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More on Arhangel'skiĭ Sheaf Amalgamations

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Abstract. We proceed to consider the topic of ideal-convergence. In particular, we introduce the ideal-version of Arhangel'skiĭ sheaf amalgamations, and exam the relations among the ideal-version of Arhangel'skiĭ sheaf amalgamations, ideal-version of QN-spaces and ideal-version of covering properties. Some characterizations of ideal-version of Arhangel'skiĭ sheaf amalgamations will be given. Our observations extend some classic results.

1. Introduction

All spaces are assumed to be infinite completely regular Hausdorff. For a given space X, and $x \in X$, the following notation, named as the *sheaf* at x, will be used throughout this note (see, [23]).

- Γ_x denotes the set of all nontrivial countable infinite sequences that converge to x.

In 1972, A.V. Arhangel'skiĭ [1] introduced the following local properties, which are named as Arhangel'skiĭ sheaf amalgamations (or Arhangel'skiĭ α_i-properties).

Definition 1.1. Let X be a topological space,

α_1 -space: X is an α_1-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ of elements of Γ_x, there is a single element $O \in \Gamma_x$ such that $O_n \setminus O$ is finite for each $n \in \mathbb{N}$.

α_2 -space: X is an α_2-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ of elements of Γ_x, there is a single element $O \in \Gamma_x$ such that $O_n \cap O$ is infinite for each $n \in \mathbb{N}$.

α_3 -space: X is an α_3-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ of elements of Γ_x, there is a single element $O \in \Gamma_x$ such that $O_n \cap O$ is infinite for infinitely many $n \in \mathbb{N}$.

α_4 -space: X is an α_4-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ of elements of Γ_x, there is a single element $O \in \Gamma_x$ such that $O_n \cap O$ is nonempty for infinitely many $n \in \mathbb{N}$.

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Lots of interesting notions (e.g, QN-spaces, wQN-spaces, covering properties) although defined for essentially different purpose, are closely related to these α_i properties. The idea of α_i-properties was applied to general selection principles theory (see, [18], [19]). The references dealing with α_i-properties are too numerous to be listed here. When the ideal convergence appeared, for every theorem which deals with convergence of sequences there is a natural question whether this theorem can be generalized in some sense to the corresponding ideal-version? If not, then for which class of ideals such generalization is possible. The ideal version of QN-spaces and wQN-spaces have been considered recently (see, [5], [9], [26]). There may be interesting questions to ask, for example, how about the ideal-version of Arhangel'skiǐ sheaf amalgamations? Is there any relation among ideal-QN spaces, ideal-covering properties and ideal-Arhangel'skiǐ sheaf amalgamations?

This paper is devoted to researching these questions. To beginning, we need to recall some necessary notions.

Let $\mathbb{N}$ be the set of all natural numbers. An ideal on $\mathbb{N}$ is a nonempty family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements. By *Fin* we denote the ideal of all finite subsets of $\mathbb{N}$. If not explicitly said we assume that all considered ideals contain *Fin* and are proper (not contain $\mathbb{N}$). Let X be a topological space. The set $\mathbb{R}^X$ of all real functions: $X \to \mathbb{R}$ is endowed with the Tychonoff product topology. Let $C_p(X)$ be the set of all continuous real functions: $X \to \mathbb{R}$ endowed with the topology which it inherits as subset of $\mathbb{R}^X$.

Definition 1.2. Let X be a topological space, $x \in X$, $\mathcal{I}$ being an ideal on $\mathbb{N}$.

- A sequence $(x_n : n \in \mathbb{N})$ of X is $\mathcal{I}$-convergent to x if for every open neighborhood U of x, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.
- A sequence $(f_n : n \in \mathbb{N})$ from $\mathbb{R}^X$ is $\mathcal{I}$-convergent to f if $(f_n(x) : n \in \mathbb{N})$ is $\mathcal{I}$-convergent to $f(x)$ for every $x \in X$.

It is good to notice that the notion of ideal convergence is a generalization of the classical one. It was first considered in the case of the ideal of sets of statistical density 0 by Steinhaus and Fast [14].

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, $x \in X$. we denote by $\mathcal{I}$-Γ_x the set of all all nontrivial countable infinite sequences that $\mathcal{I}$-converge to x. Now, we introduce the following key definitions.

Definition 1.3. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$, X being a topological space.

$(\mathcal{I},\mathcal{J})\alpha_1$ -space: X is an $(\mathcal{I},\mathcal{J})\alpha_1$-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ from $\mathcal{I}$-Γ_x, there is a single element $O \in \mathcal{J}$-Γ_x such that $O_n \setminus O$ is finite for each $n \in \mathbb{N}$.

$(\mathcal{I},\mathcal{J})\alpha_2$ -space: X is an $(\mathcal{I},\mathcal{J})\alpha_2$-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ from $\mathcal{I}$-Γ_x, there is a single element $O \in \mathcal{J}$-Γ_x such that $O_n \cap O$ is infinite for each $n \in \mathbb{N}$.

$(\mathcal{I},\mathcal{J})\alpha_3$ -space: X is an $(\mathcal{I},\mathcal{J})\alpha_3$-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ from $\mathcal{I}$-Γ_x, there is a single element $O \in \mathcal{J}$-Γ_x such that $O_n \cap O$ is infinite for infinitely many $n \in \mathbb{N}$.

$(\mathcal{I},\mathcal{J})\alpha_4$ -space: X is an $(\mathcal{I},\mathcal{J})\alpha_4$-space if for each $x \in X$, each sequence $(O_n : n \in \mathbb{N})$ from $\mathcal{I}$-Γ_x, there is a single element $O \in \mathcal{J}$-Γ_x such that $O_n \cap O$ is nonempty for infinitely many $n \in \mathbb{N}$.

