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Further Results of Special Elements in Rings With Involution

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Abstract. In this paper, we consider Moore-Penrose invertible, group invertible, and core invertible elements in rings with involution to characterize EP, generalized normal, generalized Hermitian elements and generalized partial isometries. As a consequence, we obtain new characterizations for elements in rings with involution to be normal and Hermitian elements.

1. Introduction

Throughout this paper, R will be a unital ring with involution, that is, a ring with unity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$. We will also use the following notations: $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, ${}^\circ a = \{x \in R \mid xa = 0\}$, $a^\circ = \{x \in R \mid ax = 0\}$ and $[a, b] = ab - ba$.

An element $a \in R$ will be called *generalized normal* [13] if there exists $n \in \mathbb{N}$ such that $a^n a^* = a^* a^n$. An element $a \in R$ will be called *generalized Hermitian* if there exists $n \in \mathbb{N}$ such that $a^n = (a^*)^n$. These notions are generalizations of the notions of normal elements ($aa^* = a^*a$) and Hermitian elements ($a^* = a$), respectively.

Let $a \in R$ (R is not necessary to be a ring with involution). Then a is *group invertible* if there exists $b \in R$ such that

$$aba = a, \quad bab = b, \quad ab = ba.$$

The element b is called a *group inverse* of a, it is unique (if exists) and denoted by $a^\#$. We will use $R^\#$ to denote the set of all group invertible elements of R.

Lemma 1.1. [7, Lemma 1.4.5] *Let $b \in R$ and $a \in R^\#$. If $ab = ba$, then $a^\# b = ba^\#$.*

A *Moore-Penrose inverse* of $a \in R$ is an element $b \in R$ satisfying

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

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If such an element b exists, it is uniquely determined by these equations and denoted by $a^\dagger$ [18]. The set of all Moore-Penrose invertible elements of R will be denoted by $R^\dagger$.

We now present the definitions of generalized partial isometry and generalized star-dagger element, which are extensions of partial isometry ($a^* = a^\dagger$) and star-dagger element ($a^\dagger a^* = a^* a^\dagger$), respectively. Let $a \in R^\dagger$. If there exists $n \in \mathbb{N}$ such that $(a^*)^n = (a^\dagger)^n$, then a will be called a *generalized partial isometry*. If there exists $n \in \mathbb{N}$ such that $(a^\dagger)^n a^* = a^*(a^\dagger)^n$, then a will be called a *generalized star-dagger* element.

An element $\tilde{a} \in R$ is called a $\{1,3\}$-*inverse* of a if $a\tilde{a}a = a$, $(a\tilde{a})^* = a\tilde{a}$. The set of all $\{1,3\}$-invertible elements of R will be denoted by $R^{\{1,3\}}$. Similarly, an element $\hat{a} \in R$ is called a $\{1,4\}$-*inverse* of a if $a\hat{a}a = a$, $(\hat{a}a)^* = \hat{a}a$. The set of all $\{1,4\}$-invertible elements of R will be denoted by $R^{\{1,4\}}$.

The core inverse for matrices were recently introduced by Baksalary and Trenkler in [3]. Rakić et al. [19] generalized the core inverse of a complex matrix to the case of an element in a ring with involution: let $a \in R$, if there exists $x \in R$ such that

$$axa = a, \quad xR = aR \quad \text{and} \quad Rx = Ra^*,$$

then x is called a *core inverse* of a, and a is called *core invertible*. It can be proved that this element x is unique [19] and it will be denoted by $a^{\text{#}}$. The set of all core invertible elements of R will be denoted by $R^{\text{#}}$. An useful equality is that for $a \in R^{\text{#}}$, one has $a^{\text{#}}aa^{\text{#}} = a^{\text{#}}$ ([19, Theorem 2.14]).

Furthermore, in [19], Rakić et al. introduced another generalized inverse (very related with the core inverse): let $a \in R$, if there exists $x \in R$ such that

$$axa = a, \quad xR = a^*R \quad \text{and} \quad Rx = Ra,$$

then x is called a *dual core inverse* of a, and a is said to be *dual core invertible*. Also, this element x is unique and it will denoted by $a_{\text{#}}$. The set of all dual core invertible elements of R will be denoted by $R_{\text{#}}$. The following trivial relation between the core and dual core inverse permits to prove results of the dual core inverse using analogous results of the core inverse: If $a \in R$, then $a \in R_{\text{#}}$ if and only if $a^* \in R^{\text{#}}$. In this case, one has $(a_{\text{#}})^* = (a^*)^{\text{#}}$.

Lemma 1.2. [22, Theorem 2.6 and Theorem 3.1] *Let $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^{\text{#}}$.
(2) $a \in R^\# \cap R^{\{1,3\}}$.
(3) *There exists $x \in R$ such that $(ax)^* = ax$, $xa^2 = a$ and $ax^2 = x$.*

In this case, $x = a^{\text{#}} = a^\# aa^{(1,3)}$, where $a^{(1,3)}$ is any $\{1,3\}$-inverse of a.

An element $a \in R$ is said to be an *EP element* if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$ [11]. The set of all EP elements of R will be denoted by R^{EP}.

Lemma 1.3. [19, Theorem 3.1] *Let $a \in R$. Then the following statements are equivalent:*

(1) $a \in R^{\mathrm{EP}}$.
(2) $a \in R^{\text{#}}$ *and* $[a, a^{\text{#}}] = 0$.
(3) $a \in R^\dagger \cap R^\#$ *and* $a^\dagger = a^{\text{#}}$.
(4) $a \in R^\dagger \cap R^\#$ *and* $a^\dagger = a_{\text{#}}$.
(5) $a \in R^{\text{#}}$ *and* $a^\# = a^{\text{#}}$.
(6) $a \in R_{\text{#}}$ *and* $a^\# = a_{\text{#}}$.

EP, normal and Hermitian matrices, as well as EP, normal and Hermitian linear operators on Banach or Hilbert spaces have been investigated by many authors [2, 5, 6, 8, 9]. In paper [1], Baksalary et al. used the representation of complex matrices provided in [10] to explore various classes of matrices, such as partial isometries, EP and star-dagger elements. Using the setting of rings with involution, EP, normal, Hermitian, partial isometries and star-dagger elements, which are Moore-Penrose invertible and group invertible, were investigated in [12, 14–16, 20]. In [13], the authors introduced and characterized generalized normal and generalized Hermitian elements in rings with involution.

The objective of the present article is to give new characterizations of EP, generalized normal, generalized Hermitian elements and generalized partial isometries for elements which are Moore-Penrose invertible, group invertible or core invertible in rings with involution. Applying these results, we get new equivalent conditions for elements in rings with involution to be normal and Hermitian elements.

2. EP elements

In this section, we present new necessary and sufficient conditions for an element of a ring with involution to be EP.

Recall that Patrício and Puystjens [17, Proposition 2] showed, for $a \in R$, $a \in R^{\text{EP}}$ if and only if $a \in R^{\#}$ and $aR = a^*R$ if and only if $a \in R^{\#}$ and $Ra = Ra^*$. Recently, in [20], the authors proved that the condition $aR = a^*R$ can be relaxed.

Lemma 2.1. [20, Theorem 3.6] *Let $a \in R$. Then the following statements are equivalent:*

(1) $a \in R^{\text{EP}}$.
(2) $a \in R^{\#}$ *and* $aR \subseteq a^*R$.
(3) $a \in R^{\#}$ *and* $Ra \subseteq Ra^*$.
(4) $a \in R^{\#}$ *and* $a^*R \subseteq aR$.
(5) $a \in R^{\#}$ *and* $Ra^* \subseteq Ra$.

