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The Turán Number of the Graph $3P_5$

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Abstract. The Turán number $ex(n, H)$ of a graph H, is the maximum number of edges in a graph of order n which does not contain H as a subgraph. Let $Ex(n, H)$ denote all H-free graphs on n vertices with $ex(n, H)$ edges. Let P_i denote a path consisting of i vertices, and mP_i denote m disjoint copies of P_i. In this paper, we give the Turán number $ex(n, 3P_5)$ for all positive integers n, which partly solve the conjecture proposed by L. Yuan and X. Zhang [7]. Moreover, we characterize all extremal graphs of $3P_5$ denoted by $Ex(n, 3P_5)$.

1. Introduction

The graphs considered in this paper are simple and undirected. For a graph $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. Let the Turán number $ex(n, H)$ denote the maximum number of edges in a simple graph of order n which does not contain H as a subgraph. Let P_i denote a path of order i and C_q denote a cycle of order q, mP_i denote m disjoint copies of P_i. For two vertex disjoint graphs G and F by $G \cup F$ we denote the vertex disjoint union of G and F, and by $G + F$ the graph obtained from $G \cup F$ by joining all vertices between G and F. By $\overline{G}$ we denote the complement of the graph G. We denote by $N_G(v)$ the set of vertices adjacent to v in G, if $V' \subseteq V(G)$, then $N_G(V') = \bigcup_{v \in V'} N_G(v)$, and $deg_G(v) = |N_G(v)|$. For $u, v \in V(G)$, (u, v) is the edge between u and v, and for $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let $E(A, B) = \{e \in E(G) | e \cap A \neq \emptyset, e \cap B \neq \emptyset\}$, $G|_A$ denote the subgraph of G induced by A. For $\{v_1, v_2, \ldots, v_m\} \subseteq V(G)$, u is adjacent to $\{v_1, v_2, \ldots, v_m\}$ means that u is adjacent to each vertex in $\{v_1, v_2, \ldots, v_m\}$. The basic notions not defined in this paper can be found in [1].

In 1941, Turán [2] proved that the Turán graph $T_{r-1}(n)$ (balanced complete $(r-1)$-partite graph on n vertices) is the extremal graph without containing K_r as a subgraph. Later, Moon [3] and Simonovits [4] showed that $K_{k-1} + T_{r-1}(n - k + 1)$ is the unique extremal graph containing no kK_r for sufficient large n. In 1959, Erdős and Gallai [5] proved that $ex(n, P_k) \leq (k-2)n(1/2)$ with equality if and only if $n = (k-1)t$. In 2011, N. Bushaw and N. Kettle [6] determined $ex(n, kP_l)$ for arbitrary l, and n appropriately large relative to k and l.

For $F_m = P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_m}$, $k_1 \geq k_2 \geq \cdots \geq k_m$, Liu, Lidický and Palmer [7] extended N. Bushaw and N. Kettle's result and determined $ex(n, F_m)$ for n sufficiently large. But they didn't solve the case for n with minor conditions. In 2014, H. Bielak and S. Kieliszek [8] determined $ex(n, 3P_4)$ for all n. L. Yuan and X. Zhang [9, 10] determined the value of $ex(n, kP_3)$ for all n, and characterized all extremal graphs. Later, for

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small n, they determined $ex(n, F_m)$ for $k_1, k_2, \ldots, k_m$ are all even or there is at most one odd. If there are two odds, they just obtained a result for $ex(n, P_{2l+1} \cup P_3)$. Finally, they proposed an important conjecture.

For convenience, we introduce the following definitions first.

Definition 1.1. [12] Let $n \geq m \geq l \geq 3$ be given three positive integers. Then n can be written as $n = (m-1) + t(l-1) + r$, where $t \geq 0$ and $0 \leq r < l - 1$. Denote by

$$[n, m, l] \equiv \binom{m-1}{2} + t\binom{l-1}{2} + \binom{r}{2}.$$

Moreover, if $n \leq m - 1$, denote by $[n, m, l] \equiv \binom{n}{2}$.

Definition 1.2. [12] Let $s = \sum_{i=1}^{m} \lfloor \frac{k_i}{2} \rfloor$ and k_i be positive integers. If $n \geq s$, then we denote

$$[n, s] \equiv \binom{s-1}{2} + (s-1)(n-s+1).$$

Conjecture 1.3. *[10] Let $k_1 \geq k_2 \geq \cdots \geq k_m \geq 3$ and $k_1 > 3$. $F_m = P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_m}$, then*

$$ex(n, F_m) = \max\{[n, k_1, k_1], [n, k_1 + k_2, k_2], \ldots, [n, \sum_{i=1}^{m} k_i, k_m], [n, \sum_{i=1}^{m} \lfloor \frac{k_i}{2} \rfloor] + c\},$$

where $c = 1$ if all of $k_1, k_2, \ldots, k_m$ are odd, and $c = 0$ for otherwise. Moreover, the extremal graphs are

$$Ex(n, P_{k_1}), \ldots, K_{\sum_{i=1}^{m} k_i - 1} \cup Ex(n - \sum_{i=1}^{m} k_i + 1, P_{k_m})$$

and

$$K_{\sum_{i=1}^{m} \lfloor \frac{k_i}{2} \rfloor - 1} + (K_2 \cup \overline{K}_{n - \sum_{i=1}^{m} \lfloor \frac{k_i}{2} \rfloor - 1}) \quad \textit{if all of } k_1, k_2, \ldots, k_m \textit{ are odd,}$$

$$K_{\sum_{i=1}^{m} \lfloor \frac{k_i}{2} \rfloor - 1} + (\overline{K}_{n - \sum_{i=1}^{m} \lfloor \frac{k_i}{2} \rfloor + 1}) \quad \textit{otherwise.}$$

Later, H. Bielak and S. Kieliszek [11] partly confirmed the conjecture 1.3, they determined $ex(n, 2P_5)$ for all positive integers n and gave the extremal graph. In 2017, $ex(n, 2P_7)$ was determined by Y. Lan, Z. Qin and Y. Shi [12]. And $ex(n, P_5 \cup P_{2l+1})$ was proved by Y. Hu and H. Tian [13] recently. However, all of them studied two disjoint paths with odd vertices. In this paper, we consider three disjoint odd paths, and propose the result as follows.

Theorem 1.4. *Let n be a positive integer.*

$$ex(n, 3P_5) = \max\{[n, 15, 5], 5n - 14\}.$$

Moreover, the extremal graphs are K_n for $n < 15$, $K_{14} \cup H$ where $H \subset Ex(n - 14, P_5)$ for $15 \leq n < 24$ and $K_5 + (K_2 \cup \overline{K_{n-7}})$ for $n \geq 24$.

This result determine the value of $ex(n, 3P_5)$ for all positive integers n that partly confirm the conjecture 1.3, and characterize all extremal graphs of $3P_5$ denoted by $Ex(n, 3P_5)$. We will prove it detailedly in the next section.

2. Proof of Theorem 1.4

For convenience, we first present the following important lemma which is used to prove our result.

Lemma 2.1. *(Faudree and Schelp [14]). If G is a graph with* $|V(G)| = kn+r (0 \le k, 0 \le r < n)$ *and G contains no* P_{n+1}*, then* $|E(G)| \le kn(n-1)/2+r(r-1)/2$ *with equality if and only if* $G = kK_n \cup K_r$ *or* $G = tK_n \cup (K_{(n-1)/2} + \overline{K}_{(n+1)/2+(k-t-1)n+r})$*, for some* $0 \le t < k$*, where n is odd, and* $k > 0, r = (n \pm 1)/2$*.*

Corollary 2.2. *Let n be a positive integer and* $n \equiv r \pmod 4$*. Then* $ex(n, P_5) = \lfloor \frac{n}{4} \rfloor \binom{4}{2} + \binom{r}{2} = \frac{3n+r(r-4)}{2}$*.*

Lemma 2.3. *(Erdős, Gallai [5]). Suppose that* $|V(G)| = n$*. If the following inequality*

$$\frac{(n-1)(l-1)}{2} + 1 \le |E(G)|$$

is satisfied for some $l \in N$*, then there exists a cycle* $C_q \subset G$ *for some* $q \ge l$*.*

Proof. [Proof of Theorem 1.4] Obviously, the extremal graph K_n gives the lower and upper bounds of $ex(n, 3P_5)$ for $n < 15$. Thus, $ex(n, 3P_5) = \binom{n}{2}$ for $n < 15$.

For $15 \le n < 24$ (see Table 1), $\mathcal{H}$ does not contain $3P_5$ as a subgraph, so $E(\mathcal{H})$ gives the lower bounds on $ex(n, 3P_5)$ for respective n. For $n \ge 24$, note that the graph $G = K_5 + (K_2 \cup \overline{K_{n-7}})$ dose not contain $3P_5$ as a subgraph, this also gives us the lower bounds, $ex(n, 3P_5) \ge 5n - 14$. Let $\delta = |E(\mathcal{H})| - (5n - 14)$, and $\delta = 0$ for $n \ge 24$.