It is easy to see that the $(Fin, Fin)\alpha_i$-spaces coincide with the corresponding α_i-spaces. For convenience, we denote by $\mathcal{I}\alpha_i$-spaces the $(Fin, \mathcal{I})\alpha_i$-spaces. The following is obvious,

$$(\mathcal{I},\mathcal{J})\alpha_1 \Rightarrow (\mathcal{I},\mathcal{J})\alpha_2 \Rightarrow (\mathcal{I},\mathcal{J})\alpha_3 \Rightarrow (\mathcal{I},\mathcal{J})\alpha_4$$

Note that for any ideals $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_1$ and $\mathcal{J}_2$ on $\mathbb{N}$ such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$, $\mathcal{J}_1 \subseteq \mathcal{J}_2$,

$$X \text{ is an } (\mathcal{I}_2,\mathcal{J}_1)\alpha_i\text{-space} \Rightarrow X \text{ is an } (\mathcal{I}_1,\mathcal{J}_2)\alpha_i\text{-space}.$$

The paper is organized as follows: Section 2 introduces basic notions, and Section 3 discusses the $\mathcal{I}\alpha_2$-spaces and $\mathcal{I}\alpha_4$-spaces. Section 4 is devoted to investigation of $\mathcal{I}\alpha_1$-spaces. Finally, the relations among our main definitions and the $(\mathcal{I},\mathcal{J}\text{-}\alpha_1)$, $(\mathcal{I},\mathcal{J}\text{-}\alpha_4)$ introduced in [5] will be discussed in Section 5.

2. Preliminaries

We shall use standard terminology and notations of topology and set theory (see [3, 12]). For any set X, $\mathcal{P}(X)$ denotes the power set of X; $|X|$ denotes the cardinality of X. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, $\mathcal{I}^+ = \{A \subseteq \mathbb{N} : A \notin \mathcal{I}\}$, $\mathcal{I}^* = \{A \subseteq \mathbb{N} : A^c \in \mathcal{I}\}$, where A^c is $\mathbb{N} \setminus A$.

2.1. Selection Principles on Covers

Let X be a topological space. A collection $\mathcal{U}$ of subsets of X is a cover of X if $\bigcup \mathcal{U} = X$ and $X \notin \mathcal{U}$. Recall that a cover $\mathcal{U}$ of X is a γ-cover if for every $x \in X$, x is contained in all but finitely many members of $\mathcal{U}$. Let $\mathcal{O}(X)$ denote the family of all open covers of X, and $\Gamma(X)$ be the collection of all open γ-covers. When X is clear from the context, we shall write $\mathcal{O}$, Γ instead of $\mathcal{O}(X)$, $\Gamma(X)$. Similarly, let Γ_F, Γ_{cl}, Γ_B denote the families of all countable closed γ-covers of X, all clopen γ-covers and all Borel γ-covers respectively.

In [24], Marion Scheepers began a systematic study of selection principles in topology and their relations to game theory and Ramsey theory. Let $\mathcal{A}$, B be families of subsets of X, let's recall the following types of selection principles:

- $S_1(\mathcal{A}, \mathcal{B})$: For any sequence $(\mathcal{U}_n \in \mathcal{A} : n \in \mathbb{N})$, there is a $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$.
- $S_{fin}(\mathcal{A}, \mathcal{B})$: For any sequence $(\mathcal{U}_n \in \mathcal{A} : n \in \mathbb{N})$, there is a finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \mathcal{B}$.
- $U_{fin}(\mathcal{A}, \mathcal{B})$: For any sequence $(\mathcal{U}_n \in \mathcal{A} : n \in \mathbb{N})$, there is a finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\{\cup \mathcal{V}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

2.2. Sequence Selection Property

A space X has the *sequence selection property* if for every $x \in X$, $S_1(\Gamma_x, \Gamma_x)$ holds. The sequence selection property was introduced by M. Scheepers in [22]. It is well known that X is an α_2-space iff X has the $S_1(\Gamma_x, \Gamma_x)$-property. We modify this notion by ideals on $\mathbb{N}$ as follow.

Definition 2.1. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A space X has *$\mathcal{I}$-sequence selection property* if for every $x \in X$, $S_1(\Gamma_x, \mathcal{I}\text{-}\Gamma_x)$ holds.

In the space $C_p(X)$, let $\bar{0}$ denote the zero function.

- $\Gamma_{\bar{0}}$ denotes all countable infinite sequences from $C_p(X)$ that converge pointwise to $\bar{0}$;
- $\mathcal{I}$-$\Gamma_{\bar{0}}$ denotes all countable infinite sequences from $C_p(X)$ that $\mathcal{I}$-converge point-wise to $\bar{0}$.

Note that $C_p(X)$ is homogeneous (as a topological group), so $C_p(X)$ has property $S_1(\Gamma_{\bar{0}}, \Gamma_{\bar{0}})$ if and only if $C_p(X)$ has the sequence selection property.

2.3. QN-Spaces and wQN-Spaces

The notions of QN-spaces and wQN-spaces are introduced in [7], and they are extended in [9] as follow (see also, [26]).

Definition 2.2. Let $\mathcal{J}$ be an ideal on $\mathbb{N}$, and X be a topological space.

(1) X is called a $\mathcal{J}$QN-space if any sequence $(f_n : n \in \mathbb{N}) \in \Gamma_{\bar{0}}$ is $\mathcal{J}$QN-convergent (there exists a sequence $(\varepsilon_n > 0 : n \in \mathbb{N}) \in \mathcal{J}\text{-}\Gamma_0$ such that for every $x \in X$, $\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon_n\} \in \mathcal{J}$).

(2) X is called a $\mathcal{J}$wQN-space if for any sequence $(f_n : n \in \mathbb{N}) \in \Gamma_{\bar{0}}$, there exists a sequence $(n_k : k \in \mathbb{N})$ of natural numbers such that $(f_{n_k} : k \in \mathbb{N})$ is $\mathcal{J}$QN-convergent to $\bar{0}$ (there exists sequence $(\varepsilon_k > 0 : n \in \mathbb{N}) \in \mathcal{J}\text{-}\Gamma_0$ such that for every $x \in X$, $\{k \in \mathbb{N} : |f_{n_k}(x)| \geq \varepsilon_k\} \in \mathcal{J}$)

The notions of $(\mathcal{I},\mathcal{J})$QN-spaces, $(\mathcal{I},\mathcal{J})$wQN-spaces are defined analogously. The relations among α_i-spaces, QN-spaces and wQN-spaces were revealed by the following result.

Theorem 2.3. ([4, 21, 23]) *For any Tychonoff space X, the following hold.*

(1) X is a QN-space if, and only if, $C_p(X)$ is an α_1-space.

(2) X is a wQN-space if, and only if, $C_p(X)$ is an α_2-space.

2.4. Orderings

Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. For a map $\varphi:\mathbb{N}\to\mathbb{N}$, the image of $\mathcal{J}$ is defined by $\varphi(\mathcal{J})=\{A\subseteq\mathbb{N}:\varphi^{-1}(A)\in\mathcal{J}\}$. Clearly, $\varphi(\mathcal{J})$ is closed under subsets and finite unions and $\mathbb{N}\notin\varphi(\mathcal{J})$. Moreover, if φ is finite-to-one then $\varphi(\mathcal{J})$ is an ideal.