Lemma 2.2. [4, Proposition 8.22] *Let a be an element of a ring R. Then $a \in R^{\#}$ if and only if $a^2x = a$ and $ya^2 = a$ both have solutions.*

In the following theorem, which is motivated by [5, Theorem 4.2], we study equivalent conditions for a Moore-Penrose invertible element to be EP.

Theorem 2.3. *Let $a \in R^{\dagger}$. Then the following statements are equivalent:*

(1) $a \in R^{\text{EP}}$.
(2) *There exists an invertible element $u \in R$ such that $a^*a = uaa^*$.*
(3) *There exists a left invertible element $u \in R$ such that $a^*a = uaa^*$.*
(4) *There exist elements $p, q \in R$ such that $a^*a = paa^*$ and $qa^*a = aa^*$.*
(5) *There exists an invertible element $v \in R$ such that $a^*a = aa^*v$.*
(6) *There exists a right invertible element $u \in R$ such that $a^*a = aa^*v$.*
(7) *There exist elements $m, n \in R$ such that $a^*a = aa^*m$ and $a^*an = aa^*$.*

Proof. (1) ⇒ (2). Assume that $a \in R^{\text{EP}}$. We define

$$r = a^*a(a^{\dagger})^*a^{\dagger}, \quad s = aa^*a^{\dagger}(a^{\dagger})^*, \quad p = 1 - aa^{\dagger} = 1 - a^{\dagger}a, \quad u = r + p, \quad v = s + p.$$

It is simple to prove $rp = 0$ and $ps = 0$, which leads to $uv = rs + p$. Now, use $aa^{\dagger} = a^{\dagger}a$, $a^{\dagger}aa^* = a^*$ and $a^*aa^{\dagger} = a^*$ to get

$$\begin{aligned} rs &= a^*a(a^{\dagger})^*a^{\dagger}aa^*a^{\dagger}(a^{\dagger})^* = a^*a(a^{\dagger})^*a^*a^{\dagger}(a^{\dagger})^* = a^*a(aa^{\dagger})^*a^{\dagger}(a^{\dagger})^* = a^*aa^{\dagger}(a^{\dagger})^* \\ &= a^*(a^{\dagger})^* = a^{\dagger}a = 1 - p. \end{aligned}$$

This computation leads to $uv = 1$. In a similar way we have $vu = 1$. Now,

$$\begin{aligned} uaa^* &= (r + p)aa^* = raa^* = a^*a(a^{\dagger})^*a^{\dagger}aa^* = a^*a(a^{\dagger})^*a^* = a^*a(aa^{\dagger})^* = a^*a(aa^{\dagger}) \\ &= a^*a(a^{\dagger}a) = a^*a. \end{aligned}$$

(2) ⇒ (3) is trivial.

(3) ⇒ (4). Let $p = u$ and $q = u_l^{-1}$, where u_l^{-1} is a left inverse of u.

(4) ⇒ (1). Assume that there exist elements $p, q \in R$ such that $a^*a = paa^*$ and $qa^*a = aa^*$. Then $a^*aaa^\dagger = paa^*aa^\dagger = paa^* = a^*a$. Multiplying the last equality by $(a^\dagger)^*$ from the left side, we deduce that $a = a^2a^\dagger$, therefore $a \in a^2R$ and $Ra \subseteq Ra^*$.

Since $aa^*a^\dagger a = qa^*aa^\dagger a = qa^*a = aa^*$, multiplying the last equality by $(a^\dagger)^*a^\dagger$ from the left side and by a from the right side, we deduce that $a = (a^\dagger)^*a^\dagger aa^*a^\dagger a^2$, then $a \in Ra^2$. Therefore, $a \in R^\#$ by Lemma 2.2, and thus $a \in R^{\mathrm{EP}}$ by Lemma 2.1.

The equivalence (1) ⇔ (5) ⇔ (6) ⇔ (7) is similar to what has been proved. □

Several equivalent conditions for a core invertible element to be EP are presented in the following result.

Theorem 2.4. *Let $a \in R^{\#\!\!\!\bigcirc}$. Then the following statements are equivalent:*

(1) $a \in R^{\mathrm{EP}}$.

(2) *There exists an invertible element $u \in R$ such that $a^*a = uaa^*$.*

(3) *There exists a left invertible element $u \in R$ such that $a^*a = uaa^*$.*

(4) *There exists an element $p \in R$ such that $a^*a = paa^*$.*

(5) *There exists an element $b \in R$ such that $a^*a = ba^*$.*

(6) *There exists an element $b_1 \in R$ such that $a^*a = b_1a^{\#\!\!\!\bigcirc}$.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) follows from Theorem 2.3.

(4) ⇒ (5) is obvious.

(5) ⇒ (6). In [19] it was proved $aa^{\#\!\!\!\bigcirc} = aa^\dagger$, hence $a^*aa^{\#\!\!\!\bigcirc} = a^*aa^\dagger = a^*$.

(6) ⇒ (1). Suppose that there exists an element $b_1 \in R$ such that $a^*a = b_1a^{\#\!\!\!\bigcirc}$. Recall that $a^{\#\!\!\!\bigcirc}$ is an outer inverse of a, i.e., $a^{\#\!\!\!\bigcirc}aa^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}$ (see [19, Theorem 2.14]). Then

$$a^*aaa^{\#\!\!\!\bigcirc} = b_1a^{\#\!\!\!\bigcirc}aa^{\#\!\!\!\bigcirc} = b_1a^{\#\!\!\!\bigcirc} = a^*a. \tag{1}$$

Also we use $(aa^{\#\!\!\!\bigcirc})^* = aa^{\#\!\!\!\bigcirc}$ (Lemma 1.2) and $aa^{\#\!\!\!\bigcirc}a = a$ [19, Theorem 2.14]. Multiplying (1) by $(a^{\#\!\!\!\bigcirc})^*$ from the left side, we deduce that $a = a^2a^{\#\!\!\!\bigcirc}$. Therefore, $Ra \subseteq Ra^{\#\!\!\!\bigcirc} = Ra^*$, that is, $a \in R^{\mathrm{EP}}$, because any core invertible element is group invertible (Lemma 1.2) and Lemma 2.1. □

Observe that for $a \in R$ one has $a \in R^{\mathrm{EP}} \Leftrightarrow a^* \in R^{\mathrm{EP}}$ (this can be deduced from [17, Proposition 2]). By employing Theorem 2.4 and the aforementioned relation between the core inverse and the dual core inverse, we give next result.

Theorem 2.5. *Let $a \in R_{\#\!\!\!\bigcirc}$. Then the following statements are equivalent:*

(1) $a \in R^{\mathrm{EP}}$.

(2) *There exists an invertible element $u \in R$ such that $aa^* = ua^*a$.*

(3) *There exists a left invertible element $u \in R$ such that $aa^* = ua^*a$.*

(4) *There exists an element $p \in R$ such that $aa^* = pa^*a$.*

(5) *There exists an element $b \in R$ such that $aa^* = ba$.*

(6) *There exists an element $b_2 \in R$ such that $aa^* = a_{\#\!\!\!\bigcirc}b_2$.*

Now, we give characterizations of EP elements through factorizations of the form $a = ucv$.

