocompact space. Then every function of $C(X)$ is bounded. First we claim that there is no $f \in C(X)$ such that $Z(f) = \emptyset$ but $Z(f^\beta) \neq \emptyset$, where f^β represents the extension of f to βX. If not and if possible let there exist such an $f \in C(X)$. Since $Z(f) = \emptyset$, f is invertible in $C(X)$ and hence $(\frac{1}{f})^\beta = \frac{1}{f^\beta}$. But $Z(f^\beta) \neq \emptyset$ implies that there exist point in βX where $\frac{1}{f^\beta}$ is infinite and hence $\frac{1}{f}$ is unbounded which contradicts the fact that X is pseudocompact.

The consequence of the above claim is that $\beta X \setminus X$ contains no zero set of βX. Hence one can conclude that all the zero set of βX are of the form $Z(f^\beta)$ for $f \in C(X)$, i.e., all the zero sets are basic closed sets when X is pseudocompact. Again, for X being pseudocompact, $C(X)$ is ring isomorphic to $C(\beta X)$ which comprise with Theorem 5.7 shows that $\Gamma(C(X))$ and $\Gamma(C(\beta X))$ are graph isomorphic and hence their maximal clique spaces are homeomorphic, i.e., $\lambda(X, Z[X])$ and $\lambda(\beta X, Z[\beta X])$ are homeomorphic. □

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