Definition 2.4. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$.

$\leq_K$: For a function $\varphi:\mathbb{N}\to\mathbb{N}$ we write $\mathcal{I}\leq_\varphi\mathcal{J}$ if $\mathcal{I}\subseteq\varphi(\mathcal{J})$, i.e, $\varphi^{-1}(A)\in\mathcal{J}$ for any $A\in\mathcal{I}$ ([17]);

$\leq_{KB}$: $\mathcal{I}\leq_{KB}\mathcal{J}$ if there is a finite-to-one function $\varphi:\mathbb{N}\to\mathbb{N}$ such that $\mathcal{I}\leq_\varphi\mathcal{J}$ ([15]).

3. $\mathcal{I}\alpha_2$-Spaces and $\mathcal{I}\alpha_4$-Spaces

This section is devoted to studying $\mathcal{I}\alpha_2$-spaces and $\mathcal{I}\alpha_4$-spaces. In particular, we shall put effort to find characterizations of these properties.

We say that a countable set converges to x if it has a bijective enumeration converges to x. When saying $\Lambda\in\Gamma_x$ or $\mathcal{I}$-Γ_x, we always assume that the index set of Λ is $\mathbb{N}$ (that is, bijective enumeration).

Theorem 3.1. *Let $\mathcal{I}_1,\mathcal{I}_2,\mathcal{J}_1,\mathcal{J}_2$ be ideals on $\mathbb{N}$ such that $\mathcal{I}_1\leq_{KB}\mathcal{I}_2$, X being a topological space.*

(1) If $C_p(X)$ is an $\mathcal{I}_1\alpha_2$-space, then it is an $\mathcal{I}_2\alpha_2$-space.

(2) Moreover, assume that $\mathcal{J}_1\leq_{KB}\mathcal{J}_2$. If $C_p(X)$ is an $(\mathcal{I}_2,\mathcal{J}_1)\alpha_2$-space, then it is an $(\mathcal{I}_1,\mathcal{J}_2)\alpha_2$-space.

Proof. Since $\mathcal{I}_1\leq_{KB}\mathcal{I}_2$, assume that $\varphi:\mathbb{N}\to\mathbb{N}$ is a finite-to-one function such that $\mathcal{I}_1\leq_\varphi\mathcal{I}_2$. Let $(\Lambda_n:n\in\mathbb{N})$ be a sequence valued in $\Gamma_{\bar{0}}$. Assume that for each $n\in\mathbb{N}$,

$$\Lambda_n=(f_k^n:k\in\mathbb{N}).$$

Since $C_p(X)$ is an $\mathcal{I}_1\alpha_2$-space, there exists $\Lambda\in\mathcal{I}_1$-$\Gamma_{\bar{0}}$ such that $\Lambda\cap\Lambda_n$ are infinite for all $n\in\mathbb{N}$. Assume that

$$\Lambda=(f_k:k\in\mathbb{N}).$$

Now, we rearrange via φ each Λ_n as $(f^n_{\varphi(k)}:k\in\mathbb{N})$, and Λ as $(f_{\varphi(k)}:k\in\mathbb{N})$. Note that for each Λ_n, its indexed set is $\mathbb{N}$, so $\{f^n_{\varphi(k)}:k\in\mathbb{N}\}\subseteq\{f^n_k:k\in\mathbb{N}\}$. This ensures that the sequence $(f_{\varphi(k)}:k\in\mathbb{N})$ intersects with each Λ_n infinitely. It is enough to show that $(f_{\varphi(k)}:k\in\mathbb{N})\in\mathcal{I}_2$-$\Gamma_{\bar{0}}$. Note that, for each $x\in X$, each $\varepsilon>0$,

$$\{\varphi(k):|f_{\varphi(k)}(x)|\geq\varepsilon\}\subseteq\varphi^{-1}(\{k:|f_k(x)|\geq\varepsilon\})$$

We finished the proof.

(2) We also assume that the finite-to-one function φ: $\omega\to\omega$ witnesses that $\mathcal{I}_1\leq_\varphi\mathcal{I}_2$, and the finite-to-one function ψ: $\mathbb{N}\to\mathbb{N}$ witnesses that $\mathcal{J}_1\leq_\psi\mathcal{J}_2$. Let $(\Lambda_n:n\in\mathbb{N})$ be a sequence from $\mathcal{I}_1$-$\Gamma_{\bar{0}}$. Assume that

$$\Lambda_n=(f_k^n:k\in\mathbb{N}).$$

We rearrange each Λ_n via φ as

$$\widetilde{\Lambda_n}=(f^n_{\varphi(k)}:k\in\mathbb{N}).$$

Then for each $n \in \mathbb{N}$, $\widetilde{\Lambda_n} \in \mathcal{I}_2\text{-}\Gamma_{\bar{0}}$. Indeed, for each $x \in X$ and each $\varepsilon > 0$

$$\{\varphi(k) : |f^n_{\varphi(k)}(x)| \geq \varepsilon\} \subseteq \varphi^{-1}(\{k : |f^n_k(x)| \geq \varepsilon\}).$$

Since $C_p(X)$ is an $(\mathcal{I}_2, \mathcal{J}_1)\alpha_2$-space, there exists $\Lambda \in \mathcal{J}_1\text{-}\Gamma_{\bar{0}}$ such that it intersects with each $\widetilde{\Lambda_n}$ infinitely. Assume that $\Lambda = (f_k : k \in \mathbb{N})$. Again, we rearrange $\widetilde{\Lambda_n}$ by

$$\widetilde{\widetilde{\Lambda_n}} = (f^n_{\psi(\varphi(k))} : k \in \mathbb{N}).$$

and

$$\widetilde{\Lambda} = (f_{\psi(k)} : k \in \mathbb{N}).$$

Then $\widetilde{\Lambda} \in \mathcal{J}_2\text{-}\Gamma_{\bar{0}}$ for each $n \in \mathbb{N}$ since for each $x \in X$ and ε,

$$\{\psi(k) : |f_{\psi(k)}(x)| \geq \varepsilon\} \subseteq \psi^{-1}(\{k : |f_k(x)| \geq \varepsilon\}).$$

Since ψ is finite-to-one, $\widetilde{\Lambda}$ intersects with each $\widetilde{\widetilde{\Lambda_n}}$ infinitely. Note also that $\{f^n_{\psi(\varphi(k))} : k \in \mathbb{N}\} \subseteq \{f^n_k : k \in \mathbb{N}\}$, so $\widetilde{\Lambda}$ intersects with each Λ_n infinitely, and then we complete the proof. □

With the same argument of the proof of Lemma 1 in [23], it is easy to get the following result.