Theorem 2.6. *Let $a \in R^\#$. Then the following statements are equivalent:*

(1) $a \in R^{\mathrm{EP}}$.

(2) *There exist elements $c, d, u, v \in R$ such that $a = ucv = v^*d^*u^*$, $R = Ru$ and $Rc \subseteq Rd$.*

(3) $a \in R^{\{1,3\}}$ *and there exist elements $c, d, u, v \in R$ such that $a = ucv$, $a^{\#\!\!\!\bigcirc} = udv$, $R = Ru$ and $Rc \subseteq Rd$.*

(4) $a \in R^{\{1,3\}}$ *and there exist elements $c, d, u, v \in R$ such that $a^*a = ucv$, $aa^* = udv$, $R = Ru$ and $Rc \subseteq Rd$.*

Proof. (1) ⇒ (2). If a is EP, then, by Lemma 2.1, $Ra \subseteq Ra^*$. For $u = v = 1$, $c = a$ and $d = a^*$, we have that $R = Ru$ and $Rc = Ra \subseteq Ra^* = Rd$.

(2) ⇒ (1). Since $a = ucv = v^*d^*u^*$, for some $c, d, u, v \in R$ satisfying $R = Ru$ and $Rc \subseteq Rd$, we get $a^* = udv$ and

$$a = ucv \in Rucv = Rcv \subseteq Rdv = Rudv = Ra^*.$$

Using Lemma 2.1, we deduce that a is EP.

(1) ⇒ (3). It follows by choosing $u = v = 1$, $c = a$ and $d = a^{\circledast}$, by $Rc = Ra = Ra^2a^\# \subseteq Ra^\# = Ra^{\circledast} = Rd$.

(3) ⇒ (1). As in the proof of (2) ⇒ (1), we conclude that $a \in Ra^{\circledast} = Ra^*$ and, by Lemma 2.1, (1) is satisfied.

(1) ⇒ (4). Set $u = v = 1$, $c = a^*a$ and $d = aa^*$.

(4) ⇒ (1). By Lemma 1.2, $a \in R^{\circledast}$. As in the proof of (2) ⇒ (1), we get $a^*a \in Raa^*$. By Theorem 2.4, a is EP. □

3. Generalized normal elements

In this section, new characterizations for generalized normal elements in rings with involution are presented.

First, we state some well-known results.

Lemma 3.1. [13, Lemma 3.1] *Let $a \in R^\dagger \cap R^\#$ and $n \in \mathbb{N}$. Then $a^na^* = a^*a^n$ if and only if $a^na^\dagger = a^\dagger a^n$ and $(a^*)^na^\dagger = a^\dagger(a^*)^n$.*

Lemma 3.2. [13, Lemma 3.2] *Let $a \in R^\dagger \cap R^\#$ and $n \in \mathbb{N}$. Then $a^na^* = a^*a^n$ if and only if $a \in R^{\mathrm{EP}}$ and $(a^*)^na^\dagger = a^\dagger(a^*)^n$.*

Definition 3.3. [20, Definition 4.2] *Let $n \in \mathbb{N}$. The element $a \in R$ is* called n-EP *if $a \in R^\dagger$ and $[a^\dagger a^n, a^na^\dagger] = 0$.*

Lemma 3.4. [20, Theorem 4.3] *Let $a \in R$ and $n \in \mathbb{N}$. Then $a \in R^{\mathrm{EP}}$ if and only if $a \in R^\dagger \cap R^\#$ and a is n-EP.*

By Lemma 3.1 and Lemma 3.4, we deduce the following proposition.

Proposition 3.5. *Let $a \in R^\dagger \cap R^\#$ and $n \in \mathbb{N}$. Then $a^na^* = a^*a^n$ if and only if $[a^na^\dagger, a^\dagger a^n] = 0$ and $(a^*)^na^\dagger = a^\dagger(a^*)^n$.*

In the following theorem, we give equivalent conditions for both Moore-Penrose invertible and group invertible elements in rings with involution to be generalized normal elements.

Theorem 3.6. *Let $a \in R^\dagger \cap R^\#$, $n \in \mathbb{N}$ and $p, q \in \mathbb{N}$ with $p + q = n + 1$. Then $a^na^* = a^*a^n$ if and only if one of the following equivalent conditions holds:*

(1) $[(a^\#)^n, a^*a^\dagger] = 0$.
(2) $[(a^\#)^n, a^\dagger a^*] = 0$.
(3) $a^\dagger a^*(a^\#)^n = a^\#(a^\dagger)^na^*$.
(4) $a^\dagger(a^\#)^na^* = a^\#a^*(a^\dagger)^n$.
(5) $a^*(a^\dagger)^na^\# = (a^\#)^na^*a^\dagger$.
(6) $a^*(a^\#)^na^\dagger = (a^\dagger)^na^*a^\#$.
(7) $a^\dagger a^*(a^\#)^n = (a^\#)^p(a^\dagger)^qa^*$.
(8) $a^\dagger a^*(a^\#)^n = (a^\dagger)^p(a^\#)^qa^*$.
(9) $a^\dagger a^*(a^\#)^n = (a^{\circledast})^p(a_{\circledast})^qa^*$.
(10) $a^\dagger a^*(a^\#)^n = (a_{\circledast})^p(a^{\circledast})^qa^*$.
(11) $a^\dagger a^*(a^\#)^n = (a^\dagger)^{n+1}a^*$.
(12) $a^\dagger a^*(a^\#)^n = (a^\#)^{n+1}a^*$.
(13) $a^\dagger a^*(a^\#)^n = (a^{\circledast})^{n+1}a^*$.
(14) $a^\dagger a^*(a^\#)^n = (a_{\circledast})^{n+1}a^*$.

Proof. If $a^n a^* = a^* a^n$, by Lemma 3.2 and Lemma 1.3, we can easily verify that conditions (1)-(14) hold.

Conversely, we will show that the condition $a^n a^* = a^* a^n$ is satisfied.

(1). The hypothesis $[(a^\#)^n, a^* a^\dagger] = 0$ yields

$$\left[(a^\#)^n a^* a^\dagger\right] a^{n+1} a^* (a^\#)^* = \left[a^* a^\dagger (a^\#)^n\right] a^{n+1} a^* (a^\#)^* = a^* a^\dagger a a^* (a^\#)^* = a^* a^* (a^\#)^* = a^*.$$

Thus, $a^* R \subseteq a^\# R = aR$ and, by Lemma 2.1, we deduce that $a \in R^{\mathrm{EP}}$, i.e., $aa^\dagger = a^\dagger a$ (or $a^\dagger = a^\#$). Therefore, using $[(a^\#)^n, a^* a^\dagger] = 0$, we get

$$(a^\#)^n a^* a^\dagger = a^* a^\dagger (a^\#)^n = a^* a^\# (a^\#)^n = a^* (a^\#)^{n+1}.$$

Multiplying the last equality by a from the right side, we obtain

$$a^* (a^\#)^n = (a^\#)^n a^* a^\dagger a = (a^\#)^n a^* a a^\dagger = (a^\#)^n a^*,$$

which implies $a^* a^n = a^n a^*$ by Lemma 1.1.