Therefore, we would like to prove that $5n - 14 + \delta$ is the upper bound for $n \ge 15$. Let us assume that there exists a graph G such that $|V(G)| = n, |E(G)| = 5n - 13 + \delta$ and without a subgraph $3P_5$. Applying Lemma 2.3 to the graph G, we obtain

$$\frac{(n-1)(l-1)}{2} + 1 \le 5n - 13 + \delta,$$

$$l \le 11 - \frac{18 - 2\delta}{n-1}.$$

We get G contains a C_q, table 1 gives the value of q for $15 \le n < 24$; for $n \ge 24, \delta = 0$, we get $l \le 10$, then $q \ge 10$. Let $0, 1, 2, \ldots, q-1$ be the consecutive vertices in C_q.

n	$\mathcal{H}$	$\lvert E(\mathcal{H})\rvert$	q	$5n-14$	δ
15	$K_{14} \cup K_1$	91	14	61	30
16	$K_{14} \cup K_2$	92	13,14	66	26
17	$K_{14} \cup K_3$	94	12,13,14	71	23
18	$K_{14} \cup K_4$	97	12,13,14	76	21
19	$K_{14} \cup K_4 \cup K_1$	97	11,12,13,14	81	16
20	$K_{14} \cup K_4 \cup K_2$	98	11,12,13,14	86	12
21	$K_{14} \cup K_4 \cup K_3$	100	11,12,13,14	91	9
22	$K_{14} \cup 2K_4$	103	10,11,12,13,14	96	7
23	$K_{14} \cup 2K_4 \cup K_1$	103	10,11,12,13,14	101	2

Table 1: The lower bounds on $ex(n, 3P_5)$ for $15 \le n < 24$, with the cycle $C_q \subset G$.

We should consider the following cases:

case 1. $q \ge 15$. We have P_{15} in C_q, then $3P_5$ is a subgraph of G, a contradiction.

case 2. $q = 14$. Let $F = G - V(C_{14})$. Note that there are no edges between C_{14} and F, otherwise for some $f \in V(F)$, without loss of generality, let $(f, 0) \in E(G)$, then we get a $P_{15} = f\ 0\ 1\ 2\ \ldots\ 11\ 12\ 13$, so $3P_5$ is a subgraph of G. The minimum number of edges in F is equal to $5n-13+\delta-\binom{14}{2}=5n-104+\delta$. By Corollary 2.2,

$$ex(n-14,P_5)=\frac{3(n-14)+r(r-4)}{2},$$

where $n-14\equiv r \pmod 4$. We get $ex(n-14,P_5)<5n-104+\delta$ for $n>\frac{166}{7}-\frac{r(4-r)}{7}-\frac{2\delta}{7}$. Therefore, for $n\geq 15$ with different δ, we get P_5 in F, then there exists $3P_5$ in G, a contradiction.

Remark 2.4 If we connect a vertex to two adjacent vertices in cycle simultaneously, we will get a longer cycle. For example, there is a cycle $C=v_0v_1\dots v_nv_0$, without loss of generality, let the vertex u be adjacent to v_0 and v_1, then we get a longer cycle $C'=v_0uv_1\dots v_nv_0$. When a vertex is adjacent to some vertices in a complete graph, some edges in complete graph should be deleted to avoid creating a longer cycle. For instance, there is a complete graph K_n, $V(K_n)=\{v_1,v_2,\dots,v_n\}$, the longest cycle is C_n. Let the vertex u be adjacent to v_i and v_j, $i,j\in\{1,2,\dots,n\}$, $j>i+1$. Then we need to delete the edge (v_{i+1},v_{j+1}), since otherwise we get a longer cycle $C_{n+1}=uv_iv_{i-1}\dots 0\dots v_{j+1}v_{i+1}v_{i+2}\dots v_ju$. In the same way, the edge (v_{i-1},v_{j-1}) also should be deleted.

case 3. $q=13$. By table 1, for $n=15$ with 91 edges, G does not contain C_{13}, so we just consider the situation for $n\geq 16$. Let $F=G-V(C_{13})$. If there does not exist any edge between C_{13} and F, then similar to the case 2, $|E(F)|\geq 5n-13+\delta-\binom{13}{2}>ex(n-13,P_5)$ for $n>\frac{143}{7}-\frac{r(4-r)}{7}-\frac{2\delta}{7}$, where $(n-13)\equiv r \pmod 4$. Therefore, for $n\geq 16$ with different δ, there exists P_5 in F, and we get $3P_5$ in G, a contradiction.

Let V_i denote the vertex set that each vertex from V_i has exactly i neighbors in C_q, $i=0,1,2,\dots,q-1$, $V_{>0}=V(F)-V_0$. Note that in this case, $V_{>0}$ is an independent set, and the vertices from V_0 can be connected only between themselves.

Without loss of generality, let $(f,0)\in E(G)$ for some $f\in V_{>0}$. Then for all $f'\in V_{>0}-f$, $N_{C_{13}}(f')\subseteq\{0,3,5,8,10\}$, otherwise, if f' is adjacent to 1 or 4, we get P_5 with $\{f',1,2,3,4\}$, and $P_{10}=5\ 6\ \dots\ 11\ 12\ 0\ f$; if $(f',2)\in E(G)$, we get $P_5=f\ 0\ 1\ 2\ f', P_5'=3\ 4\ 5\ 6\ 7, P_5''=8\ 9\ 10\ 11\ 12$; if $(f',6)\in E(G)$, we get $P_5=1\,2\,3\,4\,5, P_5'=f'\ 6\,7\,8\,9, P_5''=10\ 11\ 12\ 0\ f$; the other situations are similar to above with symmetry. If f is adjacent to other vertex on cycle, the property is preserved, that is if f is adjacent to v_i, $v_i\in V(C_{13})$, then $N_{C_{13}}(f')\subseteq S_{v_i}=\{v_i,v_{i+3},v_{i+5},v_{i+8},v_{i+10}\}$ (If $i+3\geq 13$, then f' is adjacent to v_{i+3-13}, the rest may be deduced by analogy). With the property, if f is adjacent to $\{v_{i_1},v_{i_2},\dots,v_{i_t}\}\subset V(C_{13})$, then $N_{C_{13}}(f')\subseteq S_{v_{i_1}}\cap S_{v_{i_2}}\cap\dots\cap S_{v_{i_t}}$. By Remark 2.4, f can be adjacent to nonadjacent vertices on C_{13}, so $|N_{C_{13}}(f)|\leq 6$. Now we consider the following subcases:

case 3.1. For all $f\in V_{>0}$, $|N_{C_{13}}(f)|\leq 3$ (see Figure 1). To make the edges of G as more as possible, let $|N_{C_{13}}(f)|=3$, the only situation is that $N_{C_{13}}(f)=\{0,3,8\}$. By Remark 2.4, there are at least 6 edges should be deleted from K_{13}, the dotted lines in the figure denote the edges in $E(\overline{G})$. We get

$$|E(G)|\leq\binom{13}{2}+3|V_{>0}|-6+ex(|V_0|,P_5)\leq\binom{13}{2}+3(n-13)-6-\frac{3}{2}|V_0|-\frac{r(4-r)}{2}<5n-13+\delta,$$

for $n>23-\frac{r(4-r)}{4}-\frac{\delta}{2}$, where $|V_0|\equiv r \pmod 4$. Therefore, for $n\geq 16$ with different δ, there exists $3P_5$ in G, a contradiction.

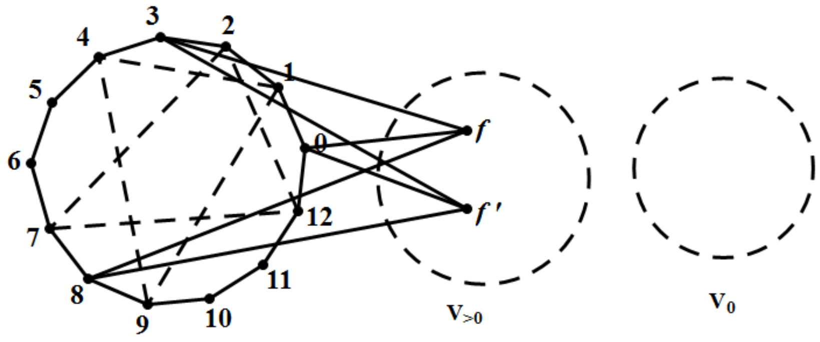

Figure 1: A graph with C_{13}, and all vertices in $V_{>0}$ have three neighbors in C_{13}.

case 3.2. There exists some $f \in V_{>0}$ that $|N_{C_{13}}(f)| = 4$ (see Figure 2). Then $|N_{C_{13}}(f')| \le 2$ for all $f' \in V_{>0} - f$, such as $N_{C_{13}}(f) = \{0, 3, 8, 11\}$ (or $\{0, 5, 8, 10\}$), $N_{C_{13}}(f') = \{3, 8\}$ (or $\{0, 5\}$). By Remark 2.4, there are at least 12 edges should be deleted from K_{13}. We get

$$|E(G)| \le \binom{13}{2} + 2(|V_{>0}| - 1) + 4 - 12 + ex(|V_0|, P_5) \le \binom{13}{2} + 2(n - 13) - 10 - \frac{1}{2}|V_0| - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{55}{3} - \frac{r(4-r)}{6} - \frac{\delta}{3}$, where $|V_0| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

