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Schatten-von Neumann Characteristic of Tensor Product Operators

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Abstract. In this paper, the relations between Schatten-von Neumann property of the tensor product of operators and Schatten-von Neumann property of its coordinate operators are studied.

1. Introduction

The general theory of singular numbers and operator ideals was given by Pietsch [7], [8] and the case of linear compact operators was investigated by Gohberg and Krein [6]. However, the first result in this area can be found in the works of Schmidt [9] and von Neumann, Schatten [10] who used these concepts in the theory of non-selfadjoint integral equations.

Later on, the main aim of mini-workshop held in Oberwolfach (Germany) was to present and discuss some modern applications of the functional-analytic concepts of $s-$numbers and operator ideals in areas like numerical analysis, theory of function spaces, signal processing, approximation theory, probability of Banach spaces and statistical learning theory (see [3]).

Let $\mathcal{H}$ be a Hilbert space, $S_\infty(\mathcal{H})$ be a class of linear compact operators in $\mathcal{H}$ and $A \in S_\infty(\mathcal{H})$. The eigenvalues of the operator $(A^*A)^{1/2} \in S_\infty(\mathcal{H})$ are called the $s-numbers$ of the operator A. We shall enumerate the nonzero s-numbers in decreasing order, taking account of their multiplicities, so that

$$s_n(A) = \lambda_n((A^*A)^{1/2}),\ n = 1, 2, ...$$

(see [6]).

The Schatten-von Neumann operator ideals are defined as

$$S_p(\mathcal{H}) = \left\{ A \in S_\infty(\mathcal{H}) : \sum_{n=1}^{\infty} s_n^p(A) < \infty \right\},\ 1 \le p < \infty$$

in [7], [8].

Throughout this paper, the algebra of linear bounded operators from any Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$ is denoted by $L(\mathcal{H}_1, \mathcal{H}_2)$. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, it is denoted by $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$.

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Now give few main definitions from [1].

Let $(H_k)_{k=1}^n$ be a finite sequence of separable Hilbert spaces and let $(e_j^{(k)})_{j=0}^{\infty}$ be an orthonormal basis in H_k. Consider the formal product

$$e_\alpha = e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes ... \otimes e_{\alpha_n}^{(n)} \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in Z_+^n = Z_+ \times Z_+ \times ... \times Z_+$ (n-times), i.e., we consider the ordered sequence $\left(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, ..., e_{\alpha_n}^{(n)}\right)$ and construct a Hilbert space spanned by the formal vectors (1) which are assumed to be an orthonormal basis of this space. The separable Hilbert space thus constructed is called the tensor product of the spaces $H_1, H_2, ..., H_n$ and is denoted by $H_1 \otimes H_2 \otimes ... H_n = \bigotimes_{k=1}^n H_k$. Its vectors have the form

$$f = \sum_{\alpha \in Z_+^n} f_\alpha e_\alpha \ (f_\alpha \in \mathbb{C}), \ \|f\|^2_{\bigotimes_{k=1}^n H_k} = \sum_{\alpha \in Z_+^n} |f_\alpha|^2 < \infty, \tag{2}$$

$$(f, g)_{\bigotimes_{k=1}^n H_k} = \sum_{\alpha \in Z_+^n} f_\alpha \overline{g_\alpha}, \ g = \sum_{\alpha \in Z_+^n} g_\alpha e_\alpha \in \bigotimes_{k=1}^n H_k.$$

Let $f^{(k)} = \sum_{j=0}^{\infty} f_j^{(k)} e_j^{(k)} \in H_k \ (k = 1, 2, ..., n)$ be some vectors. By definition,

$$f = f^{(1)} \otimes f^{(2)} \otimes ... \otimes f^{(n)} = \sum_{\alpha \in Z_+^n} f_{\alpha_1}^{(1)} f_{\alpha_2}^{(2)} ... f_{\alpha_n}^{(n)} e_\alpha. \tag{3}$$

