Filomat 34:10 (2020), 3299–3310
https://doi.org/10.2298/FIL2010299F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some New Generalized Result of Gronwall-Bellman-Bihari Type Inequality With Some Singularity

Djamal Foukrach[a,b], Badreddine Meftah[c]

[a]*Department of Mathematics, Faculty of Exact Sciences and Computer Science, University Hassiba Benbouali of Chlef, Algeria*
[b]*Laboratory of Fixed Point Theory and Applications, Department of Mathematics, E.N.S., Kouba, Algiers, Algeria*
[c]*Laboratoire des télécommunications, Faculté des Sciences et de la Technologie, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria*

Abstract. The aim purpose of the present work is to investigate some new nonlinear Gronwall-Bellman-Bihari type inequalities with singular kernel via k-fractional integral of Riemann-Liouville. This investigation generalizes some integral inequalities obtained in the literature and extends some other existing types of fractional integral inequalities. The obtained findings can be used to study some properties of solution for fractional differential equations.

1. Introduction

Fractional calculus is one of the disciplines of mathematical analysis which deals with arbitrary derivation and integration, the scope of which extends to several fields in addition to mathematics such as biological, economic and engineering sciences and other areas see [8, 12–16, 23, 49] and references therein.

Beginning with the classical Riemann-Liouville fractional integral and derivative operators, a large number of fractional integral and derivative operators as well as their generalizations and extensions have been presented by numerous mathematicians with a slightly different formulas see [2, 24, 26, 27, 33].

The above mentioned fractional operators has been widely used mostly in the fields of integral inequalities by many researchers see [22, 31, 32, 35–44].

The famous Gronwall inequality [18], can be declared as follows: if u be a continuous and nonnegative function defined on the interval $I = [a, a + h]$, and if

$$u(t) \leq \int_a^t (\alpha u(s) + \beta)\, ds \tag{1}$$

for all $t \in I$, where a, α, β and h are nonnegative constants, then

$$u(t) \leq bh \exp \alpha h.$$

The inequality (1) has been largely studied by considerable number of authors during the last century and has motivated some important lines of study which are currently active. Over the last decades a

2010 *Mathematics Subject Classification*. 26D15, 26D20, 26A33.
Keywords. Gronwall inequality; k-fractional integral operators; integral inequalities.
Received: 27 October 2019; Accepted: 29 July 2020
Communicated by Miodrag Spalević
Email addresses: foukrach_djamal@yahoo.fr (Djamal Foukrach), badrimeftah@yahoo.fr (Badreddine Meftah)

large number of papers have been appeared in the literature. These articles deal with the simple proofs, generalizations, refinements, extensions and improvements. Among the most important and imposing one, are: Bellman [4, 5], Gollwitzer [17], Bihari [6], Ou-Yang [46] , Györi [19], Beesack [3], Dafermos [7], Martynyuk and Kosolapov [28], Norbury et al. [45], Pachpatte [47, 48], Jiang et Meng [25], Abdeldaim and Yakout [1], and many others that we have not mentioned.

Indeed, these inequalities are not enough to treat all the problems because some of them in theory and in practice call upon the resolution of integral inequalities with singular kernels. Among the results which subsiste in this direction we mention the works of: Henry [21], Ye et al. [51], Medvedˇ [30] and Nisar et al. [34].

The objective of this study is to establish some new nonlinear Gronwall-Bellman-Bihari type inequalities with a singular kernel via the k-fractional Riemann-Liouville integral. The obtained inequalities can be used as practical tools to study it of certain properties of the solutions of differential and integral equations.

2. Preliminaries

For the reader's convenience, let us recall some basic definitions and preliminary results of fractional calculus which we'll use in the next section.

Definition 2.1. *([49], [50]) For a continuous function $u : [0,\infty) \to \mathbb{R}$, the Rimann-Liouville derivative of fractional order $\alpha > 0$ is defined as*

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n-1}}\, ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2. *([49], [50]) The Riemann-Liouville fractional integral of order α is defined as*

$$I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}\, ds, \quad \alpha > 0,$$

provided the integral exists.