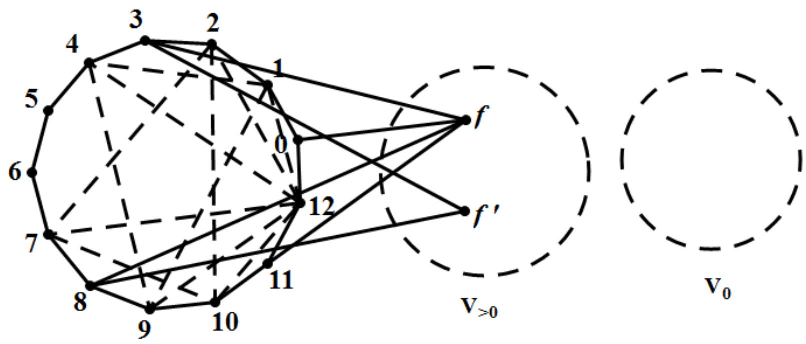

Figure 2: A graph with C_{13}, and for some $f \in V_{>0}$, $|N_{C_{13}}(f)| = 4$.

case 3.3. There exists some $f \in V_{>0}$ that $|N_{C_{13}}(f)| = 5$. Then $|N_{C_{13}}(f')| \le 1$ for all $f' \in V_{>0} - f$, such as $N_{C_{13}}(f) = \{0, 2, 5, 7, 10\}$ (or $\{0, 3, 5, 8, 10\}$), $N_{C_{13}}(f') = 10$ (or 0). By Remark 2.4, there are at least 19 edges should be deleted from K_{13}. We get

$$|E(G)| \le \binom{13}{2} + |V_{>0}| - 1 + 5 - 19 + ex(|V_0|, P_5) \le \binom{13}{2} + \frac{3}{2}(n - 13) - 15 - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{113}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 3.4. There exists some $f \in V_{>0}$ that $|N_{C_{13}}(f)| = 6$. Then $N_{C_{13}}(f') = \emptyset$ for all $f' \in V_{>0} - f$. By Remark 2.4, there are at least 20 edges should be deleted from K_{13}, we get

$$|E(G)| \le \binom{13}{2} + 6 - 20 + ex(n - 14, P_5) \le \binom{13}{2} - 14 + \frac{3}{2}(n - 14) - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{112}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $(n - 14) \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 4. $q = 12$. By table 1, we just consider the situation for $n \geq 17$. Let $F = G - V(C_{12})$. Note that $V_{>0} \neq \emptyset$, since otherwise $E(F) \geq 5n - 13 + \delta - \binom{12}{2} > ex(n-12, P_5)$ for $n > \frac{122}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $(n-12) \equiv r \pmod 4$. Therefore, for $n \geq 17$ with different δ, there exists P_5 in F, and we get $3P_5$ in G, a contradiction.

case 4.1. Suppose that there exists $P_2 = f_1 f_2$ in $G|_{V_{>0}}$.

Since $f_1, f_2 \in V_{>0}$, there are no more edges in $G|_{V_{>0}}$. Without loss of generality, let $(f_1, 0) \in E(G)$. For $f' \in V_{>0} - \{f_1, f_2\}$, $N_{C_{12}}(f') \subseteq \{0, 2, 5, 7, 10\}$, otherwise, if f' is adjacent to 1 or 4, we get P_5 with $\{f', 1, 2, 3, 4\}$, and $P_{10} = 5\ 6\ \ldots\ 11\ 0\ f_1\ f_2$; if f' is adjacent to 3 or 6, we get P_5 with $\{f', 3, 4, 5, 6\}$, and $P'_5 = 7\ 8\ 9\ 10\ 11, P''_5 = f_2, f_1, 0, 1, 2$; the other situations are similar to above with symmetry. Moreover, f_1 and f_2 are symmetric, if f_2 is adjacent to the vertices on cycle, the property is preserved, that is if f_2 is adjacent to v_i, $v_i \in V(C_{12})$, then $N_{C_{12}}(f') \subseteq S_{v_i} = \{v_i, v_{i+2}, v_{i+5}, v_{i+7}, v_{i+10}\}$ (If $i + 2 \geq 12$, then f' is adjacent to v_{i+2-12}, the rest may be deduced by analogy). With the property, if $N_{C_{12}}(f_2) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\} \subset V(C_{12})$, then $N_{C_{12}}(f') \subseteq S_{v_{i_1}} \cap S_{v_{i_2}} \cap \cdots \cap S_{v_{i_t}}$. What's more, by Remark 2.4, f_2 can be adjacent to nonadjacent vertices in C_{12}. And f_2 can be adjacent to nonadjacent vertices in $\{0, 3, 4, 5, 6, 7, 8, 9\}$, otherwise, if $(f_2, 2) \in E(G)$, there exists a longer cycle $C_{13} = 0\ f_1\ f_2\ 2\ 3\ \ldots\ 10\ 11\ 0$, the situations of vertices 1, 10, 11 are similar. So $|N_{C_{12}}(f_2)| \leq 5$. Now we consider the following subcases:

case 4.1.1. $|N_{C_{12}}(f_2)| = 1$ (see Figure 3). Without loss of generality, let $N_{C_{12}}(f_1) = N_{C_{12}}(f_2) = 0$, as previously mentioned, f' has at most five neighbors in C_{12}. When $|N_{C_{12}}(f')| = 5$, by Remark 2.4, there are at least 17 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + 5(|V_{>0}| - 2) + 3 - 17 + ex(|V_0|, P_5) \leq \binom{12}{2} + 5(n-12) - 24 - \frac{1}{2}|V_0| - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $-5 - \frac{r(4-r)}{2} - \delta < 0$, where $|V_0| \equiv r \pmod 4$. So there exists $3P_5$ in G, a contradiction.

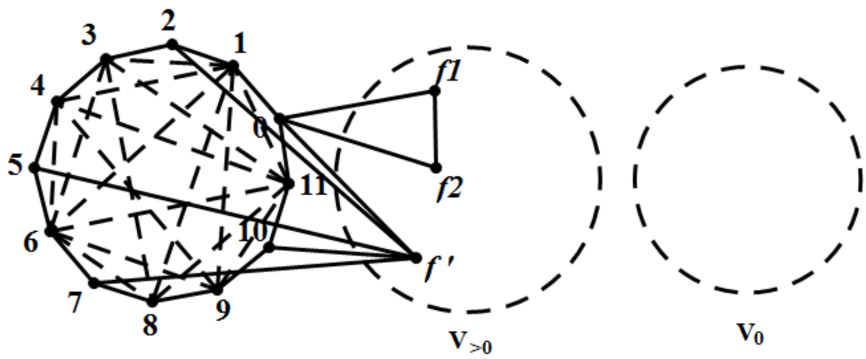

Figure 3: A graph with C_{12} and P_2 in $V_{>0}$, $|N_{C_{12}}(f_2)| = 1$.

case 4.1.2. $|N_{C_{12}}(f_2)| = 2$ (see Figure 4). Then $|N_{C_{12}}(f')| \leq 4$. When $|N_{C_{12}}(f')| = 4$: if $N_{C_{12}}(f_2) = \{0, 5\}$(or $\{0, 7\}$), then $N_{C_{12}}(f') = \{0, 5, 7, 10\}$(or $\{0, 2, 5, 7\}$), meanwhile, $N_{C_{12}}(f_1) = \{0, 5\}$(or $\{0, 7\}$). By Remark 2.4, there are at least 11 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + 4(|V_{>0}| - 2) + 5 - 11 + ex(|V_0|, P_5) \leq \binom{12}{2} + 4(n-12) - 14 - \frac{5}{2}|V_0| - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > 17 - \frac{r(4-r)}{2} - \delta$, where $|V_0| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

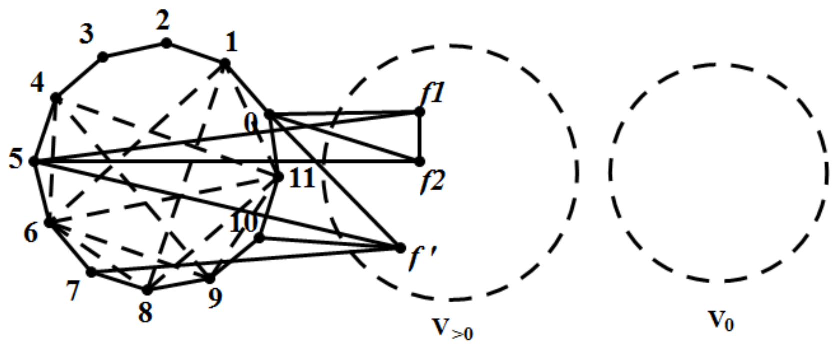

Figure 4: A graph with C_{12} and P_2 in $V_{>0}$, $|N_{C_{12}}(f_2)| = 2$.

case 4.1.3. $|N_{C_{12}}(f_2)| = 3$ (see Figure 5). Then $|N_{C_{12}}(f')| \leq 3$. When $|N_{C_{12}}(f')| = 3$: if $N_{C_{12}}(f_2) = \{0,5,7\}$, then $N_{C_{12}}(f') = \{0,5,7\}$, meanwhile, $N_{C_{12}}(f_1) = \{0,5\}$(or $\{0,7\}$). By Remark 2.4, there are at least 6 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + 3(|V_{>0}| - 2) + 6 - 6 + ex(|V_0|, P_5) \leq \binom{12}{2} + 3(n-12) - 6 - \frac{3}{2}|V_0| - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{37}{2} - \frac{r(4-r)}{4} - \frac{\delta}{2}$, where $|V_0| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