(3). Suppose that $a^\dagger a^* (a^\#)^n = a^\# (a^\dagger)^n a^*$. Then

$$\begin{aligned} a &= a^{n+1} (a^\#)^n = a^{n+1} [a a^\dagger (a^\#)^n] = a^{n+1} (a^\dagger)^* a^* (a^\#)^n \\ &= a^{n+1} (a^\dagger)^* (a^2 a^\#)^* (a^\#)^n = a^{n+1} (a^\dagger)^* (a^\#)^* (a a^\dagger a)^* a^* (a^\#)^n \\ &= a^{n+1} (a^\dagger)^* (a^\#)^* a^* a [a^\dagger a^* (a^\#)^n] = a^{n+1} (a^\dagger)^* (a^\#)^* a^* a a^\# (a^\dagger)^n a^*, \end{aligned}$$

i.e., $Ra \subseteq Ra^*$, which gives that $a \in R^{\mathrm{EP}}$, by Lemma 2.1. Hence, we can use $a^\# = a^\dagger$ and $aa^\dagger = a^\dagger a$. Employing the condition (3), we have

$$(a^\#)^n a^* = a (a^\#)^{n+1} a^* = a a^\dagger a^* (a^\#)^n = (a^2 a^\dagger)^* (a^\#)^n = a^* (a^\#)^n,$$

and, by Lemma 1.1, $a^* a^n = a^n a^*$.

(5). Using the assumption $a^* (a^\dagger)^n a^\# = (a^\#)^n a^* a^\dagger$, we get

$$\begin{aligned} a^* (a^\dagger)^* a^\dagger (a^\#)^n a^{n+2} &= a^\dagger (a^\#)^n a^{n+2} = a^\dagger (a^\#)^n a a^{n+1} = a^\dagger (a^\#)^n a a^\dagger a a^{n+1} \\ &= a^\dagger a (a^\#)^n a^\dagger a a^{n+1} = a^\dagger a (a^\#)^n a^* (a^\dagger)^* a^{n+1} \\ &= a^\dagger a (a^\#)^n (a^\# a^2)^* (a^\dagger)^* a^{n+1} = a^\dagger a (a^\#)^n a^* a^* (a^\#)^* (a^\dagger)^* a^{n+1} \\ &= a^\dagger a (a^\#)^n a^* (a a^\dagger a)^* (a^\#)^* (a^\dagger)^* a^{n+1} \\ &= a^\dagger a \left[(a^\#)^n a^* a^\dagger\right] a a^* (a^\#)^* (a^\dagger)^* a^{n+1} \\ &= a^\dagger a a^* (a^\dagger)^n a^\# a a^* (a^\#)^* (a^\dagger)^* a^{n+1} \\ &= \left[a^* (a^\dagger)^n a^\#\right] a a^* (a^\#)^* (a^\dagger)^* a^{n+1} \\ &= (a^\#)^n a^* a^\dagger a a^* (a^\#)^* (a^\dagger)^* a^{n+1} = (a^\#)^n a^* a^* (a^\#)^* (a^\dagger)^* a^{n+1} \\ &= (a^\#)^n a^* (a^\dagger)^* a^{n+1} = (a^\#)^n a^\dagger a a^{n+1} = (a^\#)^n a^{n+1} = a. \end{aligned}$$

So, $aR \subseteq a^* R$ and $a \in R^{\mathrm{EP}}$, by Lemma 2.1. The rest of the proof of (5) is analogous as the proof of (1).

The proofs of conditions (2), (4) and (6) are similar to the proofs of conditions (1), (3) and (5), respectively. We can verify conditions (7)-(14) in the same way as condition (3). □

Necessary and sufficient conditions for core invertible elements to be generalized normal elements are investigated now.

Theorem 3.7. *Let $a \in R^{\circledast}$ and $n \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $a^n a^* = a^* a^n$.

(2) *a is EP and* $(a^*)^n a^{\circledast} = a^{\circledast} (a^*)^n$.

(3) $a^n a^{\circledast} = a^{\circledast} a^n$ *and* $(a^*)^n a^{\circledast} = a^{\circledast} (a^*)^n$.

(4) $a^n a^{\circledast} a^{\circledast} a^n = a^{\circledast} a^n a^n a^{\circledast}$ *and* $(a^*)^n a^{\circledast} = a^{\circledast} (a^*)^n$.

Proof. Recall that by Lemma 1.2, the element a is group invertible.

(1) ⇒ (2). Recall that the definition of the core inverse contains that the core inverse is an inner inverse. From item (3) of Lemma 1.2 and the equality $a^n a^* = a^* a^n$, we get

$$a = (a^\#)^{n-1}a^n = (a^\#)^{n-1}aa^{n-1} = (a^\#)^{n-1}aa^{\circledast}aa^{n-1} = (a^\#)^{n-1}aa^{\circledast}a^n$$
$$= (a^\#)^{n-1}(a^{\circledast})^*a^*a^n = (a^\#)^{n-1}(a^{\circledast})^*a^na^*.$$

Hence, $Ra \subseteq Ra^*$. By Lemma 2.1, we get that a is EP. Applying involution to $a^na^* = a^*a^n$, we obtain $a(a^*)^n = (a^*)^na$, which gives, by Lemma 1.1 and Lemma 1.3, $(a^*)^na^{\circledast} = a^{\circledast}(a^*)^n$.

(2) ⇔ (3). It follows by [21, Theorem 3.1].

(3) ⇒ (4). This is clear.

(4) ⇒ (1). Recall that $aa^{\circledast}a = a$. The condition $a^na^{\circledast}a^{\circledast}a^n = a^{\circledast}a^na^na^{\circledast}$ implies

$$a^{2n-1}a^{\circledast} = a^\#a^{2n}a^{\circledast} = a^\#aa^{\circledast}a^{2n}a^{\circledast} = a^\#aa^na^{\circledast}a^{\circledast}a^n = a^na^{\circledast}a^{\circledast}a^n$$
$$= a^na^{\circledast}a^{\circledast}a^naa^\# = a^{\circledast}a^na^na^{\circledast}aa^\# = a^{\circledast}a^{2n-1}.$$

By [21, Theorem 3.4], we conclude that a is EP. Using Lemma 1.3 and Proposition 3.5, we have that $a^na^* = a^*a^n$. □

Remark that we can get a similar result as Lemma 3.7 for dual core invertible elements.

Theorem 3.8. *Let $a \in R^{\circledast}$, $n \in \mathbb{N}$, and $p, q \in \mathbb{N}$ with $p + q = n + 1$. Then $a^na^* = a^*a^n$ if and only if one of the following equivalent conditions holds:*

(1) $\left[(a^\#)^n, a^*a^{\circledast}\right] = 0.$
(2) $\left[(a^\#)^n, a^{\circledast}a^*\right] = 0.$
(3) $a^{\circledast}a^*(a^\#)^n = a^\#(a^{\circledast})^na^*.$
(4) $a^*(a^{\circledast})^na^\# = (a^\#)^na^*a^{\circledast}.$
(5) $a^*(a^\#)^na^{\circledast} = (a^{\circledast})^na^*a^\#.$
(6) $a^{\circledast}a^*(a^\#)^n = (a^\#)^p(a^{\circledast})^qa^*.$
(7) $a^{\circledast}a^*(a^\#)^n = (a^{\circledast})^p(a^\#)^qa^*.$
(8) $a^{\circledast}a^*(a^\#)^n = (a^{\circledast})^{n+1}a^*.$
(9) $a^{\circledast}a^*(a^\#)^n = (a^\#)^{n+1}a^*.$

Proof. The assumption $a^na^* = a^*a^n$, by Theorem 3.7, implies that conditions (1)-(9) are satisfied.