Definition 2.3. *([9]) The k-gamma function is defined by*

$$\Gamma_k(\alpha) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{\alpha-1} dt \qquad (\mathcal{R}(\alpha) > 0; k \in \mathbb{R}^+).$$

We note that the k-gamma function enjoy the following properties

1- $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$

2- $\Gamma_k(k) = 1$

3- $\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right)$, where Γ is the usual function gamma.

Definition 2.4. *([9]) The k-beta function is defined by*

$$B_k(\alpha,\beta) = \begin{cases} \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\, dt & (\min\{\mathcal{R}(\alpha),\mathcal{R}(\beta)\} > 0), \\ \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} & (\alpha,\beta \in \mathbb{C}\backslash k\mathbb{Z}_0^-), \end{cases}$$

where $k\mathbb{Z}_0^-$ denotes the set of k-multiples of the elements in $\mathbb{Z}_0^-$.

Definition 2.5. *([11]) The k-Mittag-Leffler function is defined by*

$$E_{k,\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma_k(n\alpha+\beta)} \qquad (\alpha,\beta \in \mathbb{C}, \mathcal{R}(\alpha) > 0, k \in \mathbb{R}^+).$$

Definition 2.6. *([33]) The k-fractional Riemann-Liouville integral of order α is defined as*

$$I_k^{\alpha}u(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\frac{\alpha}{k}}}\, ds.$$

Definition 2.7. *([29]) A function $\varphi(u)$ is said to be subadditive, if*

$$\varphi(u+v) \leq \varphi(u) + \varphi(v)$$

for $u, v \geq 0$.

Definition 2.8. *([20]) A function $\varphi(u)$ is said to be submultiplicative, if*

$$\varphi(uv) \leq \varphi(u)\varphi(v)$$

for $u, v \geq 0$.

Definition 2.9. *([10]) A function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to belong to the class $\mathcal{F}$ if $\varphi(u) > 0$ is nondecreasing and continuous for $u \geq 0$ and*

$$\frac{1}{p}\varphi(u) \leq \varphi\left(\frac{u}{p}\right)$$

for $p > 0$.

Lemma 2.10. *([25]) Assume that $a \geq 0, p \geq q \geq 0$ and $p \neq 0$, then for any $\varepsilon > 0$ we have*

$$a^{\frac{q}{p}} \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}}a + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}.$$

3. Main results

Theorem 3.1. *Let h and u be nonnegative and locally integrable functions defined on $[0, T)$ with $T \leq +\infty$, and f be a nonnegative, nondecreasing, and continuous function on $[0, T)$ such that f is bounded on $[0, T)$ i.e. $|f(t)| \leq M$ for all $t \in [0, T)$. Let $p, q, \varepsilon, \lambda, k$ positive numbers such that $p \geq q$. If*

$$u^p(t) \leq h(t) + f(t)\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} u^q(\rho)\, d\rho \tag{2}$$

then

$$u(t) \leq \left\{h_1(t) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (f_1(t))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} h_1(\rho)\, d\rho\right\}^{\frac{1}{p}},$$

where

$$h_1(t) = h(t) + \frac{k}{\lambda}\frac{p-q}{p}\varepsilon^{\frac{q}{p}} t^{\frac{\lambda}{k}} f(t) \tag{3}$$

and

$$f_1(t) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} f(t). \tag{4}$$

Proof. Define a function z by

$$z(t) = u^p(t). \tag{5}$$

Substituting (5) in (2) and using Lemma 2.10, we obtain

$$z(t) \leq h_1(t) + f_1(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} z(\rho)\, d\rho \tag{6}$$

where $h_1$and f_1 are defined as in (3) and (4) respectively.