Lemma 3.2. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following hold true.*

(a) $(f_n : n \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$ *if and only if* $(|f_n| : n \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$.

(b) If $(f^n_m : m \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$ *for every* $n \in \mathbb{N}$, *then* $(\sum_{n\leq k} f_{n,m} : m \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$ *for any* $k \in \mathbb{N}$.

Let $C \subseteq \mathbb{N}$, and $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$. Recall that C is called a pseudounion of $\mathcal{A}$ if

- $\mathbb{N} \setminus C$ is infinite;
- $A \subseteq^* C$ for any $A \in \mathcal{A}$,

where $A \subseteq^* C$ means $A \setminus C$ is finite.

It is well known that $\mathcal{I}$ has a pseudounion if and only if $\mathcal{I} \leq_{KB} Fin$ (see, e.g, [26, Remark 4.2]). Recall that if $\mathcal{I}$ has a pseudounion, then X is a wQN-space if and only if X is an $\mathcal{I}wQN$-space ([5], Corollary 3.4). So, under the assumption of $\mathcal{I} \leq_{KB} Fin$, $\mathcal{I}\alpha_i$ property is equal to α_i for $i = 2, 3, 4$.

Definition 3.3. ([26]) Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. $\mathcal{J}$ is called a weak $P(\mathcal{I})$ ideal if for each sequence $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$, there exist $A \in \mathcal{I}^+$ such that $A \cap A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Lemma 3.4. ([26, Lemmaa 2.2]) *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent:*

(1) $\mathcal{J}$ is not a weak $P(\mathcal{I})$-ideal.

(2) There is a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that for any $\{B_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ we have

$$\bigcup_{n\in\mathbb{N}} (B_n \cap A_n) \in \mathcal{J}.$$

Theorem 3.5. *Let $\mathcal{I}, \mathcal{J}$ be ideals. If $\mathcal{J}$ is not a weak $P(\mathcal{I})$-ideal, then for each topological space X, $C_p(X)$ is an $(\mathcal{I}, \mathcal{J})\alpha_4$-space.*

Proof. By the assumption there is a partition $\{A_k : k \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that for any $\{B_k : k \in \mathbb{N}\} \subseteq \mathcal{I}$

$$(\dagger) \quad \bigcup_{k\in\mathbb{N}} (B_k \cap A_k) \in \mathcal{J}.$$

Let $(\Lambda_n : n \in \mathbb{N})$ be a sequence from $\mathcal{I}\text{-}\Gamma_{\bar{0}}$. We may assume that for each $n \in \mathbb{N}$,

$$\Lambda_n = (f^n_m : m \in \mathbb{N}).$$

Claim 3.6. $(f^n_m : m \in A_n, n \in \mathbb{N}) \in \mathcal{J}\text{-}\Gamma_{\bar{0}}$.

Proof. For each $x \in X$ and each $\varepsilon > 0$, we have that $\{m \in \mathbb{N} : |f_{n,m}(x)| \geq \epsilon\} \in \mathcal{I}$ since $\Lambda_n \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$. Note that

$$\bigcup_{n\in\mathbb{N}} \{m \in A_n : |f_{n,m}(x)| \geq \epsilon\} = \bigcup_{n\in\mathbb{N}} (A_n \cap \{m \in \mathbb{N} : |f_{n,m}(x)| \geq \epsilon\})$$

By (†), the right side of the equation belongs to $\mathcal{J}$, and then $\{m \in A_n, n \in \mathbb{N} : |f_{n,m}(x)| \geq \epsilon\} \in \mathcal{J}$. This implies that $(f^n_m : m \in A_n, n \in \mathbb{N}) \in \mathcal{J}\text{-}\Gamma_{\bar{0}}$. □

It is easy to see that the sequence $(f^n_m : m \in A_n, n \in \mathbb{N})$ intersects with infinitely many Λ_n. □

Corollary 3.7. *If $\mathcal{I}$ is not a weak P-ideal, then for every topological space X, $C_p(X)$ is an $\mathcal{I}\alpha_4$-space.*

4. $\mathcal{I}\alpha_1$-Space

In [27], Tsaban-Zdomskyy obtained an elegant characterization of α_1-spaces via Borel images in $\mathbb{N}^{\mathbb{N}}$: For any perfectly normal space X, $C_p(X)$ is an α_1 space if, and only if each Borel image of X into $\mathbb{N}^{\mathbb{N}}$ is bounded. We are interested in finding a similar characterization for $\mathcal{I}\alpha_1$-spaces. The natural candidate for the bound is the following key definition.

Definition 4.1. ([13]) Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. For any $f, g \in \mathbb{N}^{\mathbb{N}}$, we say that $f \leq_{\mathcal{I}} g$ if $[f > g] = \{n : f(n) > g(n)\} \in \mathcal{I}$. $X \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathcal{I}$-bounded if there exists $g \in \mathbb{N}^{\mathbb{N}}$ such that $(\forall f \in X)(f \leq_{\mathcal{I}} g)$.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Recall that $\mathcal{B} \subseteq \mathcal{I}$ is a basis if $(\forall I \in \mathcal{I})(\exists B \in \mathcal{B})(I \subseteq B)$.

Theorem 4.2. *Let $\mathcal{I}$ be an ideal with a pseudounion, X being a perfectly normal space. Then the following statements are equivalent.*

(1) $C_p(X)$ is an $\mathcal{I}\alpha_1$-space.

(2) Each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is $\mathcal{I}$-bounded.

(3) X satisfies $U_{fin}(\mathcal{B}, \mathcal{I}\text{-}\Gamma_{\mathcal{B}})$.

According to Corollary 3.4 in [5], if $\mathcal{I}$ is an ideal with a pseudounion, in the realm of perfectly normal spaces, $C_p(X)$ is an $\mathcal{I}\alpha_1$-space if, and only if it is an α_1-space. Among the result of Tsaban-Zdomskky mentioned above, the implication (1) ⇒ (2) is obvious. The following lemma shows (2) ⇒ (1).