Assume that the condition (1) holds. By using $a^{\circledast}a^2 = a$, $(aa^{\circledast})^* = aa^{\circledast}$, and $aa^{\circledast}a = a$, we have

$$aa^\#(a^\#)^na^*a^{n+1}a^{\circledast} = \left[(a^\#)^na^*a^{\circledast}\right]a^{n+2}a^{\circledast} = a^*a^{\circledast}(a^\#)^na^{n+2}a^{\circledast}$$
$$= a^*a^{\circledast}a^2a^{\circledast} = a^*aa^{\circledast} = a^*,$$

hence $a^*R \subseteq aR$ and, by Lemma 2.1, $a \in R^{\mathrm{EP}}$. Using Lemma 1.3 and (1), $a^*(a^\#)^{n+1} = (a^\#)^na^*a^\#$. By multiplying the last equality by a from the right side, we get $a^*(a^\#)^n = (a^\#)^na^*a^\#a$. Recall that $a^\dagger = a^\#$ since a is EP, which leads to $a^*a^\#a = a^*$. Therefore, $a^*(a^\#)^n = (a^\#)^na^*$, which implies $a^*a^n = a^na^*$ by Lemma 1.1.

The rest of the proof follows similarly as in the proof of Theorem 3.6. □

For $n = 1$ in Theorem 3.8, we get new characterizations of normal elements in rings with involution.

Corollary 3.9. *Let $a \in R^{\circledast}$. Then $aa^* = a^*a$ if and only if one of the following equivalent conditions holds:*

(1) $\left[a^\#, a^*a^{\circledast}\right] = 0.$
(2) $\left[a^\#, a^{\circledast}a^*\right] = 0.$
(5) $a^*a^\#a^{\circledast} = a^{\circledast}a^*a^\#.$

(6) $a^{\oplus}a^*a^\# = a^\# a^{\oplus}a^*$.
(7) $a^{\oplus}a^*a^\# = a^{\oplus}a^\# a^*$.
(8) $a^{\oplus}a^*a^\# = (a^{\oplus})^2a^*$.
(9) $a^{\oplus}a^*a^\# = (a^\#)^2a^*$.

More characterizations for both Moore-Penrose invertible and group invertible element to be normal element, can be obtained if we let $n = 1$ in Theorem 3.6.

4. Generalized Hermitian elements

This section will be consisted of several characterizations of generalized Hermitian elements in a ring with involution.

In the first result of this section, we prove that a group invertible generalized Hermitian element is an EP element.

Lemma 4.1. *Let $a \in R^\#$. If a is a generalized Hermitian element, then $a \in R^{\mathrm{EP}}$.*

Proof. Since $a^n = (a^*)^n$, for some $n \in \mathbb{N}$, then $a = (a^\#)^{n-1}a^n = (a^\#)^{n-1}(a^*)^n = (a^\#)^{n-1}(a^*)^{n-1}a^*$, which implies $Ra \subseteq Ra^*$. Thus, $a \in R^{\mathrm{EP}}$ by Lemma 2.1. □

In the following theorem, we will investigate some equivalent conditions for a group invertible element to be a generalized Hermitian element.

Theorem 4.2. *Let $a \in R^\#$ and $m, n \in \mathbb{N}$. Then $a^n = (a^*)^n$ if and only if one of the following equivalent conditions holds:*

(1) $a^{n+m} = (a^*)^n a^m$.
(2) $a^n(a^\#)^m = (a^*)^n(a^\#)^m$.
(3) $a^n(a^\#)^m = (a^\#)^m(a^*)^n$.
(4) $(a^*)^n(a^\#)^{n+1} = a^\#$.
(5) $(a^*)^{n+1}(a^\#)^n = a^*$.

Proof. If $a^n = (a^*)^n$, then the conditions (1)–(4) are obviously true, and by Lemma 4.1, $a^\# = a^\dagger$. Hence $(a^*)^{n+1}(a^\#)^n = a^*(a^*)^n(a^\#)^n = a^*a^n(a^\#)^n = a^*aa^\# = a^*aa^\dagger = a^*$, thus the condition (5) holds.

On the other hand, we will show that the condition $a^n = (a^*)^n$ holds when a satisfies one of the conditions (1)–(5).

(1). Multiplying the equality $a^{n+m} = (a^*)^n a^m$ by $(a^\#)^{n+m-1}$ from the right side, we get $a = (a^*)^n a^m (a^\#)^{n+m-1} \in a^*R$. By Lemma 2.1, $a \in R^{\mathrm{EP}}$ and, by Lemma 1.3, $a^\# = a^{\oplus}$. Therefore,

$$a^n = a^{n+m}(a^\#)^m = (a^*)^n a^m (a^\#)^m = (a^*)^n aa^\# = (a^*)^n aa^{\oplus} = (a^*)^n.$$

(2). If we multiply $a^n(a^\#)^m = (a^*)^n(a^\#)^m$ by a^{2m} from the right side, we observe that (1) is satisfied.

(3). The hypothesis $a^n(a^\#)^m = (a^\#)^m(a^*)^n$ gives

$$a = (a^\#)^{n-1}a^n = (a^\#)^{n-1}a^m(a^n(a^\#)^m) = (a^\#)^{n-1}a^m(a^\#)^m(a^*)^n \in Ra^*,$$

that is, a is EP. So, $a^\dagger = a^\#$, and

$$a^n = a^m(a^n(a^\#)^m) = a^m(a^\#)^m(a^*)^n = a^\# a(a^*)^n = a^\dagger aa^*(a^*)^{n-1} = (a^*)^n.$$

(4). Multiplying $(a^*)^n(a^\#)^{n+1} = a^\#$ by a^{n+m+1} from the right side, we obtain that (1) holds.

(5). Applying involution to $(a^*)^{n+1}(a^\#)^n = a^*$, we have that $a = \left[(a^\#)^n\right]^* a^{n+1} = a^*[(a^\#)^{n+1}]^* a^{n+1} \in a^*R$. Hence, a is EP and then $a^* \in Ra$, which implies $a^*aa^\# = a^*$. Now, multiplying $(a^*)^{n+1}(a^\#)^n = a^*$ by a^n from the right side, we get $(a^*)^{n+1}a^\# a = a^*a^n$, therefore, $(a^*)^{n+1} = a^*a^n$. Now,

$$a^n = aa^\dagger a^n = (a^\dagger)^*a^*a^n = (a^\dagger)^*(a^*)^{n+1} = (a^{n+1}a^\dagger)^* = (a^{n+1}a^\#)^* = (a^n)^* = (a^*)^n.$$

□

If we suppose that $n = 1$ in Theorem 4.2, we obtain next characterizations for a group invertible element to be Hermitian.

Corollary 4.3. *Let $a \in R^{\#}$ and $m \in \mathbb{N}$. Then $a = a^*$ if and only if one of the following equivalent conditions holds:*

(1) $a^{m+1} = a^* a^m$.
(2) $a(a^{\#})^m = a^*(a^{\#})^m$.
(3) $a(a^{\#})^m = (a^{\#})^m a^*$.
(4) $a^*(a^{\#})^2 = a^{\#}$.
(5) $(a^*)^2 a^{\#} = a^*$.

Remark that characterizations (1), (2), (4) and (5) of Corollary 4.3 recover corresponding conditions in [21, Theorem 4.2], where $a \in R^{\textcircled{\#}}$.

Further, necessary and sufficient conditions for a core invertible element to be a generalized Hermitian element are given.