We can also choose a function $\chi : [0, T) \to \mathbb{R}^+$ satisfying

$$\chi z(t) = f_1(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} z(\rho)\, d\rho. \tag{7}$$

From (6) and (7) we deduce

$$z(t) \leq h_1(t) + \chi z(t). \tag{8}$$

substituting $z(t)$ in the right side of (8) we get

$$z(t) \leq h_1(t) + \chi h_1(t) + \chi^2 z(t).$$

By repeating n-times with $n > 1$, the substitution process we can conclude that

$$z(t) \leq \sum_{i=0}^{n-1} \chi^i h_1(t) + \chi^n z(t), \text{ for } n \geq 1. \tag{9}$$

Let's show now that

$$\chi^n z(t) \leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)} (f_1(t))^n \int_0^t (t-\rho)^{n\frac{\lambda}{k}-1} z(\rho)\, d\rho \tag{10}$$

for $n \geq 1$ and $t \in [0, T)$.

It is clear that relation (10) is true for $n = 1$. Suppose this is true for all $k \leq n$, for $k = n+1$ the induction hypothesis and using the fact that f_1 is nondecreasing, we have

$$\begin{aligned}
\chi^{n+1} z(t) &= \chi\chi^n z(t) \\
&= f_1(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} \chi^n z(\rho)\, d\rho \\
&\leq f_1(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} \left(\frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)} (f_1(\rho))^n \int_0^\rho (\rho-\tau)^{n\frac{\lambda}{k}-1} z(\tau)\, d\tau \right) d\rho \\
&\leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)} (f_1(t))^{n+1} \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} \left(\int_0^\rho (\rho-\tau)^{n\frac{\lambda}{k}-1} z(\tau)\, d\tau \right) d\rho.
\end{aligned}$$

By interchanging the order of integration taking into account that $0 \leq \tau \leq \rho \leq t$, and making the change of variable $\rho = w(t-\tau) + \tau$, we get

$$\begin{aligned}
\chi^{n+1}z(t) &\leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)}(f_1(t))^{n+1}\int_0^t\left(\int_\tau^t (t-\rho)^{\frac{\lambda}{k}-1}(\rho-\tau)^{n\frac{\lambda}{k}-1}d\rho\right)z(\tau)\,d\tau \\
&\leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)}(f_1(t))^{n+1} \\
&\quad\times\int_0^t\left(\int_\tau^t (t-\tau)^{\frac{\lambda}{k}-1}(1-w)^{\frac{\lambda}{k}-1}w^{n\frac{\lambda}{k}-1}(t-\tau)^{n\frac{\lambda}{k}-1}(t-\tau)\,dw\right)z(\tau)\,d\tau \\
&\leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)}(f_1(t))^{n+1}\int_0^t (t-\tau)^{(n+1)\frac{\lambda}{k}-1}\left(\int_0^1 (1-w)^{\frac{\lambda}{k}-1}w^{n\frac{\lambda}{k}-1}dw\right)z(\tau)\,d\tau \\
&\leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)}(f_1(t))^{n+1}\int_0^t (t-\tau)^{(n+1)\frac{\lambda}{k}-1}(kB_k(n\lambda,\lambda))\,z(\tau)\,d\tau \\
&\leq \frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)}\left(\frac{k\Gamma_k(n\lambda)\Gamma_k(\lambda)}{\Gamma_k((n+1)\lambda)}\right)(f_1(t))^{n+1}\int_0^t (t-\tau)^{(n+1)\frac{\lambda}{k}-1}z(\tau)\,d\tau \\
&\leq \frac{k^{n}(\Gamma_k(\lambda))^{n+1}}{\Gamma_k((n+1)\lambda)}(f_1(t))^{n+1}\int_0^t (t-\tau)^{(n+1)\frac{\lambda}{k}-1}z(\tau)\,d\tau.
\end{aligned}$$