The coefficients $f_\alpha = f_{\alpha_1}^{(1)} f_{\alpha_2}^{(2)} ... f_{\alpha_n}^{(n)}$ of decomposition (3) satisfy condition (2). Therefore, vector (3) belongs to $\bigotimes_{k=1}^n H_k$ and, in addition,

$$\|f\|_{\bigotimes_{k=1}^n H_k} = \prod_{k=1}^{\infty} \|f_k\|_{H_k}.$$

Let $(H_k)_{k=1}^n$ and $(G_k)_{k=1}^n$ be two sequence of Hilbert spaces and let $(A_k)_{k=1}^n$ be a sequence of operators $A_k \in L(H_k, G_k)$. The tensor product $A_1 \otimes A_2 \otimes ... \otimes A_n = \bigotimes_{k=1}^n A_k$ is defined by the formula

$$\begin{aligned} \left(\bigotimes_{k=1}^n A_k\right) f &= \left(\bigotimes_{k=1}^n A_k\right)\left(\sum_{\alpha \in Z_+^n} f_\alpha e_\alpha\right) \\ &= \sum_{\alpha \in Z_+^n} f_\alpha \left(A_1 e_{\alpha_1}^{(1)}\right) \otimes \left(A_2 e_{\alpha_2}^{(2)}\right) \otimes ... \otimes \left(A_n e_{\alpha_n}^{(n)}\right), \ f \in \bigotimes_{k=1}^n H_k. \end{aligned} \tag{4}$$

It is stated that the series on the right-hand side of (4) is weakly convergent in $\bigotimes_{k=1}^n G_k$ and defines the operator $\bigotimes_{k=1}^n A_k \in L\left(\bigotimes_{k=1}^n H_k, \bigotimes_{k=1}^n G_k\right)$. Furthermore,

$$\|\bigotimes_{k=1}^n A_k\| = \prod_{k=1}^n \|A_k\|.$$

Our aim in this paper is to study the relations between Schatten-von Neumann property of the tensor product of operators and Schatten-von Neumann property of its coordinate operators.

2. Schatten-von Neumann properties of tensor product operators

Let H_k be a Hilbert space, $A_k \in L(H_k)$ for $1 \leq k \leq n,\ n \in \mathbb{N}$ and

$$A = A_1 \otimes A_2 \otimes ... \otimes A_n : H \to H,$$

where $H = \bigotimes_{k=1}^{n} H_k$, be a tensor product of operators $A_k,\ k = 1, 2, ..., n$.

Throughout this paper, for the simplicity we assume that:
(1) if for some $j \geq 1,\ s_j(A_k) > 0$, then $s_j(A_k) < s_{j-1}(A_k)$ for any $1 \leq k \leq n$;
(2) if for different two vectors

$$\left(m_1^1, m_2^1, ..., m_n^1\right) \text{ and } \left(m_1^2, m_2^2, ..., m_n^2\right)$$

at least one of the numbers

$$s_{m_1^1}(A_1)s_{m_2^1}(A_2)...s_{m_n^1}(A_n) \text{ and } s_{m_1^2}(A_1)s_{m_2^2}(A_2)...s_{m_n^2}(A_n)$$

is not zero, then

$$s_{m_1^1}(A_1)s_{m_2^1}(A_2)...s_{m_n^1}(A_n) \neq s_{m_1^2}(A_1)s_{m_2^2}(A_2)...s_{m_n^2}(A_n).$$

Firstly, using the method in [11] we can generalize the following result.

Theorem 2.1. *The tensor product $A \in S_\infty(H)$ is nonzero and compact if and only if for $k = 1, 2, ..., n$ $A_k \in S_\infty(H_k)$ are both nonzero and compact.*

Now we give the main results of this paper.