Proposition 4.4. *Let $a \in R^{\textcircled{\#}}$ and $m, n \in \mathbb{N}$. Then $a^n = (a^*)^n$ if and only if one of the following equivalent conditions holds:*

(1) $a^n a a^{\textcircled{\#}} = (a^*)^n$.
(2) $a^n (a^{\textcircled{\#}})^m = (a^*)^n (a^{\textcircled{\#}})^m$.
(3) $a^n (a^{\#})^m = (a^*)^n (a^{\textcircled{\#}})^m$.
(4) $a^n (a^{\#})^m = (a^{\textcircled{\#}})^m (a^*)^n$.
(5) $a^n (a^{\textcircled{\#}})^m = (a^{\textcircled{\#}})^m (a^*)^n$.
(6) $(a^*)^n a^{\textcircled{\#}} (a^{\#})^n = a^{\#}$.
(7) $(a^*)^n (a^{\textcircled{\#}})^{n+1} = a^{\textcircled{\#}}$.
(8) $(a^*)^n (a^{\textcircled{\#}})^{n+1} = a^{\#}$.
(9) $(a^{\textcircled{\#}})^n (a^*)^{n+1} = a^*$.
(10) $(a^*)^n a^{\textcircled{\#}} (a^{\#})^n = a^{\textcircled{\#}}$.
(11) $a^{\textcircled{\#}} (a^*)^n (a^{\#})^n = a^{\textcircled{\#}}$.
(12) $a(a^*)^n a^{\textcircled{\#}} = a^n$.

Proof. We will only prove that condition (1) implies $a^n = (a^*)^n$. The rest follows similarly as in the proof of Theorem 4.2.

(1). Applying the involution to $a^n a a^{\textcircled{\#}} = (a^*)^n$, we have $a a^{\textcircled{\#}} (a^*)^n = a^n$, implying $a^n = a a^{\textcircled{\#}} (a^*)^n = a a^{\textcircled{\#}} (a^n a a^{\textcircled{\#}}) = a^n a a^{\textcircled{\#}} = (a^*)^n$. □

As a consequence of Proposition 4.4, we get equivalent conditions for core invertible elements to be Hermitian. Observe that conditions (1), (2), (6), (8), (10) and (12) of the next result appeared in [21, Theorem 4.2].

Corollary 4.5. *Let $a \in R^{\textcircled{\#}}$ and $m \in \mathbb{N}$. Then a is Hermitian if and only if one of the following equivalent conditions holds:*

(1) $a^2 a^{\textcircled{\#}} = a^*$.
(2) $a(a^{\textcircled{\#}})^m = a^*(a^{\textcircled{\#}})^m$.
(3) $a(a^{\#})^m = a^*(a^{\textcircled{\#}})^m$.
(4) $a(a^{\#})^m = (a^{\textcircled{\#}})^m a^*$.
(5) $a(a^{\textcircled{\#}})^m = (a^{\textcircled{\#}})^m a^*$.
(6) $a^* a^{\textcircled{\#}} a^{\#} = a^{\#}$.
(7) $a^*(a^{\textcircled{\#}})^2 = a^{\textcircled{\#}}$.
(8) $a^*(a^{\textcircled{\#}})^2 = a^{\#}$.
(9) $a^{\textcircled{\#}}(a^*)^2 = a^*$.
(10) $a^* a^{\textcircled{\#}} a^{\#} = a^{\textcircled{\#}}$.
(11) $a^{\textcircled{\#}} a^* a^{\#} = a^{\textcircled{\#}}$.
(12) $a a^* a^{\textcircled{\#}} = a$.

5. Generalized partial isometries and EP elements

We will study generalized partial isometries and EP elements in this section.

Proposition 5.1. *Let $a \in R^{\dagger} \cap R^{\#}$. If a is generalized partial isometry, then $a \in R^{\mathrm{EP}}$ if and only if a is generalized normal element.*

Proof. Suppose that a is generalized partial isometry and $a \in R^{\mathrm{EP}}$, then $a(a^*)^n = a(a^{\dagger})^n = a(a^{\#})^n = (a^{\#})^n a = (a^{\dagger})^n a = (a^*)^n a$, for some $n \in \mathbb{N}$. Applying involution to the above equality, we conclude that a is a generalized normal element.

The converse follows by Lemma 3.2. □

Motivated by [15, Theorem 2.4], more equivalent conditions such that $a \in R^{\dagger} \cap R^{\#}$ to be both generalized partial isometry and EP element are presented in the following theorem. For $n = 1$ and/or $m = 1$, we recover some conditions of [14, Theorem 2.3] and [15, Theorem 2.2].

Theorem 5.2. *Let $a \in R^{\dagger} \cap R^{\#}$ and $m, n \in \mathbb{N}$. Then $(a^*)^n = (a^{\dagger})^n$ and $a \in R^{\mathrm{EP}}$ if and only if one of the following equivalent conditions holds:*

(1) $a^m(a^*)^n = (a^{\dagger})^n a^m$.
(2) $(a^*)^n a^m = a^m (a^{\dagger})^n$.
(3) $(a^*)^n a^{\dagger} = a^{\dagger}(a^{\#})^n$ (*or* $(a^*)^n a^{\dagger} = (a^{\#})^n a^{\dagger}$).
(4) $a^{\dagger}(a^*)^n = (a^{\#})^n a^{\dagger}$ (*or* $a^{\dagger}(a^*)^n = a^{\dagger}(a^{\#})^n$).
(5) $(a^*)^n a^{n+m} = a^m$.
(6) $a^{n+m}(a^*)^n = a^m$.
(7) $(a^{\dagger})^n = a(a^*)^n a^{\dagger}$.
(8) $(a^{\dagger})^n = aa^{\dagger}(a^*)^n$.

Proof. If $(a^*)^n = (a^{\dagger})^n$ and $a \in R^{\mathrm{EP}}$, we easily show that conditions (1)–(8) are satisfied.

Conversely, we will verify that $(a^*)^n = (a^{\dagger})^n$ and $a \in R^{\mathrm{EP}}$. In the foregoing, we shall use $a^{\dagger} \in a^*R \cap Ra^*$.

(1). Using the assumption $a^m(a^*)^n = (a^{\dagger})^n a^m$, we get

$$\begin{aligned} a &= a^n(a^{\#})^{n-1} = [(a^*)^n]^*(a^{\#})^{n-1} = [a^{\dagger}a(a^*)^n]^*(a^{\#})^{n-1} \\ &= [a^{\dagger}(a^{\#})^{m-1}a^m(a^*)^n]^*(a^{\#})^{n-1} = [a^{\dagger}(a^{\#})^{m-1}(a^{\dagger})^n a^m]^*(a^{\#})^{n-1} \\ &= (a^*)^m[a^{\dagger}(a^{\#})^{m-1}(a^{\dagger})^n]^*(a^{\#})^{n-1} \in a^*R, \end{aligned}$$

which implies that $a \in R^{\mathrm{EP}}$, by Lemma 2.1. Therefore,

$$\begin{aligned} (a^*)^n &= a^{\dagger}(a^{\#})^{m-1}a^m(a^*)^n = a^{\dagger}(a^{\#})^{m-1}(a^{\dagger})^n a^m \\ &= (a^{\#})^n(a^{\#})^m a^m = (a^{\#})^n a^{\#} a = (a^{\#})^n = (a^{\dagger})^n. \end{aligned}$$

(2) is similar to (1).