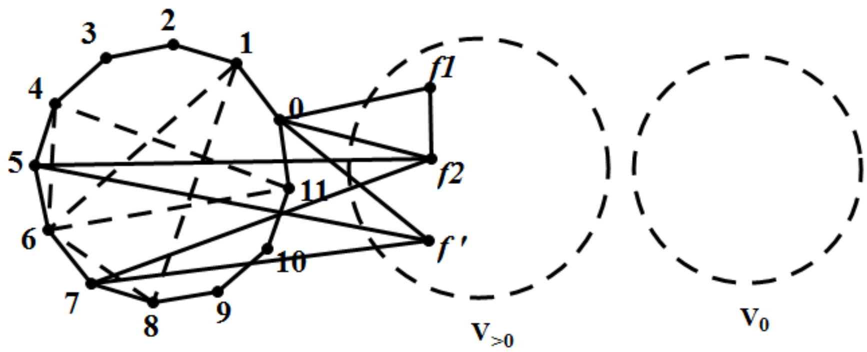

Figure 5: A graph with C_{12} and P_2 in $V_{>0}$, $|N_{C_{12}}(f_2)| = 3$.

case 4.1.4. $|N_{C_{12}}(f_2)| = 4$, $|N_{C_{12}}(f')| \leq 1$. When $|N_{C_{12}}(f')| = 1$: if $N_{C_{12}}(f_2) = \{0,3,5,7\}$ (or $\{0,5,7,9\}$), then $N_{C_{12}}(f') = 5$(or 7). By Remark 2.4, there are at least 9 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + (|V_{>0}| - 2) + 6 - 9 + ex(|V_0|, P_5) \leq \binom{12}{2} + \frac{3}{2}(n-12) - 5 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{116}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$. There exists a $3P_5$ in G, a contradiction.

case 4.1.5. $|N_{C_{12}}(f_2)| = 5$, that is $N_{C_{12}}(f_2) = \{0,3,5,7,9\}$, then $N_{C_{12}}(f') = \emptyset$. By Remark 2.4, there are at least 17 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + 7 - 17 + ex(n-14, P_5) \leq \binom{12}{2} - 10 + \frac{3}{2}(n-14) - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{96}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 4.2. Suppose that $V_{>0}$ is an independent set.

Let $V_{0+} = \{g \in V_0 | N(g) \cap V_{>0} \neq \emptyset\}$ and $V_{0-} = V_0 - V_{0+}$. Let us consider the following subcases:

case 4.2.1. $V_{0+} \neq \emptyset$.

There exists at least one vertex f in $V_{>0}$ that has neighbors in V_{0+}, then V_{0+} is an independent set and $deg(g) = 1, \forall g \in V_{0+}$, it means that $|E(V_{>0}, V_{0+})| = |V_{0+}|$.

For $|N_{C_{12}}(f)| = 1$, (see Figure 6). Similar to the case 4.1, $f' \in V_{>0} - \{f\}$ has at most five neighbors in C_{12}. If $|N_{C_{12}}(f')| = 5$, by Remark 2.4, there are at least 17 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + 5(|V_{>0}|-1)+1-17+|V_{0+}|+ex(|V_{0-}|, P_5) \leq \binom{12}{2}+5(n-12)-21-4|V_{0+}|-\frac{7}{2}|V_{0-}|-\frac{r(4-r)}{2} < 5n-13+\delta,$$

for $-2 - \frac{r(4-r)}{2} - \delta < 0$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

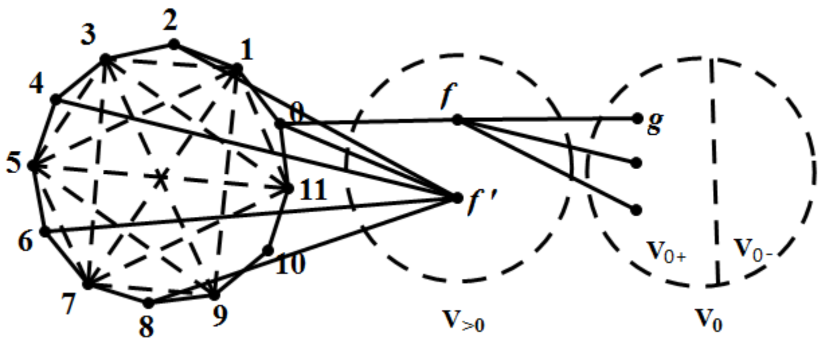

Figure 6: A graph with C_{12} and $V_{0+} \neq \emptyset$.

For $|N_{C_{12}}(f)| = 2$, $|N_{C_{12}}(f')| \leq 4$. When $|N_{C_{12}}(f')| = 4$: if $N_{C_{12}}(f) = \{0, 5\}$(or $\{0, 7\}$), then $N_{C_{12}}(f') = \{0, 5, 7, 10\}$(or $\{0, 2, 5, 7\}$). By Remark 2.4, there are at least 11 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2}+4(|V_{>0}|-1)+2-11+|V_{0+}|+ex(|V_{0-}|, P_5) \leq \binom{12}{2}+4(n-12)-13-3|V_{0+}|-\frac{5}{2}|V_{0-}|-\frac{r(4-r)}{2} < 5n-13+\delta,$$

for $n > 18 - \frac{r(4-r)}{2} - \delta$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

For $|N_{C_{12}}(f)| = 3$, $|N_{C_{12}}(f')| \leq 3$. When $|N_{C_{12}}(f')| = 3$: if $N_{C_{12}}(f_2) = \{0, 5, 7\}$, then $N_{C_{12}}(f') = \{0, 5, 7\}$. By Remark 2.4, there are at least 6 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2}+3(|V_{>0}|-1)+3-6+|V_{0+}|+ex(|V_{0-}|, P_5) \leq \binom{12}{2}+3(n-12)-6-2|V_{0+}|-\frac{3}{2}|V_{0-}|-\frac{r(4-r)}{2} < 5n-13+\delta,$$

for $n > \frac{37}{2} - \frac{r(4-r)}{4} - \frac{\delta}{2}$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

For $|N_{C_{12}}(f)| = 4$, $|N_{C_{12}}(f')| \leq 1$. When $|N_{C_{12}}(f')| = 1$: if $N_{C_{12}}(f_2) = \{0, 3, 5, 7\}$(or $\{0, 5, 7, 9\}$), then $N_{C_{12}}(f') = 5$(or 7). By Remark 2.4, there are at least 9 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + (|V_{>0}| - 1) + 4 - 9 + |V_{0+}| + ex(|V_{0-}|, P_5) \leq \binom{12}{2} + \frac{3}{2}(n-12) - 6 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{110}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

For $|N_{C_{12}}(f)| = 5$, $N_{C_{12}}(f') = \emptyset$, by Remark 2.4, there are at least 17 edges should be deleted from K_{12}. We get

$$|E(G)| \leq \binom{12}{2} + 5 - 17 + |V_{0+}| + ex(|V_{0-}|, P_5) \leq \binom{12}{2} - 12 + \frac{3}{2}(n-12) - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > 14 - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 4.2.2. $V_{0+} = \emptyset$.

By Remark 2.4, the vertices in $V_{>0}$ can be adjacent to at most 6 vertices in C_{12}. Let $V'_i = V_{>0} - V_i, i = 1,2,3,4,5,6$.

case 4.2.2.1. $V_6 \neq \emptyset$. We have $|V_6| = 1$ and $|V_1| = |V_{>0}| - 1$, or $|V_6| = 2$ and $V'_i = \emptyset$, otherwise there exists $3P_5$ in G. For the two situations, there are at least 15 edges should be deleted from K_{12}. We obtain

$$|E(G)| \leq \binom{12}{2} + 6 \times 1 - 15 + |V_{>0}| - 1 + ex(|V_0|, P_5) < 5n - 13 + \delta,$$

for $n > \frac{102}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$, or

$$|E(G)| \leq \binom{12}{2} + 6 \times 2 - 15 + ex(|V_0|, P_5) < 5n - 13 + \delta$$

for $n > \frac{110}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$. We get $3P_5$ in both situations, a contradiction.

case 4.2.2.2. $V_6 = \emptyset, V_5 \neq \emptyset$. We have $|V_5| = 1$ and $|N_{C_{12}}(V'_i)| \leq 2$, or $|V_5| = 2$ and $V'_i = \emptyset$ otherwise there exists $3P_5$ in G. For the two situations, there are at least 14 edges should be deleted from K_{12}. We obtain

$$|E(G)| \leq \binom{12}{2} + 5 \times 1 - 14 + 2(|V_{>0}| - 1) + ex(|V_0|, P_5) < 5n - 13 + \delta,$$

for $n > \frac{44}{3} - \frac{r(4-r)}{6} - \frac{\delta}{3}$, where $|V_0| \equiv r \pmod 4$, or

$$|E(G)| \leq \binom{12}{2} + 5 \times 2 - 14 + ex(|V_0|, P_5) < 5n - 13 + \delta,$$

for $n > \frac{108}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$. We get $3P_5$ in both situations, a contradiction.

case 4.2.2.3. $V_6 = V_5 = \emptyset$. For $V_i, i = 1,2,3,4$, let V_j be the first nonempty set in V_4, V_3, V_2, V_1, by Remark 2.4, there are at least $\binom{j+1}{2} - 1$ edges should be deleted from K_{12}. Then

$$|E(G)| \leq \binom{12}{2} + \sum_{i=1}^{4} i \cdot |V_i| - (\binom{j+1}{2} - 1) + ex(|V_0|, P_5) < 5n - 13 + \delta,$$

for $n \geq 17$ with different δ, we get $3P_5$ in G, a contradiction.

case 5. $q = 11$. By table 1, we just consider the situation for $n \geq 19$. Let $F = G - V(C_{11})$. Note that $V_{>0} \neq \emptyset$, since otherwise $E(F) \geq 5n - 13 + \delta - \binom{11}{2} > ex(n - 11, P_5)$ for $n > \frac{103}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $(n - 11) \equiv r \pmod 4$. Therefore, for $n \geq 19$ with different δ, there exists P_5 in F, and we get $3P_5$ in G, a contradiction.

case 5.1. Suppose that there exists $P_3 = f_1 f_2 f_3$ in $G|_{V_{>0}}$.