Lemma 4.3. *Let X be a perfectly normal space, $\mathcal{I}$ being an ideal on $\mathbb{N}$ with a pseudounion. If each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is $\mathcal{I}$-bounded, then $C_p(X)$ is an $\mathcal{I}\alpha_1$-space.*

Proof. Let $C \subsetneq \mathbb{N}$ be a pseudounion of $\mathcal{I}$. That is, $\mathbb{N} \setminus C$ is infinite and $A \subset^* C$ for every $A \in \mathcal{I}$. Let $(\Lambda_n : n \in \mathbb{N})$ be a sequence from $\Gamma_{\bar{0}}$. Assume that for each $n \in \mathbb{N}$,

$$\Lambda_n = (f^n_m : m \in \mathbb{N}).$$

Define a map ψ: $X \to \mathbb{N}^{\mathbb{N}}$ by

$$\psi(x)(0) = \min\{k : \wedge(\forall m \geq k) \mid f^0_m(x) \mid < 1\}.$$

For each $n \geq 1$,

$$\psi(x)(n+1) = \min\{k : k > \psi(x)(n) \wedge (\forall m \geq k) \mid f^n_m(x) \mid < \tfrac{1}{n+1}\}.$$

This is a Borel map, so $\psi[X]$ is $\mathcal{I}$-bounded in $\mathbb{N}^{\mathbb{N}}$, and assume that $g \in \mathbb{N}^{\mathbb{N}}$ witnesses this. Moreover, we may assume that g is nondecreasing (Otherwise, we can replace g by $\tilde{g}$: $\tilde{g}(n) = \sum_{i\leq n} g(i)$). Note that for each $x \in X$, $\mathbb{N} \setminus C \subset^* [\psi(x) \leq g]$. Set $h = g \circ (\mathbb{N} \setminus C)$, we have that

$$\psi(x) \leq^* \psi(x) \circ (\mathbb{N} \setminus C) \leq^* h.$$

The first inequality holds due to the fact that each $\psi(x)$ is nondecreasing, and the second one thanks to the assumption that g is nondecreasing. Put

$$\Lambda = (f^n_m : m \geq h(n), n \in \mathbb{N}).$$

Therefore, $\psi[X] \leq^* h$, which implies that Λ is convergent to $\bar{0}$. □

Lemma 4.4. *X satisfies $U_{fin}(\mathcal{B}, \mathcal{I}\text{-}\Gamma_{\mathcal{B}})$ if and only if each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is $\mathcal{I}$-bounded.*

Proof. (⇒) For each n, m, let

$$U^n_m = \{x : f(x)(n) \leq m\}.$$

Then for each n, $\{U^n_m : m \in \mathbb{N}\}$ is a Borel cover of X. Since X satisfies $U_{fin}(\mathcal{B}, \mathcal{I}\text{-}\Gamma_{\mathcal{B}})$, there exists a $g \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$$

is an $\mathcal{I}$-B_Γ-cover of X, where $\mathcal{V}_n = \{U^n_m : m \leq g(n)\}$. For each x,

$$[g \leq f(x)] \subseteq \{n : x \notin \cup\mathcal{V}_n\} \in \mathcal{I}.$$

This shows that $f[X]$ is $\mathcal{I}$-bounded in $\mathbb{N}^{\mathbb{N}}$.

(⇐) For any sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of Borel covers of X. Assume that for each n,

$$\mathcal{U}_n = \{U^n_m : n \in \mathbb{N}\}.$$

Without loss of generality, we may assume that for each n, $\mathcal{U}_n$ is consisting of pairwise disjoint Borel sets. Define a function f: $X \to \mathbb{N}^{\mathbb{N}}$ by

$$f(x)(n) = m \text{ if } m = \min\{k : x \in U^n_k\} \text{ for each } n \in \mathbb{N}.$$

Then f is a Borel map, so there exists $g \in \mathbb{N}^{\mathbb{N}}$ that is an $\mathcal{I}$-bound for $f[X]$. Put

$$\mathcal{V}_n = \{U^n_k : k \leq f(n)\}.$$

For each $x \in X$,

$$\{n : n \notin \cup\mathcal{V}_n\} \subseteq [g \leq f(x)] \in \mathcal{I}$$

Then $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is an $\mathcal{I}$-B_Γ cover of X. □

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Denote by $\mathfrak{b}_{\mathcal{I}}$ the minimal size of $\mathcal{I}$-unbounded families in $\mathbb{N}^{\mathbb{N}}$ ([13]).

A sequence $\Lambda = (f_n : n \in \mathbb{N}) \in \Gamma_{\bar{0}}$ is called *monotonically convergens* to $\bar{0}$ if for each $x \in X$, $f_n(x) \geq f_{n+1}(x)$ for each $n \in \mathbb{N}$. Modifying the monotonic sequence selection property introduced in [22], we define the following notion.

Definition 4.5. ([8]) Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, and X being a perfectly normal space. We say that X has the $\mathcal{I}$-monotonic sequence selection property (written, $S^M_1(\Gamma_{\bar{0}}, \mathcal{I}\text{-}\Gamma_{\bar{0}})$) if for each sequence $(\Lambda_n : n \in \mathbb{N})$ such that each Λ_n is monotonically converges to $\bar{0}$. There exists $\Lambda \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$ such that $|\Lambda \cap \Lambda_n| = 1$ for each n.

Definition 4.6. ([10]) Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, X being a topological space. X is called has the $\mathcal{I}$-Hurewicz property if for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of γ-covers, there exists for each n a finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is an $\mathcal{I}$-γ-cover (that is, for each $x \in X$, $\{n : x \notin \cup\mathcal{V}_n\} \in \mathcal{I}$).

Theorem 4.7. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The minimal size of $X \subset \mathbb{R}$ such that $C_p(X)$ is not an $\mathcal{I}\alpha_1$-space is $\mathfrak{b}_{\mathcal{I}}$.*

Note that the minimal size of $X \subset \mathbb{R}$ such that X does not have $\mathcal{I}$-Hurewicz is $\mathfrak{b}_{\mathcal{I}}$ ([11], Theorem 3.1). In addition, it is easy to see that if $C_p(X)$ is an $\mathcal{I}\alpha_1$-space, then it has the $S(\Gamma_{\bar{0}}, \mathcal{I}\text{-}\Gamma_{\bar{0}})$ property, and then has the $S_1^M(\Gamma_{\bar{0}}, \mathcal{I}\text{-}\Gamma_{\bar{0}})$ property.

For any topological space X, if X has the $\mathcal{I}$-Hurewicz property then $C_p(X)$ has the $S_1^M(\Gamma_{\bar{0}}, \mathcal{I}\text{-}\Gamma_{\bar{0}})$ ([8], Theorem 3.4). Moreover, almost literal repetition of the proof of Theorem 1 in [22], give us the following. For the reader's convenience, we present here the proof.