Now, we will prove that

$$\lim_{n\to+\infty}\chi^n z(t)=0 \text{ for each } t\in[0,T).$$

It's obvious when n tends towards $+\infty$, $n\frac{\lambda}{k}-1$ will be very big. Hence $(t-\tau)^{n\frac{\lambda}{k}-1}\leq t^{n\frac{\lambda}{k}-1}$ for all $\tau\in[0,t]$. Since z is nonnegative and locally integrable on $[0,T)$, z is integrable on $[0,t]$. So, z is bounded on $[0,t]$ there exist $L>0$ such that for all $\tau\in[0,t]:|z(\tau)|\leq L$. From these arguments and the assumptions given, we conclude from (10)

$$\begin{aligned}
\chi^n z(t) &\leq L\frac{k^{n-1}(\Gamma_k(\lambda))^n}{\Gamma_k(n\lambda)}t^{n\frac{\lambda}{k}}\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}M\right)^n \\
&\leq L\frac{k^{n-1}\left(k^{\frac{\lambda}{k}-1}\Gamma\left(\frac{\lambda}{k}\right)\right)^n}{k^{\frac{n\lambda}{k}-1}\Gamma\left(\frac{n\lambda}{k}\right)}t^{n\frac{\lambda}{k}}\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}M\right)^n \\
&\leq L\frac{\left(\Gamma\left(\frac{\lambda}{k}\right)\right)^n}{\Gamma\left(\frac{n\lambda}{k}\right)}\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}t^{\frac{\lambda}{k}}M\right)^n. \qquad (11)
\end{aligned}$$

By virtue of Stirling's formula, (11) gives

$$\chi^n z(t)\leq L\left(\frac{k}{\lambda 2\pi}\right)^{\frac{1}{2}}\frac{\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}\left(\frac{tek}{\lambda}\right)^{\frac{\lambda}{k}}M\,\Gamma\left(\frac{\lambda}{k}\right)\right)^n}{n^{n\frac{\lambda}{k}-\frac{1}{2}}},$$

which implies

$$\chi^n z(t)\to 0 \text{ as } n\to+\infty. \qquad (12)$$

Taking the limit on both sides of (9) and using (12) we obtain

$$z(t)\leq h_1(t)+\sum_{i=1}^{\infty}\frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)}(f_1(t))^i\int_0^t (t-\rho)^{i\frac{\lambda}{k}-1}h_1(\rho)\,d\rho. \qquad (13)$$

Combining (5) and (13), we get the desired result. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, and additionally if we choose $p = q = 1$, if*

$$u(t) \le h(t) + f(t)\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} u(\rho)\,d\rho$$

then

$$u(t) \le h(t) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (f(t))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} h(\rho)\,d\rho.$$

Remark 3.3. *Corollary 3.2 will be reduced to Theorem 1 from [51], if we take $k = 1$.*

Remark 3.4. *In Corollary 3.2 if we put $f(t) = k\phi(t)$, we obtain*

$$u(t) \le \left\{ h(t) + \sum_{i=1}^{+\infty} \frac{\left(k^2\Gamma_k(\lambda)\phi(t)\right)^i}{k\Gamma_k(i\lambda)} \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} h(\rho)\,d\rho \right\},$$

which is the correct expression of Theorem 2.1 from [34].

Corollary 3.5. *Under the assumptions of Theorem 3.1, and additionally $h(t)$ is a nondecreasing function on $[0, T)$. If (2) is satisfied, then one has the following estimate*

$$u(t) \le \left\{ h_1(t)\, E_{k,\lambda,k}\left(k\Gamma_k(\lambda)\, f_1(t)\, t^{\frac{\lambda}{k}}\right) \right\}^{\frac{1}{p}},$$

where $h_1$and f_1 are defined as in (3) and (4) respectively.