(3). Suppose that $(a^*)^n a^{\dagger} = a^{\dagger}(a^{\#})^n$. Then, by

$$\begin{aligned} aa^{\dagger} &= a^n a(a^{\#})^n a^{\dagger} = a^n aa^{\dagger}a(a^{\#})^n a^{\dagger} = a^{n+1}a^{\dagger}(a^{\#})^n aa^{\dagger} \\ &= a^{n+1}(a^*)^n a^{\dagger}aa^{\dagger} = a^{n+1}(a^*)^n a^{\dagger} = a^{n+1}a^{\dagger}(a^{\#})^n \\ &= a^n aa^{\dagger}(a^{\#})^n = a^n(a^{\#})^n = aa^{\#}, \end{aligned}$$

we deduce that $a = a^2a^{\#} = a^2a^{\dagger} \in Ra^*$. Hence, $a \in R^{\mathrm{EP}}$ by Lemma 2.1. Now,

$$(a^{\dagger})^n = (a^{\#})^n = (a^{\#})^{n+1}a = a^{\dagger}(a^{\#})^n a = (a^*)^n a^{\dagger}a = (a^*)^n aa^{\dagger} = (a^*)^n.$$

(4) is similar to (3).

(5). Multiplying $(a^*)^n a^{n+m} = a^m$ by $(a^\#)^{m-1}$ from the right side, we obtain $(a^*)^n a^{n+1} = a$ which yields $aR \subseteq a^*R$ and so $a \in R^{\mathrm{EP}}$ by Lemma 2.1. Thus,

$$(a^*)^n = (a^*)^{n-1}(aa^\dagger a)^* = (a^*)^n aa^\dagger = (a^*)^n a^{n+m}(a^\dagger)^{n+m} = a^m(a^\dagger)^{n+m} = (a^\dagger)^n.$$

(6) is similar to (5).

(7). The hypothesis $(a^\dagger)^n = a(a^*)^n a^\dagger$ gives

$$a = aa^*(a^\dagger)^* = a(a^*)^{n+1}[(a^\#)^n]^*(a^\dagger)^* = [a(a^*)^n a^\dagger]aa^*[(a^\#)^n]^*(a^\dagger)^* = (a^\dagger)^n aa^*[(a^\#)^n]^*(a^\dagger)^*,$$

which yields $a \in a^\dagger R$, hence $aR \subseteq a^*R$. So, $a \in R^{\mathrm{EP}}$ and

$$(a^\dagger)^n = a^\dagger(a^\#)^n a = a^\dagger[a(a^*)^n a^\#]a = (a^*)^n a^\# a = (a^*)^n.$$

(8) is similar to (7). □

Sufficient conditions for a to be generalized star-dagger are given now. If $n = 1$, then we recover [14, Theorem 3.1].

Proposition 5.3. *Let $a \in R^\dagger$ and $n \in \mathbb{N}$. Then each of the following conditions is sufficient for $(a^\dagger)^n a^* = a^*(a^\dagger)^n$:*

(1) $a^* = a^*(a^\dagger)^n$.
(2) $a^* = (a^\dagger)^n a^*$.
(3) $a^\dagger = (a^\dagger)^{n+1}$.
(4) $a^* = (a^\dagger)^{n+1}$.
(5) $(a^\dagger)^n = a^*a^*$.

Proof. First, we check that (3) implies that $(a^\dagger)^n a^* = a^*(a^\dagger)^n$:

$$(a^\dagger)^n a^* = (a^\dagger)^{n+1}aa^* = a^\dagger aa^* = a^* = (aa^\dagger a)^* = a^*aa^\dagger = a^*a(a^\dagger)^{n+1} = a^*(a^\dagger)^n.$$

Further, notice that (1) ⇒ (3), by $a^\dagger = a^\dagger aa^\dagger = a^\dagger(a^\dagger)^*a^* = a^\dagger(a^\dagger)^*a^*(a^\dagger)^n = (a^\dagger)^{n+1}$.

(2) ⇒ (3) *is similar to* (1) ⇒ (3).

(4) *and* (5) *are trivial.* □

Remark 5.4. In the proof of the Proposition 5.3 it is shown that (1) ⇒ (3). We will prove that (1) is equivalent to (3). In fact, assume that $a^\dagger = (a^\dagger)^{n+1}$ and let $b = a^\dagger$. Now, by using $b = b^{n+1}$ and $b^\dagger = a$, we have

$$a^*(a^\dagger)^n = (b^\dagger)^*b^n = (b^\dagger bb^\dagger)^*b^n = (b^\dagger)^*b^\dagger bb^n = (b^\dagger)^*b^\dagger b = (b^\dagger)^* = a^*.$$

The following example shows that the conditions (1)–(5) of the Proposition 5.3 are not, in general, equivalent for $(a^\dagger)^n a^* = a^*(a^\dagger)^n$.

Example 5.5. We consider $\mathbb{C}$ as a ring whose involution is the identity (notice that $\mathbb{C}$ is an Abelian ring). The condition $(z^\dagger)^n z^* = z^*(z^\dagger)^n$ is satisfied for all $z \in \mathbb{C}$.

The unique complex numbers satisfying the condition (1) are $\{0\} \cup \{z \in \mathbb{C} : z^n = 1\}$, and the same holds for conditions (2) and (3). We have that $z \in \mathbb{C}$ satisfies (4) if and only if $z \in \{0\} \cup \{z \in \mathbb{C} : z^{n+2} = 1\}$, and the same holds for condition (5).

6. Characterizations of $(a^*)^n = (a^{\#})^n$

In the beginning of this section, we observe that if a core invertible element a satisfies $(a^*)^n = (a^{\#})^n$, then a is EP.

Lemma 6.1. *Let $a \in R^{\#}$ and $n \in \mathbb{N}$. If $(a^*)^n = (a^{\#})^n$, then $a \in R^{\mathrm{EP}}$.*

Proof. Multiplying $(a^*)^n = (a^{\#})^n$ by a^{n+1} from the right side, we have that $(a^*)^n a^{n+1} = (a^{\#})^n a^{n+1} = a$. Thus, $aR \subseteq a^*R$ which gives $a \in R^{\mathrm{EP}}$, by Lemma 2.1. □

Next, some necessary and sufficient conditions for an element a in a ring with involution to satisfy $(a^*)^n = (a^{\#})^n$ are presented.

Theorem 6.2. *Let $a \in R^{\#}$ and $m, n \in \mathbb{N}$. Then $(a^*)^n = (a^{\#})^n$ if and only if one of the following equivalent conditions holds:*

(1) $(a^*)^n a^m = (a^{\#})^n a^m$.
(2) $(a^*)^n a^m = a^m (a^{\#})^n$.
(3) $(a^*)^n (a^{\#})^m = (a^{\#})^{n+m}$.
(4) $(a^*)^n a^n = aa^{\#}$.
(5) $(a^*)^n a^{n+m} = a^m$.
(6) $(a^*)^n (a^{\#})^m = (a^{\#})^n (a^{\#})^m$.
(7) $(a^*)^n a^m = a^m (a^{\#})^n$.
(8) $(a^*)^n (a^{\#})^m = (a^{\#})^n (a^{\#})^m$.
(9) $(a^*)^n (a^{\#})^m = (a^{\#})^m (a^{\#})^n$.
(10) $(a^*)^n (a^{\#})^m = (a^{\#})^m (a^{\#})^n$.
(11) $(a^*)^n (a^{\#})^m = (a^{\#})^{n+m}$.

Proof. In the case that $(a^*)^n = (a^{\#})^n$, by Lemma 6.1, it is easy to check that conditions (1)–(11) hold.