Lemma 4.8. *Let X be a perfectly normal space. If $C_p(X)$ has the $S_1^M(\Gamma_{\bar{0}}, \mathcal{I}\text{-}\Gamma_{\bar{0}})$, then X has the $\mathcal{I}$-Hurewicz property.*

Proof. Let $(\mathcal{U}_m : m \in \mathbb{N})$ be a sequence of γ-covers of X, and assuming

$$\mathcal{U}_m = (U_n^m : n \in \mathbb{N}).$$

Thanks to the perfect normality of X, we further assume that for each m, each n,

$$U_n^m = \bigcup_{k \in \mathbb{N}} U_{n,k}^m,$$

where $(U_{n,k}^m : k \in \mathbb{N})$ is a increasing sequence of closed sets. Define for each m, each n, each k, a continuous function $f_{n,k}^m$: $X \to [0,1]$ by

$$f_{n,k}^m(x) = \begin{cases} 0, & if\ x \in U_{n,k}^m \\ 1, & if\ x \notin U_n^m \end{cases} \tag{1}$$

Define for each m, n a function $g_n^m(x)$: $X \to [0,1]$ by

$$g_n^m(x) = |\prod_{j \le n} f_{j,n}^m(x)|.$$

Then for each m, let $\Lambda_m = (g_n^m : n \in \mathbb{N})$, then $\Lambda_m \in \Gamma_{\bar{0}}$ and is monotonically. By the property $S_1^M(\Gamma_{\bar{0}}, \mathcal{I}\text{-}\Gamma_{\bar{0}})$, there exists $\Lambda \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$ such that $|\Lambda \cap \Lambda_m| = 1$. Assume that

$$\Lambda = (g_{n_m}^m : m \in \mathbb{N}), \text{ where } g_{n_m}^m \in \Lambda_m.$$

Now, let $\mathcal{V}_m = \{U_j^m : j \le n_m\}$. Then $\{\cup\mathcal{V}_m : m \in \mathbb{N}\} \in \mathcal{I}\text{-}\Gamma$. Indeed, for each $x \in X$, let $A_x = \{m : |g_{n_m}^m(x)| \ge 1\}$, then $A_x \in \mathcal{I}$ since $\Lambda \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$. So

$$(\forall m \in (\mathbb{N} \setminus A_x))(\exists j \le n_m)(|f_{j,n_m}^m(x)| < 1).$$

This implies that $x \in U_j^m$, and then $x \in \cup\mathcal{V}_m$. Therefore, $\{m : x \notin \cup\mathcal{V}_m\} \subseteq A_x \in \mathcal{I}$. □

Proof of Theorem 4.7 For each $X \subset \mathbb{R}$ with $|X| < \mathfrak{b}_{\mathcal{I}}$, $C_p(X)$ is an $\mathcal{I}\alpha_1$-space by Theorem 4.3. Taking $X \subset \mathbb{R}$ with size $\mathfrak{b}_{\mathcal{I}}$ such that it is not an $\mathcal{I}$-Hurewicz space. Then $C_p(X)$ is not an $\mathcal{I}\alpha_1$-space.

Remark 4.9. It was showed that for each analytic P-deal $\mathcal{I}$, $\mathfrak{b} = \mathfrak{b}_{\mathcal{I}}$ ([13], Corollary 5.5). Thus, in the realms of analytic P-ideals, the minimal size of $X \subset \mathbb{R}$ such that $C_p(X)$ is not an $\mathcal{I}\alpha_1$-space is $\mathfrak{b}_{\mathcal{I}}$.

The next work is motivated by Lemma 30 in [27]. For $h \in \mathbb{N}^{\mathbb{N}}$, let $A_h = \{(n, m); m \le h(n), n \in \mathbb{N}\}$ and

$$\mathcal{I}_b = \{B \subseteq \mathbb{N} \times \mathbb{N} : (\exists h \in \mathbb{N}^{\mathbb{N}})(B \subseteq A_h)\}.$$

This is an ideal on $\mathbb{N} \times \mathbb{N}$. The following notation can be viewed as a generalization of the $\mathcal{I}$-convergence property introduced by Tsaban and Zdomskyy in [27].

Definition 4.10. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on countable sets D_1, D_2 respectively. We say that X has $IC(\mathcal{I}, \mathcal{J})$-property if for each sequence $(f_n : n \in \mathbb{N})$ of continuous functions that $\mathcal{I}$-converges to $\bar{0}$, there exists $F \in \mathcal{I}^*$ such that $(f_n : n \in F)$ is $\mathcal{J}$-convergent to $\bar{0}$.

It is easy to see that the $IC(\mathcal{I}_b, Fin)$-property is just the bounded-ideals convergence property. If we require that $(f_{m_n} : n \in \mathbb{N})$ is $\mathcal{J}$-convergent to $\bar{0}$ for

$$F = \{m_0 < m_1 < m_2 < \cdots\}.$$

Then the $IC(\mathcal{I}, Fin)$-property coincides with the $IC(\mathcal{I})$-property defined in [16].

Theorem 4.11. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, and X being a perfectly normal space. Then the following conditions are equivalent.*

(1) X has $IC(\mathcal{I}_b, \mathcal{I})$-property;

(2) $C_p(X)$ is an $\mathcal{I}\alpha_1$-space.

Proof. (1) ⇒ (2) Let $(\Lambda_n : n \in \mathbb{N})$ be a sequence from $\Gamma_{\bar{0}}$. Assume that for each $n \in \mathbb{N}$,

$$\Lambda_n = (f^n_m : m \in \mathbb{N}).$$

Putting all these functions together, and viewing it as a double indexed sequence, we have that

Claim 4.12. $(f^n_m : (n, m) \in \mathbb{N} \times \mathbb{N}) \in \mathcal{I}_b\text{-}\Gamma_{\bar{0}}$.