Theorem 2.2. *Let $p \in [1, \infty)$. If $A_k \in S_p(H_k)$ for $k = 1, 2, ..., n$, then $A \in S_p(H)$.*

Proof. By Theorem 2.1, since $A_k \in S_\infty(H_k)$ for $k = 1, 2, ..., n$, then $A = A_1 \otimes A_2 \otimes ... \otimes A_n \in S_\infty(H)$. In this case it is clear

$$A^* = A_1^* \otimes A_2^* \otimes ... \otimes A_n^*,$$
$$A^*A = A_1^*A_1 \otimes A_2^*A_2 \otimes ... \otimes A_n^*A_n,$$
$$\sqrt{A^*A} = \sqrt{A_1^*A_1} \otimes \sqrt{A_2^*A_2} \otimes ... \otimes \sqrt{A_n^*A_n}.$$

On the other hand it is known that for the spectrum of the tensor product operator A the following relation

$$\sigma\left(\sqrt{A^*A}\right) = \sigma\left(\sqrt{A_1^*A_1}\right)\sigma\left(\sqrt{A_2^*A_2}\right)...\sigma\left(\sqrt{A_n^*A_n}\right).$$

is true (see [2]). Therefore

$$\{s_m(A) : m \geq 1\} = \{s_{m_1}(A_1)s_{m_2}(A_2)...s_{m_n}(A_n) :\ m_1 \geq 1,\ m_2 \geq 1, ...,\ m_n \geq 1\}.$$

It is easily to see that the series $\sum_{m=1}^{\infty} s_m^p(A)$ is equal to one of rearrangement series of

$$\sum_{\substack{m=1 \\ m=(m_1,...,m_n)}}^{\infty} s_{m_1}^p(A_1)s_{m_2}^p(A_2)...s_{m_n}^p(A_n).$$

Now we will investigate the convergence of last multiple series. From the knowing algebraic equality of the multiple series

$$\begin{aligned}\sum_{m=1}^{\infty} s_m^p(A) &= \sum_{\substack{m=1 \\ m=(m_1,...,m_n)}}^{\infty} s_{m_1}^p(A_1)s_{m_2}^p(A_2)...s_{m_n}^p(A_n) \\ &= \sum_{m_1=1}^{\infty} s_{m_1}^p(A_1) \sum_{m_2=1}^{\infty} s_{m_2}^p(A_2)... \sum_{m_n=1}^{\infty} s_{m_n}^p(A_n)\end{aligned}$$

we have that the series

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}...\sum_{m_n=1}^{\infty} s_{m_1}^p(A_1)s_{m_2}^p(A_2)...s_{m_n}^p(A_n)$$

is convergent (see [5]). Therefore, $A \in S_p(H)$. □

Theorem 2.3. *Let $A_k \in S_\infty(H_k)$, $p_k \in [1,\infty)$ for $k = 1,2,...,n$ and $p = \max\limits_{1\le k\le n} p_k$. If $A_k \in S_{p_k}(H_k)$ for $k = 1,2,...,n$, then $A \in S_p(H)$.*

Proof. Let $\|A_k\| \le 1$ for any $k = 1,2,...,n$. Then for $k = 1,2,...,n$ $0 \le s_{m_k}^p(A_k) \le s_{m_k}^{p_k}(A_k)$ holds. Then for any $k = 1,2,...,n$ the series $\sum\limits_{m_k=1}^{\infty} s_{m_k}^p(A_k)$ is convergent, i.e., $A_k \in S_p(H_k)$, $k = 1,2,...,n$, $p = \max\limits_{1\le k\le n} p_k$. Hence from the Theorem 2.2 implies that $A \in S_p(H)$.

Now, consider the general case of the compact operators A_k in H_k for $k = 1,2,...,n$. In this case, for the operator

$$T_k = \frac{1}{1+\|A_k\|}A_k : H_k \to H_k,\ k = 1,2,...,n$$

we have $\|T_k\| \le 1$. And also from the following relations

$$s_{m_k}^{p_k}\left(\frac{1}{1+\|A_k\|}A_k\right) \le \left(\frac{1}{1+\|A_k\|}\right)^{p_k} s_{m_k}^{p_k}(A_k) \le s_{m_k}^{p_k}(A_k),\ A_k \in S_{p_k}(H_k),\ k = 1,2,...,n$$

it is obtained that $T_k \in S_{p_k}(H_k)$, $k = 1,2,...,n$.