Since $f_1, f_2, f_3 \in V_{>0}$, there exists exactly one P_3 and no more edges in $V_{>0}$, $V_{0+} = \emptyset$. Without loss of generality, let $(f_1, 0) \in E(G)$. For $f' \in V_{>0} - \{f_1, f_2, f_3\}$, $N_{C_{11}}(f') \subseteq \{3, 8\}$, otherwise, if f' is adjacent to 0, we get $P_5 = f'\ 0\ f_1\ f_2\ f_3$, and $P_{10} = 1\ 2\ \ldots\ 10$; if f' is adjacent to 1 or 4, we get P_5 with $\{f', 1, 2, 3, 4\}$, and $P_{10} = 5\ 6\ \ldots\ 10\ 0\ f_1\ f_2\ f_3$; if f' is adjacent to 2 or 5, we get P_5 with $\{f', 2, 3, 4, 5\}$, and $P'_5 = 6\ 7\ 8\ 9\ 10, P''_5 = 1\ 0\ f_1\ f_2\ f_3$; the other situations are similar to above with symmetry. Moreover, f_1 and f_3 are symmetric, if f_3 is adjacent to other vertices on cycle, the property is preserved, that is if f_3 is adjacent to v_i, $v_i \in V(C_{11})$, then f' can be adjacent to $S_{v_i} = \{v_{i+3}, v_{i+8}\}$ (If $i + 3 \geq 11$, then f' is adjacent to v_{i+3-11}, the rest may be deduced by analogy). With the property, if f_3 is adjacent to $\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\} \subset V(C_{11})$, then $N_{C_{11}}(f') \subseteq S_{v_{i_1}} \cap S_{v_{i_2}} \cap \cdots \cap S_{v_{i_t}}$. Note that f_3 can be adjacent to nonadjacent vertices in $\{0, 4, 5, 6, 7\}$, otherwise, if $(f_3, 3) \in E(G)$, there exists a longer cycle $C_{12} = 0\ f_1\ f_2\ f_3\ 3\ \ldots\ 10\ 0$, the situations of vertices 1, 2, 8, 9, 10 are similar. So $|N_{C_{11}}(f_3)| \leq 3$. In the same way, f_2 can be adjacent to nonadjacent vertices in $\{0, 3, 4, 5, 6, 7, 8\}$. What's more, if $N_{C_{11}}(f_3) = v_i, N_{C_{11}}(f_2) = v_j$, $v_i, v_j \in V(C_{11})$, then $j = i$, or $j < i - 2$, or $j > i + 2$. Now we consider the following subcases:

case 5.1.1. For $|N_{C_{11}}(f_3)| = 1$ (see Figure 7). Without loss of generality, let $N_{C_{11}}(f_3) = N_{C_{11}}(f_1) = 0$, as previously mentioned, $|N_{C_{11}}(f_2)| \leq 4$. $N_{C_{11}}(f') \subseteq \{3, 8\}$, by Remark 2.4, there are at least 2 edges should be deleted from K_{11}. We get

$$|E(G)| \leq \binom{11}{2} + 2(|V_{>0}| - 3) + 8 - 2 + ex(|V_0|, P_5) \leq \binom{11}{2} + 2(n - 11) - \frac{1}{2}|V_0| - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $\frac{46}{3} - \frac{r(4-r)}{6} - \frac{\delta}{3} < 0$, where $|V_0| \equiv r \pmod 4$, so there exists $3P_5$ in G, a contradiction.

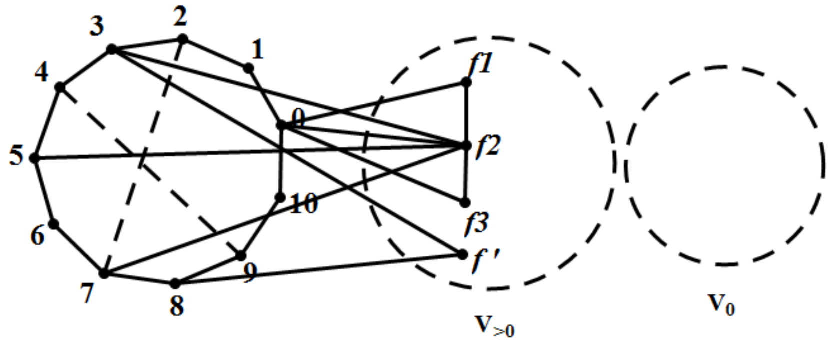

Figure 7: A graph with C_{11} and P_3 in $V_{>0}$, $|N_{C_{11}}(f_3)| = 1$.

case 5.1.2. For $|N_{C_{11}}(f_3)| = 2$ (see Figure 8). Then $N_{C_{11}}(f_3) = N_{C_{11}}(f_1) = \{0, a\}$, $a \in \{4, 5, 6, 7\}$. $|N_{C_{11}}(f_2)| \leq 3$, and $|N_{C_{11}}(f')| \leq 1$. By Remark 2.4, there are at least 2 edges should be deleted from K_{11}. We get

$$|E(G)| \leq \binom{11}{2} + (|V_{>0}| - 3) + 9 - 2 + ex(|V_0|, P_5) \leq \binom{11}{2} + \frac{3}{2}(n - 11) + 4 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{111}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $|V_0| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

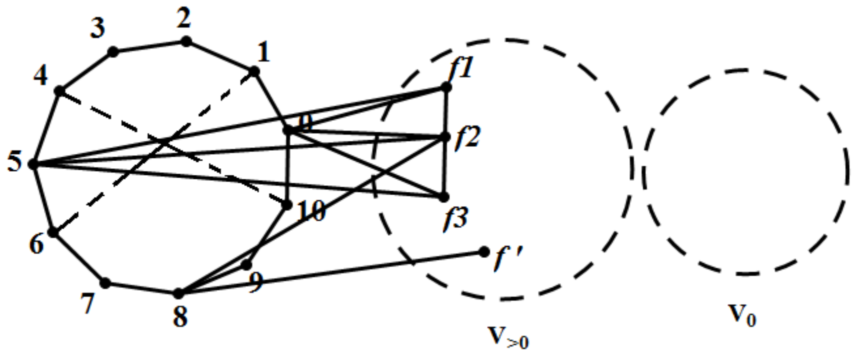

Figure 8: A graph with C_{11} and P_3 in $V_{>0}$, $|N_{C_{11}}(f_3)| = 2$.

case 5.1.3. For $|N_{C_{11}}(f_3)| = 3$, $N_{C_{11}}(f_3) = \{0, 4, 6\}$(or $\{0, 4, 7\}$ or $\{0, 5, 7\}$), then $N_{C_{11}}(f') = \emptyset$, and $N_{C_{11}}(f_1) = N_{C_{11}}(f_2) = 0$. By Remark 2.4, there are at least 6 edges should be deleted from K_{11}. We get

$$|E(G)| \leq \binom{11}{2} + 7 - 6 + ex(n - 14, P_5) \leq \binom{11}{2} + 1 + \frac{3}{2}(n - 14) - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{96}{7} - \frac{r(4-r)}{7} - \frac{2\delta}{7}$, where $(n-14) \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 5.2. Suppose that there exists $P_2 = f_1 f_2$ in $G|_{V_{>0}}$.

Note that $|E(V_{>0}, V_{0+})| \leq |V_{0+}| + 1$. And $G|_{V_{>0}}$ could contain more edges. Without loss of generality, let f_1, f_2 are adjacent to 0. Then other edges in $V_{>0}$ can be adjacent to 2,4,7,9, otherwise, if there exists $P'_2 = f_3 f_4$ in $G|_{V_{>0}}$, f_3, f_4 are adjacent to 0, we get $P_5 = f_1\ f_2\ 0\ f_3\ f_4$, and $P_{10} = 1\ 2\ \ldots\ 10$; if $P'_2 = f_3 f_4$ is adjacent to 1, we get $P_5 = f_3\ f_4\ 1\ 2\ 3$, $P'_5 = 4\ 5\ 6\ 7\ 8$, $P''_5 = 9\ 10\ 0\ f_1\ f_2$; if $P'_2 = f_3 f_4$ is adjacent to 3 or 5, we get P_5 with $\{f_3, f_4, 3, 4, 5\}$, and $P'_5 = 6\ 7\ 8\ 9\ 10$, $P''_5 = f_1\ f_2\ 0\ 1\ 2$; the situations of vertices 6, 8, 10 are similar to above with symmetry. Generally, let f_1 or f_2 be adjacent to v_i, $v_i \in V(C_{11})$, then other edges in $V_{>0}$ can be adjacent to $S_{v_i} = \{v_{i+2}, v_{i+4}, v_{i+7}, v_{i+9}\}$ (If $i+2 \geq 11$, then f' is adjacent to v_{i+2-11}, the rest may be deduced by analogy). With the property, if $N_{C_{11}}(f_1) \cup N_{C_{11}}(f_2) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\} \subset V(C_{11})$, then $N_{C_{11}}(f_3) \cup N_{C_{11}}(f_4) \subseteq S_{v_{i_1}} \cap S_{v_{i_2}} \cap \cdots \cap S_{v_{i_t}}$. Note that there are at most three independent edges in $G|_{V_{>0}}$. What's more, since $(f_1, 0) \in E(G)$, f_2 can be adjacent to nonadjacent vertices in $\{0, 3, 4, 5, 6, 7, 8\}$, otherwise we get a longer cycle. So $|N_{C_{11}}(f_2)| \leq 4$. Now we consider the following subcases:

case 5.2.1. For $|N_{C_{11}}(f_2)| \leq 2$ (see Figure 9). To make the edges of G as more as possible, let $N_{C_{11}}(f_1) = N_{C_{11}}(f_2) = \{0, 5\}$, then there exists a second edge $(f_3, f_4) \in E(G|_{V_{>0}}), N_{C_{11}}(f_3) = N_{C_{11}}(f_4) \in \{7, 9\}$, and there are at most three independent edges in $G|_{V_{>0}}$. Similar to the previous case, $|N_{C_{11}}(f')| \leq 5$ for all isolated vertices $f' \in V_{>0}$. When $|N_{C_{11}}(f')| = 5$, by Remark 2.4, there are at least 14 edges should be deleted from K_{11}. We obtain

$$|E(G)| \leq \binom{11}{2} + 5(|V_{>0}| - 2) + 5 + 2 - 14 + |V_{0+}| + 1 + ex(|V_{0-}|, P_5) \leq \binom{11}{2} + 5(n-11) - 16 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $-3 - \frac{r(4-r)}{2} - \delta < 0$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

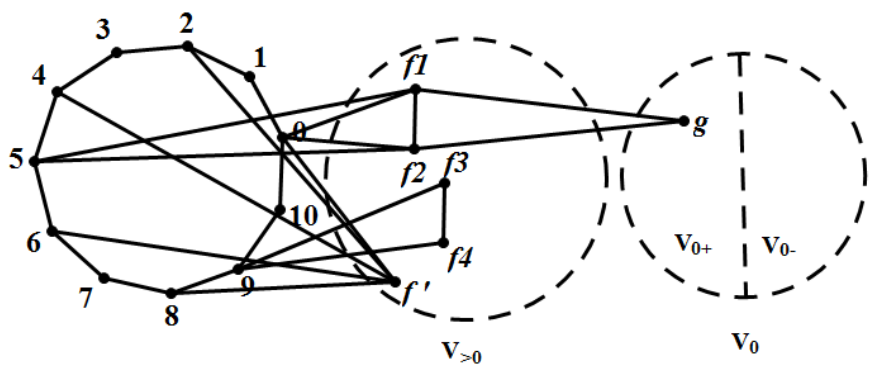

Figure 9: A graph with C_{11} and P_2 in $V_{>0}$, $|N_{C_{11}}(f_2)| = 2$.

case 5.2.2. For $3 \leq |N_{C_{11}}(f_2)| \leq 4$, then $|N_{C_{11}}(f_1)| \leq 2$ and there exists only one edge in $G|_{V_{>0}}$. Moreover, $|N_{C_{11}}(f')| \leq 5, \forall f' \in V_{>0} - \{f_1, f_2\}$. When $|N_{C_{11}}(f')| = 5$, by Remark 2.4, there are at least 14 edges should be deleted from K_{11}. We obtain

$$|E(G)| \leq \binom{11}{2} + 5(|V_{>0}| - 2) + 7 - 14 + |V_{0+}| + 1 + ex(|V_{0-}|, P_5) \leq \binom{11}{2} + 5(n-11) - 16 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $-3 - \frac{r(4-r)}{2} - \delta < 0$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 5.3. Suppose that $V_{>0}$ is an independent set.

Let $V_{0-+} = \{h \in V_{0-} | N_G(h) \cap V_{0+} \neq \emptyset\}$, $V_{0--} = V_{0-} - V_{0-+}$.

case 5.3.1. $V_{0-+} \neq \emptyset$ (see Figure 10). Without loss of generality, let $f\ g\ h$ be a path in F such that $f \in V_{>0}$, $g \in V_{0+}$, $h \in V_{0-+}$. Then V_{0-+} is an independent set and $|E(V_{0+}, V_{0-})| = |V_{0-+}|$. Then for all

$f' \in V_{>0} - \{f\}$, $N_{C_{11}}(f') \subseteq \{3, 8\}$, and $N_{V_{0+}}(f') = \emptyset$. Therefore, $N_{V_{>0}}(V_{0+}) = \{f\}$, and V_{0+} is also an independent set. By Remark 2.4, there are at least 2 edges should be deleted from K_{11}. We obtain

$$|E(G)| \leq \binom{11}{2} + 2(|V_{>0}| - 1) + 1 - 2 + |V_{0+}| + |V_{0-+}| + ex(|V_{0--}|, P_5) \leq \binom{11}{2} + 2(n-11) - 3 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $n > \frac{43}{3} - \frac{r(4-r)}{6} - \frac{\delta}{3}$, where $|V_{0--}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

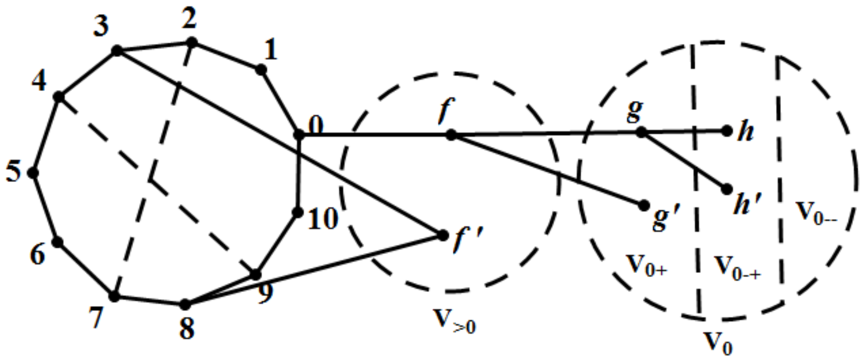

Figure 10: A graph with C_{11} and independent set $V_{>0}$, $V_{0-+} \neq \emptyset$.

case 5.3.2. $V_{0-+} = \emptyset, V_{0+} \neq \emptyset$.

case 5.3.2.1. $G|_{V_{0+}}$ contains P_2 (see Figure 11). Then there exists only one edge (g_1, g_2) in $G|_{V_{0+}}$. It is same to the previous case that $N_{C_{11}}(f') \subseteq \{3, 8\}$, $N_{V_{0+}}(f') = \emptyset$, so $N_{V_{>0}}(V_{0+}) = \{f\}$. By Remark 2.4, there are at least 2 edges should be deleted from K_{11}. We obtain

$$|E(G)| \leq \binom{11}{2} + 2(|V_{>0}| - 1) + 1 - 2 + |V_{0+}| + 1 + ex(|V_{0-}|, P_5) \leq \binom{11}{2} + 2(n-11) - 2 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

we get $n > \frac{44}{3} - \frac{r(4-r)}{6} - \frac{\delta}{3}$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

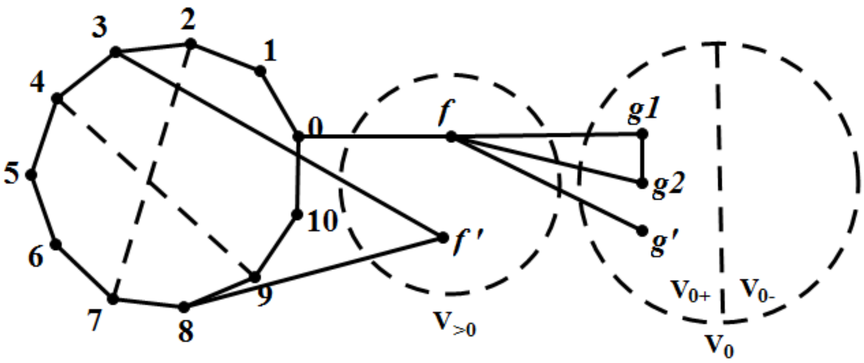

Figure 11: A graph with C_{11} and P_2 in V_{0+}.

case 5.3.2.2. V_{0+} is an independent set. In this case, $|N_G(g)| = 1$, or $|N_G(g)| = 2$ and $|V_{0+}| = 1, \forall g \in V_{0+}$. What's more, $|N_{C_{11}}(f)| \leq 5, \forall f \in V_{>0}$. When $|N_{C_{11}}(f)| = 5$, by Remark 2.4, there are at least 14 edges should be deleted from K_{11}. Then we get

$$|E(G)| \leq \binom{11}{2} + 5|V_{>0}| - 14 + |V_{0+}| + ex(|V_{0-}|, P_5) \leq \binom{11}{2} + 5(n-11) - 14 - \frac{r(4-r)}{2} < 5n - 13 + \delta,$$

for $-1-\frac{r(4-r)}{2}-\delta<0$, where $|V_{0-}|\equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 5.3.2.3. $V_{0+}=\emptyset$. $|N_{C_{11}}(f)|\le 5, \forall f\in V_{>0}$. If $|N_{C_{11}}(f)|_{max}=i$, by Remark 2.4, there are at least $\binom{i+1}{2}-1$ edges should be deleted from K_{11}. Then we obtain

$$|E(G)|\le\binom{11}{2}+i|V_{>0}|-(\binom{i+1}{2}-1)+ex(|V_0|,P_5)<5n-13+\delta,$$

for all $n\ge 19$ with different δ. Therefore we get $3P_5$ in G, a contradiction.

case 6. $q=10$. By table 1, we just consider the situation for $n\ge 22$. Let $F=G-V(C_{10})$. Note that $V_{>0}\ne\emptyset$, since otherwise $E(F)\ge 5n-13+\delta-\binom{10}{2}>ex(n-10,P_5)$ for $n>\frac{86}{7}-\frac{r(4-r)}{7}-\frac{2\delta}{7}$, where $n-10\equiv r$ (mod 4). Therefore, for $n\ge 22$ with different δ, we get P_5 in F, and there exists $3P_5$ in G, a contradiction.

case 6.1. Suppose that there exists $P_4=f_1\ f_2\ f_3\ f_4$ in $G|_{V_{>0}}$ (see Figure 12).