Proof. From Theorem 3.1 we have

$$u(t) \le \left\{ h_1(t) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (f_1(t))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} h_1(\rho)\,d\rho \right\}^{\frac{1}{p}}. \tag{14}$$

Using the fact that h is a nondecreasing function, (14) gives

$$\begin{aligned}
u(t) &\le \left\{ h_1(t)\left(1 + \sum_{i=1}^{+\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (f_1(t))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} d\rho\right)\right\}^{\frac{1}{p}} \\
&\le \left\{ h_1(t)\left(1 + \sum_{i=1}^{+\infty} \frac{k^{i}(\Gamma_k(\lambda))^i}{i\lambda\Gamma_k(i\lambda)} (f_1(t))^i\, t^{i\frac{\lambda}{k}}\right)\right\}^{\frac{1}{p}} \\
&\le \left\{ h_1(t)\left(1 + \sum_{i=1}^{+\infty} \frac{\left(k\Gamma_k(\lambda) f_1(t) t^{\frac{\lambda}{k}}\right)^i}{\Gamma_k(i\lambda+k)}\right)\right\}^{\frac{1}{p}} \\
&\le \left\{ h_1(t)\sum_{i=0}^{+\infty} \frac{\left(k\Gamma_k(\lambda) f_1(t) t^{\frac{\lambda}{k}}\right)^i}{\Gamma_k(i\lambda+k)}\right\}^{\frac{1}{p}} \\
&\le \left\{ h_1(t)\, E_{k,\lambda,k}\left(k\Gamma_k(\lambda)\, f_1(t)\, t^{\frac{\lambda}{k}}\right)\right\}^{\frac{1}{p}},
\end{aligned}$$

which is the desired result. □

Remark 3.6. *Corollary 3.5 will be reduced to Corollary 2 from [51], if we take $k = p = q = 1$.*

Remark 3.7. *In Corollary 3.5 if we put $f(t) = k\phi(t)$ and choose $p = q = 1$, we obtain*

$$u(t) \le h(t)\, E_{k,\lambda,k}(k^2 \Gamma_k(\lambda)\, \phi(t)\, t^{\frac{\lambda}{k}}),$$

which is the correct expression of Corollary 2.3 from [34].

Theorem 3.8. *Let h and u be nonnegative and locally integrable functions defined on $[0, T)$ with $T \le +\infty$, and f and w be a nonnegative, nondecreasing, and continuous function on $[0, T)$ such that f is bounded on $[0, T)$ i.e. $|f(t)| \le M$ for all $t \in [0, T)$ and $w \in \mathcal{F}$, subadditive and convex function. If*

$$u(t) \le h(t) + \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\, w(u(\rho))\, d\rho \tag{15}$$

then

$$u(t) \le w^{-1}\left\{ w(h(t)) + \sum_{i=1}^{\infty} \tfrac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (tw(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho))\, d\rho \right\},$$

where

$$\left\{ w(h(t)) + \sum_{i=1}^{\infty} \tfrac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (tw(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho))\, d\rho \right\} \in Dom\, w^{-1}.$$

Proof. By applying w in both sides of (15), we get

$$w(u(t)) \le w\left(h(t) + \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\, w(u(\rho))\, d\rho \right).$$

Using the fact that w is continous and subadditive, we get

$$w(u(t)) \le w(h(t)) + w\left(\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\, w(u(\rho))\, d\rho \right). \tag{16}$$

Applying Jensen's inequality to (16), we obtain

$$w(u(t)) \le w(h(t)) + t \int_0^t w\left((t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\, w(u(\rho)) \right) d\rho.$$

Since $w \in \mathcal{F}$, and f is bounded, then

$$\begin{aligned}
w(u(t)) &\leq w(h(t)) + t\int_0^t w\left((t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\, w(u(\rho))\right) d\rho \\
&\leq w(h(t)) + t\int_0^t w\left(\frac{1}{\frac{1}{w(u(\rho))}}(t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\right) d\rho \\
&\leq w(h(t)) + t\int_0^t \frac{1}{\frac{1}{(t-\rho)^{\frac{\lambda}{k}-1} w(u(\rho))}} w(f(\rho))\, d\rho \\
&\leq w(h(t)) + t\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} w(u(\rho))\, w(f(\rho))\, d\rho \\
&\leq w(h(t)) + t\, w(M)\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} w(u(\rho))\, d\rho.
\end{aligned}$$