Then by the first part of proof is true

$$T = T_1 \otimes T_2 \otimes ... \otimes T_n \in S_p(H),\ p = \max_{1\le k\le n} p_k.$$

Therefore from the equality

$$\begin{aligned} A &= \left((1+\|A_1\|)\,E_1 \otimes (1+\|A_2\|)\,E_2 \otimes ... \otimes (1+\|A_n\|)\,E_n\right)(T_1 \otimes T_2 \otimes ... \otimes T_n), \\ & (1+\|A_1\|)\,E_1 \otimes (1+\|A_2\|)\,E_2 \otimes ... \otimes (1+\|A_n\|)\,E_n \in L\left(\bigotimes_{k=1}^{n} H_k\right) \end{aligned}$$

where $E_k : H_k \to H_k$, $k = 1,2,...,n$ are identity operators, and by the important theorem of compact operators in [4] it is established that $A \in S_p(H)$, $p = \max\limits_{1\le k\le n} p_k$. □

Theorem 2.4. *Let $A_k \in S_\infty(H_k)$, $k = 1,2,...,n$ and $1 \le p < \infty$. If $A \in S_p(H)$, then $A_k \in S_p(H)$ for $k = 1,2,...,n$.*

Proof. The any singular number $s_j(A)$ of the operator A can not be repeated infinite times in expression

$$s_{j_1}(A_1)s_{j_2}(A_2)...s_{j_n}(A_n)$$

which is product of different singular numbers $s_{j_1}(A_1)$, $s_{j_2}(A_2)$, ..., $s_{j_n}(A_n)$ of the operator A_k, $k = 1,2,...,n$, respectively. If it had been repeated infinite times, the series $\sum\limits_{j=1}^{\infty} s_j^p(A)$, $1 \le p < \infty$ would not converge and thus $A \notin S_p(H)$ would held. That is, the j-th singular numbers $s_j(A)$ of the operator A can be repeated finite times in expression $s_{j_1}(A_1)s_{j_2}(A_2)...s_{j_n}(A_n)$ which is product of different singular numbers $s_{j_1}(A_1)$, $s_{j_2}(A_2)$, ..., $s_{j_n}(A_n)$ of the operator A_k, $k = 1,2,...,n$, respectively.

From $A \in S_p(H)$ and the following relation

$$\sum_{j=1}^{\infty} s_j^p(A) = \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}...\sum_{j_n=1}^{\infty} s_{j_1}^p(A_1)s_{j_2}^p(A_2)...s_{j_n}^p(A_n) = \prod_{k=1}^{n}\left(\sum_{j_k=1}^{\infty} s_{j_k}^p(A_k)\right) < \infty,$$

we have $\sum\limits_{j_k=1}^{\infty} s_{j_k}^p(A_k) < \infty$ for any $k = 1,2,...,n$. This means that $A_k \in S_p(H_k)$ for each $k = 1,2,...,n$. □

Theorem 2.5. *Let* $1 \leq p < \infty$. *If* $A_k \in S_p(H_k)$, $k = 1, 2, ..., j-1, j+1, ..., n$, $A_j \in S_\infty(H_j)$ *and* $A \in S_p(H)$, *then* $A_j \in S_p(H_j)$.

Proof. The validity of this claim is clear from the following equality

$$\sum_{k=1}^{\infty} s_j^p(A_k) = \frac{1}{\prod_{\substack{k\neq j \\ k=1}}^{n} \left(\sum\limits_{j_k=1}^{\infty} s_{j_k}^p(A_k) \right)} \sum_{j=1}^{\infty} s_j^p(A).$$

□

From last Theorem 2.5 it is implies the following corollary.

Corollary 2.6. *Let* $1 \leq p < \infty$. *If at least one of* $k = 1, 2, ..., n$ $A_k \notin S_p(H_k)$, *then* $A \notin S_p(H)$.

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