Proof. For each $x \in X$, each $\varepsilon > 0$. We need to show that $\{(n, m) \in \mathbb{N}\times\mathbb{N} : |f^n_m(x)| \geq \varepsilon\} \in \mathcal{I}_b$. Define h: $\mathbb{N} \to \mathbb{N}$ by

$$h(n) = \min\{k : (\forall m \geq k)(|f^n_m(x)| < \varepsilon)\}.$$

Then

$$\{(n, m) \in \mathbb{N} \times \mathbb{N} : |f^n_m(x)| \geq \varepsilon\} \subseteq A_h.$$

Note that $A_h \in \mathcal{I}_b$, we finish the proof of Claim. □

By (1), there exists $A \in \mathcal{I}^*_b$ such that $(f^n_m : m \in A_{(n)}, n \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$, where $A_n = \{m : (n, m) \in A\}$. In addition, there exists $h \in \mathbb{N}^{\mathbb{N}}$ such that $A^c \subseteq A_h$. Let

$$\Lambda = (f^n_m : m > h(n), n \in \mathbb{N}).$$

Then $\Lambda \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$, and $\Lambda_n \setminus \Lambda$ is finite for each $n \in \mathbb{N}$.

(2) ⇒ (1) For each double indexed sequence $(f^n_m : (n, m) \in \mathbb{N} \times \mathbb{N})$ that belongs to $\mathcal{I}_b\text{-}\Gamma_{\bar{0}}$. Let $\Lambda_n = (f^n_m : m \in \mathbb{N})$.

Claim 4.13. *For each $n \in \mathbb{N}$, $\Lambda_n \in \Gamma_{\bar{0}}$.*

Proof. Note that for any $x \in X$ and any $\epsilon > 0$, $\{(n, m) \in \mathbb{N}\times\mathbb{N} : |f^n_m(x)| \geq \epsilon\} \in \mathcal{I}_b$. So there exists $h_{x,\varepsilon} \in \mathbb{N}^{\mathbb{N}}$ such that $\{(n, m) \in \mathbb{N} \times \mathbb{N} : |f^n_m(x)| \geq \epsilon\} \subseteq A_{h_{x,\varepsilon}}$. Therefore, for each $n \in \mathbb{N}$, $\{m : |f^n_m(x)| \geq \varepsilon\} \subseteq \{m : m \leq h_{x,\varepsilon}(n)\}$. □

Since $C_p(X)$ is an $\mathcal{I}\alpha_1$-space, there exists $\Lambda \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$ such that $\Lambda_n \setminus \Lambda$ is finite for each $n \in \mathbb{N}$. Assume that $\Lambda = (f_m : m \in \mathbb{N})$, and define h: $\mathbb{N} \to \mathbb{N}$ by

$$h(n) = \max\{k : (\forall m \geq k)(f_m \in \Lambda_n)\} + 1.$$

Then $A_h \in \mathcal{I}_b$ and $(f^n_m : (n, m) \in A^c_h) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$. □

Theorem 4.14. *Let $\mathcal{I}$ be a P-ideal on $\mathbb{N}$, and X being a topological space. Then the following conditions each one implies the next.*

(1) X has $IC(Fin, QN)$-property;

(2) X is a QN-space;

(3) X has $IC(\mathcal{I}, QN)$-property;

(4) X has $IC(\mathcal{I})$-property.

Proof. (1) ⇒ (2) Let $(f_n : n \in \mathbb{N})$ be a sequence from $C_p(X)$ that converges to $\bar{0}$. By $IC(Fin, QN)$, there exists $F \in Fin^*$ such that $(f_n : n \in F)$ is QN-convergent to $\bar{0}$. This implies that $(f_n : n \in \mathbb{N})$ is QN-convergent to $\bar{0}$.

(2) ⇒ (3) Assume that X is a QN-space. For any sequence $(f_n : n \in \mathbb{N})$ from $C_p(X)$ which $\mathcal{I}$-converges to $\bar{0}$, since $\mathcal{I}$ is a P-ideal, there exists $F \in \mathcal{I}^*$ such that $(f_n : n \in F)$ is convergent to $\bar{0}$ (see, [20]), and then it QN-converges to $\bar{0}$ since X is a QN-space.

(3) ⇒ (4) Obvious. □

Remark 4.15. Note that every QN-space has $IC(Fin, QN)$-property, so these properties are coincide.

The following result shows that in the realm of P-ideals, the assumption of X having $IC(\mathcal{I})$-property in Proposition 6.10 in [26] can be removed.

Proposition 4.16. *Let $\mathcal{I}$ be a P-ideal on $\mathbb{N}$, X being a QN-space. Then for any ideal $\mathcal{J}$ that extends $\mathcal{I}$, X is an $(\mathcal{I}, \mathcal{J})$QN-space*

5. $(\mathcal{I}, \mathcal{J}\text{-}\alpha_1)$ and $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$

Definition 5.1. ([5]) Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$, X being a topological space.

(1) $C_p(X)$ has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_1)$ if for each sequence $((f_m^n : m \in \mathbb{N}) : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(f_m^n : m \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$. There exists a sequence $(B_n : n \in \mathbb{N})$ from $\mathcal{J}$ with $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{N}$ such that

$$(\forall \varepsilon > 0)(\forall x \in X)(\exists J \in \mathcal{J})(\forall n, m)(m \notin J \cup B_n \rightarrow |f_m^n(x)| < \varepsilon).$$

(2) $C_p(X)$ has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if for each sequence $((f_m^n : m \in \mathbb{N}) : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(f_m^n : m \in \mathbb{N}) \in \mathcal{I}\text{-}\Gamma_{\bar{0}}$. There exists a sequence $(m_n : n \in \mathbb{N})$ such that $(f_{m_n}^n : n \in \mathbb{N}) \in \mathcal{J}\text{-}\Gamma_{\bar{0}}$.

According to the definition, the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ coincides with the $S_1(\mathcal{I}\text{-}\Gamma_{\bar{0}}, \mathcal{J}\text{-}\Gamma_{\bar{0}})$ property. What's more, we have the following.

Theorem 5.2. *Let X be a topological space, and $\mathcal{I}, \mathcal{J}$ being ideals on $\mathbb{N}$. Then $C_p(X)$ has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if, and only if it is an $(\mathcal{I}, \mathcal{J})\alpha_4$-space.*

This result was implied by the following Lemma. The implication (1) ⇒ (2) is the same as Theorem 2 in [23], and the implication (2) ⇒ (3) is the same as Lemma 1(3) in [23]. To make the paper self-contained and accessible to a wide audience, we supply proofs in the follows, all of these constructions are due to Scheepers.

Lemma 5.3. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$, X being a topological space. Then the following conditions each one implies the next.*

(1) $C_p(X)$ is an $(\mathcal{I}, \mathcal{J})\alpha_4$-space;

(2) $C_p(X)$ has the $S_1(\mathcal{I}\text{-}\Gamma_{\bar{0}}, \mathcal{J}\text{-}\Gamma_{\bar{0}})$ property;

(3) $C_p(X)$ is an $(\mathcal{I}, \mathcal{J})\alpha_2$-space.