On the other hand, we will show that $(a^*)^n = (a^{\#})^n$ holds when the element a satisfies one of the conditions (1)–(11).

(1). Multiplying $(a^*)^n a^m = (a^{\#})^n a^m$ by $(a^{\#})^m$ from the right side, we obtain $(a^*)^n aa^{\#} = (a^{\#})^n aa^{\#}$, that is $(a^*)^n = (a^{\#})^n$.

(2). Since $(a^*)^n a^m = a^m (a^{\#})^n$, then $(a^*)^n = (a^*)^n a^m (a^{\#})^m = a^m (a^{\#})^{n+m} = (a^{\#})^n$.

(3). Using $(a^*)^n (a^{\#})^m = (a^{\#})^{n+m}$, we have that (1) holds:

$$(a^*)^n a^m = (a^*)^n a^{\#} a^{m+1} = (a^*)^n (a^{\#})^m a^{2m} = (a^{\#})^{n+m} a^{2m} = (a^{\#})^n a^m.$$

(4). The element a is EP because $a = aa^{\#}a = (a^*)^n a^n a^{\#} a \in a^*R$ and Lemma 2.1. Now, Multipliying $(a^*)^n a^n = aa^{\#}$ by $(a^{\#})^n$ from the right side we get $(a^*)^n aa^{\#} = (a^{\#})^n$. Recall that a is EP, and in particular, $a^{\#} = a^{\dagger} = a^{\#}$, hence

$$(a^{\#})^n = (a^{\#})^n = (a^*)^n aa^{\#} = (a^*)^n aa^{\dagger} = (a^*)^{n-1} a^* aa^{\dagger} = (a^*)^n.$$

(5). Multiplying $(a^*)^n a^{n+m} = a^m$ by $(a^{\#})^m$ from the right side, notice that the condition (4) is satisfied.

(6). If we multiply $(a^*)^n (a^{\#})^m = (a^{\#})^n (a^{\#})^m$ by a^{2m} from the right side, we get (1).

(7). The hypothesis $(a^*)^n a^m = a^m (a^{\#})^n$ implies

$$\begin{aligned}(a^*)^n &= [(a^*)^n a^m](a^{\#})^m = a^m (a^{\#})^n (a^{\#})^m = a^m (a^{\#})^n a (a^{\#})^{m+1} \\ &= a^m (a^{\#})^n a^n (a^{\#})^{m+n} = a^m a^{\#} a (a^{\#})^{m+n} = a^m (a^{\#})^{m+n} = (a^{\#})^n.\end{aligned}$$

(8). Multiplying the equality $(a^*)^n (a^{\#})^m = (a^{\#})^n (a^{\#})^m$ by $a^{m+1} a^{\#}$ from the right side, we see that $(a^*)^n = (a^{\#})^n$.

(9). Applying $(a^*)^n(a^{\oplus})^m = (a^\#)^m(a^{\oplus})^n$, we observe that

$$\begin{aligned}(a^*)^n &= [(a^*)^n(a^{\oplus})^m]a^{m+1}a^{\oplus} = (a^\#)^m(a^{\oplus})^n a^{m+1}a^{\oplus}\\ &= (a^\#)^m[(a^{\oplus})^n a^{n+1}](a^\#)^n a^m a^{\oplus} = (a^\#)^m a(a^\#)^n a^m a^{\oplus}\\ &= (a^\#)^{m+n}a^{m+1}a^{\oplus} = a(a^\#)^n a^{\oplus} = aa^{\oplus}a(a^\#)^n a^{\oplus}\\ &= a(a^{\oplus})^{n+1}[a^{n+1}(a^\#)^n a^{\oplus}] = a(a^{\oplus})^{n+1}aa^{\oplus}\\ &= a(a^{\oplus})^{n+1} = (a^{\oplus})^n.\end{aligned}$$

(10). Multiplying $(a^*)^n(a^\#)^m = (a^\#)^m(a^{\oplus})^n$ by $a^{m+1}a^{\oplus}$ from the right side, we get $(a^*)^n = (a^{\oplus})^n$ as in part (9).

(11) Notice that multiplying $(a^*)^n(a^\#)^m = (a^{\oplus})^{n+m}$ by $a^{m+1}a^{\oplus}$ from the right side, it follows that $(a^*)^n = (a^{\oplus})^n$. □

Remark that Theorem 6.2 recovers some conditions of [21, Theorem 4.3] for an element a satisfying $a^* = a^{\oplus}$.

References

[1] O.M. Baksalary, G.P.H. Styan, G. Trenkler, On a matrix decomposition of Hartwig and Spindelböck, Linear Algebra Appl. 430(10) (2009) 2798–2812.
[2] O.M. Baksalary, G. Trenkler, Characterizations of EP, normal and Hermitian matrices, Linear Multilinear Algebra 56 (2006) 299–304.
[3] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58(6) (2010) 681–697.
[4] K.P.S. Bhaskara Rao, The Theory of Generalized Inverses Over Commutative Rings, CRC Press, 2002.
[5] E. Boasso, Factorizations of EP Banach operators and EP Banach algebra elements, J. Math. Anal. Appl. 379 (2011) 245–255.
[6] S. Cheng, Y. Tian, Two sets of new characterizations for normal and EP matrices, Linear Algebra Appl. 375 (2003) 181–195.
[7] D.S. Djordjević, V. Rakočević, Lectures on Generalized Inverses, Faculty of Sciences and Mathematics, University of Niš, 2008.
[8] D.S. Djordjević, Characterization of normal, hyponormal and EP operators, J. Math. Anal. Appl. 329(2) (2007) 118–1190.
[9] D.S. Djordjević, J.J. Koliha, Characterizing Hermitian, normal and EP operators, Filomat 21(1) (2007) 39–54.
[10] R.E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commmute, Linear Multilinear Algebra 14 (1984) 241–256.
[11] J.J. Koliha, P. Patrício, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. 72(1) (2002) 137–152.
[12] D. Mosić, D.S. Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431 (2009) 732–745.
[13] D. Mosić, D.S. Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings, Applied Math. Comput. 218(12) (2012) 6702–6710.
[14] D. Mosić, D.S. Djordjević, Partial isometries and EP elements in rings with involution, Electron. J. Linear Algebra 18 (2009) 761–772.
[15] D. Mosić, D.S. Djordjević, Further results on partial isometrices and EP elements in rings with involution, Math. Computer Modelling 54 (2011) 460–465.
[16] D. Mosić, D.S. Djordjević, J.J. Koliha, EP elements in rings, Linear Algebra Appl. 431 (2009) 527–535.
[17] P. Patrício, R. Puystjens, Drazin-Moore-Penrose invertiblity in rings, Linear Algebra Appl. 389 (2004) 159–173.
[18] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955) 406–413.
[19] D.S. Rakić, N. Č. Dinčić, D.S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115–133.
[20] S.Z. Xu, J.L. Chen, J. Benítez, EP elements in rings with involution, Bull. Malays. Math. Sci. Soc. 42(6) (2019) 3409–3426.
[21] S.Z. Xu, J.L. Chen, D. Mosić, On characterizations of special elements in rings with involution, Chin. Ann. Math. Ser. B 40 (1) (2019) 65–78.
[22] S.Z. Xu, J.L. Chen, X.X. Zhang, New characterizations for core and dual core inverses in rings with involution, Front. Math. China. 12 (1) (2017) 231–246.