Now, putting $z(t) = w(u(t))$ and using Theorem 3.1 with $p = q = 1$, we get

$$z(t) \leq \left\{ w(h(t)) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (t\, w(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho))\, d\rho \right\},$$

which implies that

$$u(t) \leq w^{-1}\left\{ w(h(t)) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (t\, w(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho))\, d\rho \right\}.$$

The proof is completed. □

Theorem 3.9. *Under the hypothesis of Theorem 3.8, we have*

$$u(t) \leq h(t) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (w(1)\, f(t))^{i-1} \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(1)\, f(\rho)\, h(\rho)\, d\rho.$$

Proof. Using (15) and the fact that $w \in \mathcal{F}$, we have

$$u(t) \leq h(t) + \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho)\, u(\rho)\, w(1)\, d\rho. \tag{17}$$

Multiplying both sides of (17) by $w(1)\, f(t)$ and letting $w(1)\, f(t)\, u(t) = z(t)$, we get

$$z(t) \leq w(1)\, f(t)\, h(t) + w(1)\, f(t)\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} z(\rho)\, d\rho. \tag{18}$$

Now applying Theorem 3.1 with $p = q = 1$ for (18) we obtain

$$z(t) \leq w(1) f(t) \left\{ h(t) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (w(1) f(t))^{i-1} \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(1) f(\rho) h(\rho) d\rho \right\}.$$

Since $w(1) f(t) u(t) = z(t)$, we conclude the desired result. □

Theorem 3.10. *Let h and u be nonnegative and locally integrable functions defined on $[0, T)$ with $T \leq +\infty$, and f, g and w be a nonnegative, nondecreasing, and continuous function on $[0, T)$ such that f and g are bonded on $[0, T)$ i.e. $|f(t)| \leq M$ and $|g(t)| \leq L$ for all $t \in [0, T)$ and $w \in \mathcal{F}$, subadditive, submultiplicative and convex function. If*

$$u(t) \leq h(t) + g(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho) w(u(\rho)) d\rho \tag{19}$$

then

$$u(t) \leq w^{-1} \left\{ w(h(t)) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (t\, w(L) w(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho)) d\rho \right\},$$

where

$$\left\{ w(h(t)) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (t\, w(L) w(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho)) d\rho \right\} \in Dom\, w^{-1}.$$

Proof. By applying w in both sides of (19), we get

$$w(u(t)) \leq w\left(h(t) + g(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho) w(u(\rho)) d\rho \right).$$

Using the fact that w is continous, subadditive and submultiplicative, we get

$$w(u(t)) \leq w(h(t)) + w(g(t)) w\left(\int_0^t (t-\rho)^{\frac{\lambda}{k}-1} f(\rho) w(u(\rho)) d\rho \right).$$

By an argument similar as in Theorem 3.8 we applying Jensen's inequality and the fact that $w \in \mathcal{F}$, f and g are bounded functions, we get

$$w(u(t)) \quad \leq \quad w(h(t)) + t\, w(L)\; w(M) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} w(u(\rho)) d\rho.$$

Let $z(t) = w(u(t))$. By using Theorem 3.1 with $p = q = 1$, we get

$$z(t) \leq \left\{ w(h(t)) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (t\, w(L) w(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho)) d\rho \right\},$$

which implies that

$$u(t) \leq w^{-1} \left\{ w(h(t)) + \sum_{i=1}^{\infty} \frac{k^{i-1}(\Gamma_k(\lambda))^i}{\Gamma_k(i\lambda)} (t\, w(L) w(M))^i \int_0^t (t-\rho)^{i\frac{\lambda}{k}-1} w(h(\rho)) d\rho \right\}.$$

The proof is completed. □

4. Applications

In this section, we present some applications of the results. Let us consider the following fractional dynamic equation

$$u(t) = F\left(t, I_k^{\alpha} u(t)\right), \tag{20}$$

where I_k^{α} is the k-fractional Riemann-Liouville integral of order α and $F : [0, T) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function with $T \leq +\infty$.