Since $f_1,f_2,f_3,f_4\in V_{>0}$, then V_{0+} is an independent set, and the vertices in V_{0+} just can be adjacent to f_2 or f_3, so $E(V_{>0},V_{0+})=|V_{0+}|$. $V_{0-+}=\emptyset$. Without loss of generality, let $(f_1,0)\in E(G)$, then $N_{C_{10}}(x)\subsetneq\{0,5\}$ for $x=f_1,f_4$, otherwise we get a longer cycle. $|N_{C_{10}}(f_2)|+|N_{C_{10}}(f_3)|\le 4$: if $N_{C_{10}}(x)=0$, then $N_{C_{10}}(f_2)=N_{C_{10}}(f_3)\subsetneq\{0,b\}$, $b\in\{4,5,6\}$, or $N_{C_{10}}(f_2)=0$, $N_{C_{10}}(f_3)\subsetneq\{0,4,6\}$; if $N_{C_{10}}(x)=\{0,5\}$, then $N_{C_{10}}(f_2)=N_{C_{10}}(f_3)\subsetneq\{0,5\}$. For $f'\in V_{>0}-\{f_1,f_2,f_3,f_4\}$, f' can be adjacent to nonadjacent vertices in $\{0,2,3,5,7,8\}$, otherwise, if f' is adjacent to 1 or 4, we get P_5 with $\{f',1,2,3,4\}$, and $P_{10}=5\ 6\ \dots\ 9\ 0\ f_1\ f_2\ f_3\ f_4$; the situations of vertices 6 and 9 are similar to above with symmetry. So $|N_{C_{10}}(f')|\le 4$. Note that $G|_{V_{>0}}$ could contain more P_4, however, they just can be adjacent to $\{0,5\}$. Above all, to make the edges of G as more as possible, let $N_{C_{10}}(f_1)=N_{C_{10}}(f_2)=N_{C_{10}}(f_3)=N_{C_{10}}(f_4)=\{0,5\}$, and $|N_{C_{10}}(f')|=4$. By Remark 2.4, there are at least 11 edges should be deleted from K_{10}. We get

$$|E(G)|\le\binom{10}{2}+4(|V_{>0}|-4)+11-11+|V_{0+}|+ex(|V_{0-}|,P_5)\le\binom{10}{2}+4(n-10)-16-\frac{r(4-r)}{2}<5n-13+\delta,$$

for $n>2-\frac{r(4-r)}{2}-\delta$, where $|V_{0-}|\equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

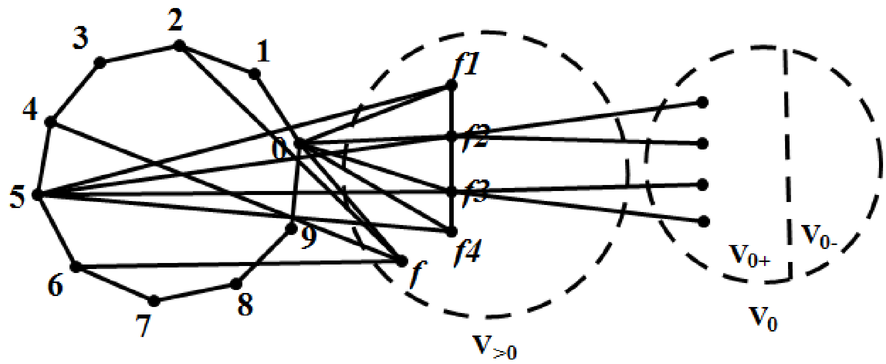

Figure 12: A graph with C_{10} and P_4 in $V_{>0}$.

case 6.2. Suppose that there exists $P_3=f_1\ f_2\ f_3$ in $G|_{V_{>0}}$.

Since $f_1,f_2,f_3\in V_{>0}$, then V_{0-+} is an independent set, and $E(V_{0+},V_{0-+})=|V_{0-+}|$. $|E(V_{>0},V_0)|\le|V_{0+}|+2$ (when $|E(V_{>0},V_0)|\le|V_{0+}|+2$, $V_{0-+}=\emptyset$ and $|V_{0+}|=1$). $G|_{V_{0+}}$ contains at most one edge. Without loss of generality, let $(f_1,0)\in E(G)$. By Remark 2.4, f_3 can be adjacent to nonadjacent vertices in $\{0,4,5,6\}$, so $N_{C_{10}}(f_3)\le 3$. f_2 can be adjacent to nonadjacent vertices in $\{0,3,4,5,6,7\}$, and if $N_{C_{10}}(f_3)=v_i,N_{C_{10}}(f_2)=v_j$, $v_i,v_j\in V(C_{10})$, then $j=i$, or $j<i-2$, or $j>i+2$. For $f'\in V_{>0}-\{f_1,f_2,f_3\}$, f' can be adjacent to nonadjacent vertices in C_{10}, so $N_{C_{10}}(f')\le 5$. Note that $G|_{V_{>0}}$ could contain more P_3 or edges, however, they just can be adjacent to $\{0,5\}$. Above all, to make the edges of G as more as possible, let $N_{C_{10}}(f_1)=N_{C_{10}}(f_3)=$

$\{0,4\}$, $N_{C_{10}}(f_2) = \{0,4,7\}$, and $|N_{C_{10}}(f')| = 5$. By Remark 2.4, there are at least 10 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 5(|V_{>0}|-3)+9-10+|V_{0+}|+2+|V_{0-+}|+ex(|V_{0--}|,P_5) \le \binom{10}{2}+5(n-10)-14-\frac{r(4-r)}{2} < 5n-13+\delta,$$

for $-6-\frac{r(4-r)}{2}-\delta<0$, where $|V_{0--}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.3. Suppose that there exists $P_2 = f_1\, f_2$ in $G|_{V_{>0}}$.

Since $f_1, f_2 \in V_{>0}$, then V_{0-+} is an independent set, and $E(V_{0+}, V_{0-+}) = |V_{0-+}|$. $|E(V_{>0}, V_0)| \le |V_{0+}| + 2$, and $G_{V_{0+}}$ contains at most one edge. Without loss of generality, let $(f_1, 0) \in E(G)$. By Remark 2.4, f_1, f_2 can be adjacent to nonadjacent vertices in $\{0,3,4,5,6,7\}$, and if $N_{C_{10}}(f_1) = v_i, N_{C_{10}}(f_2) = v_j$, $v_i, v_j \in \{0,3,4,5,6,7\}$, then $j = i$, or $j < i-2$, or $j > i+2$. So $|N_{C_{10}}(f_1)| + |N_{C_{10}}(f_2)| \le 5$, such as $N_{C_{10}}(f_1) = 0, N_{C_{10}}(f_2) = \{0,3,5,7\}$, or $N_{C_{10}}(f_1) = \{0,3\}, N_{C_{10}}(f_2) = \{0,3,6\}$. There exist at most five independent edges in $G|_{V_{>0}}$. For $f' \in V_{>0} - \{f_1, f_2\}$, f' can be adjacent to nonadjacent vertices in C_{10}, so $N_{C_{10}}(f') \le 5$. By Remark 2.4, there are at least 10 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 5(|V_{>0}|-2)+6-10+|V_{0+}|+2+|V_{0-+}|+ex(|V_{0--}|,P_5) \le \binom{10}{2}+5(n-10)-12-\frac{r(4-r)}{2} < 5n-13+\delta,$$

for $-4-\frac{r(4-r)}{2}-\delta<0$, where $|V_{0--}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.4. Suppose that $V_{>0}$ is an independent set.