Proof. (1) ⇒ (2) Let $((f_k^n : k \in \mathbb{N}) : n \in \mathbb{N})$ be a sequence from $\mathcal{I}\text{-}\Gamma_{\bar{0}}$ such that for each n, each k, each $x \in X$, $f_k^n(x) \geq 0$. Let

$$g_k^m = \sum_{0 \leq i \leq m} f_k^i.$$

By Lemma 3.2, $((g_k^m : k \in \mathbb{N}) : m \in \mathbb{N}) \subseteq \mathcal{I}\text{-}\Gamma_{\bar{0}}$. By (1), there exist $m_1 < m_2 < m_3 < \cdots$ and $n_1, n_2, n_3, \cdots$, such that $(g_{n_k}^{m_k} : k \in \mathbb{N}) \in \mathcal{J}\text{-}\Gamma_{\bar{0}}$. That is, for each x, each $\varepsilon > 0$,

$$\{k : |g_{n_k}^{m_k}(x)| \geq \varepsilon\} \in \mathcal{J}.$$

Let $m_0 = 0$, choose for each i a k_i by

$$f_{k_i}^i(x) = f_{n_j}^i \text{ for } m_{j-1} < i \leq m_j.$$

For each i let $\phi(i)$ be the unique j such that $m_{j-1} < i \leq m_j$. Then ϕ is a finite-to-one function and nondecreasing. So for each i, each $x \in X$, $0 \leq f_{k_i}^i(x) \leq g_{n_{\phi(i)}}^{m_{\phi(i)}}(x)$. Therefore, for each $x \in X$, each $\varepsilon > 0$,

$$\{i : |f_{k_i}^i(x)| \geq \varepsilon\} \subseteq \{i : |g_{n_{\phi(i)}}^{m_{\phi(i)}}(x)| \geq \varepsilon\} \in \mathcal{J}.$$

Thus, $\mathcal{J}\text{-}lim_{i\to\infty} f_{k_i}^i = \bar{0}$.

(2) $\Rightarrow$ (3) Let $((g_k^m : k \in \mathbb{N}) : m \in \mathbb{N}) \subseteq \mathcal{I}\text{-}\Gamma_{\bar{0}}$. For each $m \in \mathbb{N}$, construct a sequence $(\Lambda_{m,k} : k \in \mathbb{N}) \subseteq \mathcal{I}\text{-}\Gamma_{\bar{0}}$ by

$$\Lambda_{m,k} = (g_n^m : n \geq k).$$

We apply the $S_1(\mathcal{I}\text{-}\Gamma_{\bar{0}}, \mathcal{J}\text{-}\Gamma_{\bar{0}})$ to the sequence

$$\Lambda_{0,0}, \Lambda_{0,1}, \Lambda_{0,2}, \cdots, \Lambda_{1,0}, \Lambda_{1,1}, \Lambda_{1,2}, \cdots, \Lambda_{n,0}, \Lambda_{n,1}, \Lambda_{n,2}, \cdots.$$

There exists $\Lambda \in \mathcal{J}\text{-}\Gamma_{\bar{0}}$ such that $|\Lambda \cap \Lambda_{m,k}| = 1$ for each $m, k \in \mathbb{N}$. □

Lemma 5.4. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on $\mathbb{N}$, X being a topological space. If $C_p(X)$ is an $(\mathcal{I}, \mathcal{J})\alpha_1$-space, then it has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_1)$.*

Proof. Assume that $C_p(X)$ is an $(\mathcal{I}, \mathcal{J})\alpha_1$-space. Let $(\Lambda_n : n \in \mathbb{N})$ from $\mathcal{I}\text{-}\Gamma_{\bar{0}}$. Assume for each $n \in \mathbb{N}$ that

$$\Lambda_n = (f_m^n : m \in \mathbb{N}).$$

There exists $\Lambda \in \mathcal{J}\text{-}\Gamma_{\bar{0}}$ such that $\Lambda_n \setminus \Lambda$ is finite for each $n \in \mathbb{N}$. Let $A_n = \{m : f_m^n \in \Lambda_n \setminus \Lambda\}$, and $A = \mathbb{N} \setminus \bigcup_{n\in\mathbb{N}} A_n$ and enumerate A as $\{a_n : n \in \mathbb{N}\}$. We construct $(B_n : n \in \mathbb{N})$ as follow. If A is finite,

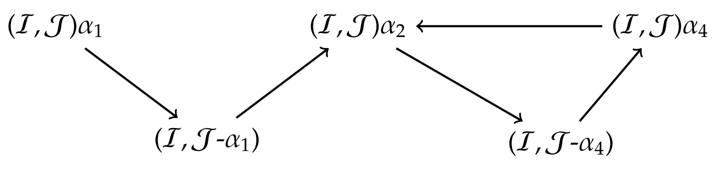
$$B_0 = A \cup A_0,\ B_n = A_n \text{ for } n > 1.$$

If A is infinite, put for each $n \in \mathbb{N}$,

$$B_n = A_n \cup \{a_n\}.$$

Then the sequence $(B_n : n \in \mathbb{N})$ witnesses that $C_p(X)$ has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_1)$. □

We have the following diagram.



(♣)

Corollary 5.5. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, X being a topological space. If $\mathcal{I} \leq_{KB} Fin$. the following are equivalent:*

(1) $C_p(X)$ is an $\mathcal{I}\alpha_2$-space;

(2) X is an $\mathcal{I}wQN$-space.

Proof. (1) $\Rightarrow$ (2) is clear from the diagram ♣ above. (2) $\Rightarrow$ (1) is also clear since $\mathcal{I} \leq_{KB} Fin$ (in this case $\mathcal{I}wQN$-spaces are $\mathcal{I}$QN-spaces([5], Corollary 3.4)). □

The following result shows that in some case, the $(\mathcal{I},\mathcal{J})\alpha_1$ property is different from the property $(\mathcal{I},\mathcal{J}\text{-}\alpha_1)$.

Proposition 5.6. *If $\mathcal{I}$ is not a weak P-ideal, then there exists $X \subseteq \mathbb{R}$ such that $C_p(X)$ is not an $\mathcal{I}\alpha_1$-space, but it has the property $(Fin,\mathcal{I}\text{-}\alpha_1)$.*

Proof. Put together Theorem 4.7, Corollary 5.3 in [26]. □

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