Proposition 4.1. *Assume that*

$$\left|F\left(t, I_k^{\alpha} u(t)\right)\right| \leq \left\{h(t) + k\Gamma_k(\alpha) f(t) I_k^{\alpha} |u(t)|^q\right\}^{\frac{1}{p}}, \tag{21}$$

where h, f, p and q are defined as in Theorem 3.1. Then the solution $u(t)$ of (20) has the following estimate

$$|u(t)| \leq \left\{h_1(t) + \sum_{i=1}^{+\infty} \frac{k^{i-1}(\Gamma_k(\alpha))^i}{\Gamma_k(i\alpha)} (f_1(t))^i \int_0^t (t-\rho)^{i\frac{\alpha}{k}-1} h_1(\rho) \, d\rho\right\}^{\frac{1}{p}}, \tag{22}$$

where h_1 and f_1 are given by (3) and (4) respectively.

Proof. Let $u(t)$ be a solution of (20), using (21) and modulus, we obtain

$$|u(t)|^p \leq h(t) + f(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} \left|u(\rho)\right|^q d\rho. \tag{23}$$

Now, an application of Theorem 3.1 for (23) we get the estimate (22). □

Proposition 4.2. *Assume that*

$$\left|F\left(t, I_k^{\alpha} u(t)\right)\right| \leq h(t) + k\Gamma_k(\alpha) f(t) I_k^{\alpha} u^q(t), \tag{24}$$

where h, f, p and q are defined as in Theorem 3.1 with $p = q$. Then (20) has at most one solution on $[0, T)$.

Proof. Let $u(t)$ and $v(t)$ be tow solutions of (20) and (24), then we have

$$u^p(t) \leq h(t) + f(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} u^p(\rho) \, d\rho, \tag{25}$$

and

$$v^p(t) \leq h(t) + f(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} v^p(\rho) \, d\rho. \tag{26}$$

Making the difference between (25) and (26), and taking the absolute value at both sides of the resulting equality, we get

$$|u^p(t) - v^p(t)| \leq f(t) \int_0^t (t-\rho)^{\frac{\lambda}{k}-1} \left(|u^p(t) - v^p(t)|\right) d\rho. \tag{27}$$

An application of Corollary 3.5 to (27) and since $h_1(t) = h(t) + \frac{k}{\lambda} \frac{p-q}{p} \varepsilon^{\frac{q}{p}} t^{\frac{\lambda}{k}} t^{\frac{\lambda}{k}} f(t) = 0$ it yields $|u^p(t) - v^p(t)| = 0$, which implies that the problem (20) and (24) admits a unique solution. □

5. Conclusion

The main contribution of this paper is the establishment of some new generalizations of nonlinear Gronwall-Bellman-Bihari type inequalities with singular kernel associated with the Riemann-Liouville-type k-fractional integral operator. By suitably choosing and/or changing the parameters in these inequalities, from our main results, we can easily obtain additional (and further) fractional integral inequalities, some of them are known in the literature, and extend other some existing ones. For example, by setting $k = 1$ or $k = p = q = 1$, some Riemann-Liouville fractional integral inequalities studied by Ye, Gao and Ding in [51] can be obtained; by choosing $f(t) = k\phi(t)$ and $p = q = 1$, some Riemann-Liouville type k-fractional integral inequalities defined by Nisar et al. in [34] can be deduced. To illustrate the benefits and importance of the obtained main findings, we note that they can be useful in the study of some properties of solutions for some classes of nonlinear fractional differential equations.

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