Let $V_{0--+} = \{w \in V_{0--} | N_G(w) \cap V_{0-+} \ne \emptyset\}$, $V_{0---} = V_{0--} - V_{0--+}$.

case 6.4.1. $V_{0--+} \ne \emptyset$ (see Figure 13). Without loss of generality, let $f\ g\ h\ w$ be a path in F such that $f \in V_{>0}$, $g \in V_{0+}$, $h \in V_{0-+}$ and $w \in V_{0--+}$. For all $f' \in V_{>0} - f$, f' is adjacent to nonadjacent vertices in $\{0,2,3,5,7,8\}$, otherwise, if f' is adjacent to 1 or 4, we get P_5 with $\{f', 1, 2, 3, 4\}$, and $P_{10} = 5\ 6\ \ldots\ 9\ 0\ f\ g\ h\ w$; the other situations are similar to above with symmetry. So $|N_{C_{10}}(f')| \le 4$. And if $N_{C_{10}}(f')$ contains vertices 2, 3, 7, or 8, then $N_{V_{0+}}(f') = \emptyset$. So to make the edges of G as more as possible, let $|N_{C_{10}}(f')| = 4$, $N_{C_{10}}(f) = \{0,5\}$. f can't be adjacent to g', for all $g' \in V_{0+} - g$, otherwise we get $P_5 = g'\ f\ g\ h\ w$ in F. Therefore, $N_{V_{0+}}(V_{0-+}) = g$, and V_{0-+} is an independent set, otherwise, if there exists an edge $h\,h'$ in $G|_{V_{0-+}}$, then there exists $P_5 = f\ g\ h'\ h\ w$ in F. so $|E(V_{0+}, V_{0-+})| = |V_{0-+}|$. V_{0--+} is an independent set and $deg_G(w) = 1$, so $|E(V_{0-+}, V_{0--})| = |V_{0--+}|$. By Remark 2.4, there are at least 11 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 4(|V_{>0}|-1)-11+3+|V_{0-+}|+|V_{0--+}|+ex(|V_{0---}|,P_5) \le \binom{10}{2}+4(n-10)-12-\frac{r(4-r)}{2} < 5n-13+\delta,$$

for $n > 6-\frac{r(4-r)}{2}-\delta$, where $|V_{0---}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

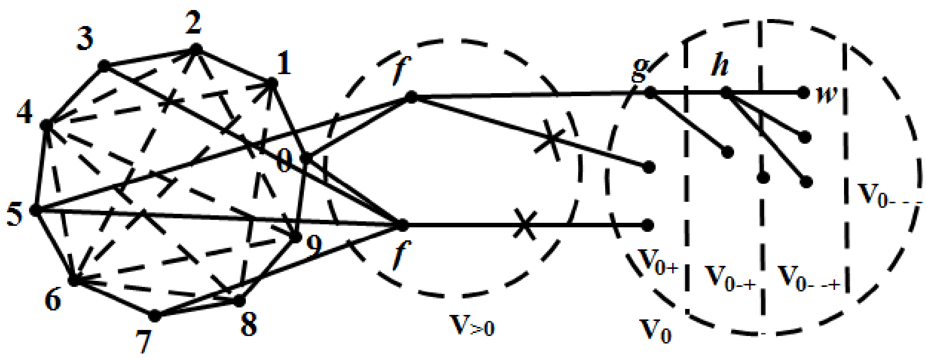

Figure 13: A graph with C_{10} and independent set $V_{>0}$, $V_{0--+} \ne \emptyset$.

case 6.4.2. $V_{0--+} = \emptyset$, $V_{0-+} \ne \emptyset$.

case 6.4.2.1. Suppose that there is a $P_2 = h_1\, h_2$ in $G|_{V_{0-+}}$.

Without loss of generality, let $f\, g\, h_1\, h_2$ be a path in F such that $f \in V_{>0}$, $g \in V_{0+}$, $h_1, h_2 \in V_{0-+}$. Similar to the previous case, for all $f' \in V_{>0} - f$, f' is adjacent to nonadjacent vertices in $\{0, 2, 3, 5, 7, 8\}$, so $|N_{C_{10}}(f')| \le 4$. And if $N_{C_{10}}(f')$ contains vertices 2, 3, 7, or 8, then $N_{V_{0+}}(f') = \emptyset$. Let $|N_{C_{10}}(f')| = 4$, $N_{C_{10}}(f) = \{0, 5\}$. So $N_{V_{0+}}(V_{0-+}) = g$. Moreover, there are at most one edge in $G|_{V_{0-+}}$, otherwise, if there exists $P_2' = h_3\, h_4$ in $G|_{V_{0-+}}$, we get $P_5 = h_1\, h_2\, g\, h_3\, h_4$ in F. By Remark 2.4, there are at least 11 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 4(|V_{>0}| - 1) - 11 + 3 + |V_{0-+}| + 1 + ex(|V_{0--}|, P_5) \le \binom{10}{2} + 4(n - 10) - 11 - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $n > 7 - \frac{r(4-r)}{2} - \delta$, where $|V_{0--}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.4.2.2. Suppose that V_{0-+} is an independent set.

Without loss of generality, let $f\, g\, h$ be a path in F such that $f \in V_{>0}$, $g \in V_{0+}$, $h \in V_{0-+}$. For all $f' \in V_{>0} - f$, f' is adjacent to nonadjacent vertices in C_{10}, $|N_{C_{10}}(f')| \le 5$. And if $N_{C_{10}}(f')$ contains vertices 1, 2, 3, 4, 6, 7, 8 or 9, then $N_{V_{0+}}(f') = \emptyset$. Let $|N_{C_{10}}(f')| = 5$, $N_{C_{10}}(f) = \{0, 5\}$. $G|_{V_{0+}}$ contains at most one P_2. If there is a $P_2 = g\, g'$ in $G|_{V_{0+}}$, then $|V_{0-+}| = 2$. If V_{0+} is an independent set, all vertices in V_{0+} can't be adjacent to $h' \in V_{0-+} - h$. So $|E(V_{>0}, V_{0+})| \le |V_{0+}|$, $|E(V_{0+}, V_{0-+})| \le |V_{0+}|$. By Remark 2.4, there are at least 10 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 5(|V_{>0}| - 1) - 10 + 1 + 2|V_{0+}| + ex(|V_{0--}|, P_5) \le \binom{10}{2} + 5(n - 10) - 14 - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $-6 - \frac{r(4-r)}{2} - \delta < 0$, where $|V_{0--}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.4.3. $V_{0-+} = \emptyset$, $V_{0+} \neq \emptyset$.

case 6.4.3.1. Suppose that there is a $P_3 = g_1\, g_2\, g_3$ in $G|_{V_{0+}}$.

Without loss of generality, let $(g_1, f) \in E(G)$. For all $f' \in V_{>0} - f$, f' is adjacent to nonadjacent vertices in $\{0, 2, 3, 5, 7, 8\}$, $|N_{C_{10}}(f')| \le 4$. And if $N_{C_{10}}(f')$ contains vertices 2, 3, 7, or 8, then $N_{V_{0+}}(f') = \emptyset$. To make the edges of G as more as possible, let $|N_{C_{10}}(f')| = 4$, $N_{C_{10}}(f) = \{0, 5\}$, then $|E(V_{>0}, V_{0+})| = |V_{0+}|$. And there are no more vertices in V_{0+}, otherwise we get P_5 in F. By Remark 2.4, there are at least 11 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 4(|V_{>0}| - 1) - 11 + 6 + ex(|V_{0-}|, P_5) \le \binom{10}{2} + 4(n - 10) - 9 - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $n > 9 - \frac{r(4-r)}{2} - \delta$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.4.3.2. Suppose that there is a $P_2 = g_1\, g_2$ in $G|_{V_{0+}}$.

For all $f' \in V_{>0} - f$, f' is adjacent to nonadjacent vertices in C_{10}, $|N_{C_{10}}(f')| \le 5$. And if $N_{C_{10}}(f')$ contains vertices 1, 2, 3, 4, 6, 7, 8 or 9, $N_{V_{0+}}(f') = \emptyset$. Let $|N_{C_{10}}(f')| = 5$, $N_{C_{10}}(f) = \{0, 5\}$. Then $|E(V_{>0}, V_{0+})| = |V_{0+}|$. $G|_{V_{0+}}$ contains at most one edge, otherwise we get P_5 in F. By Remark 2,4, there are at least 10 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 5(|V_{>0}| - 1) - 11 + 1 + |V_{0+}| + 1 + ex(|V_{0-}|, P_5) \le \binom{10}{2} + 5(n - 10) - 14 - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $-6 - \frac{r(4-r)}{2} - \delta < 0$, where $|V_{0-}| \equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.4.3.3. Suppose that V_{0+} is an independent set.

For all $f \in V_{>0}$, f can be adjacent to nonadjacent vertices in C_{10}, so $|N_{C_{10}}(f)| \le 5$. Since C_{10} contains P_{10}, then F can't contain P_5. To make the edges of G as more as possible, let $|N_{C_{10}}(f')| = 5$. By Remark 2.4, there are at least 10 edges should be deleted from K_{10}. We get

$$|E(G)| \le \binom{10}{2} + 5|V_{>0}| - 10 + ex(n - 10, P_5) \le \binom{10}{2} + 5(n - 10) - 10 - \frac{r(4 - r)}{2} < 5n - 13 + \delta,$$

for $-2-\frac{r(4-r)}{2}-\delta<0$, where $|V_{0-}|\equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

case 6.4.4. $V_{0+}=\emptyset$.

Recall that $V_{>0}$ is an independent set, for all $f\in V_{>0}$, f can be adjacent to nonadjacent vertices in C_{10}, so $|N_{C_{10}}(f)|\le 5$. When $|N_{C_{10}}(f')|=5$, by Remark 2.4, there are at least 10 edges should be deleted from K_{10}. We get

$$|E(G)|\le\binom{10}{2}+5|V_{>0}|-10+ex(|V_0|,P_5)\le\binom{10}{2}+5(n-10)-10-\frac{r(4-r)}{2}<5n-13+\delta,$$

for $-2-\frac{r(4-r)}{2}-\delta<0$, where $|V_{0-}|\equiv r \pmod 4$. There exists $3P_5$ in G, a contradiction.

In conclusion, the proof is completed. □

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