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On Pseudo $\mathcal{H}$-Symmetric Lorentzian Manifolds With Applications to Relativity

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Abstract. In this paper, we introduce a new type of curvature tensor named $\mathcal{H}$-curvature tensor of type $(1,3)$ which is a linear combination of conformal and projective curvature tensors. First we deduce some basic geometric properties of $\mathcal{H}$-curvature tensor. It is shown that a $\mathcal{H}$-flat Lorentzian manifold is an almost product manifold. Then we study pseudo $\mathcal{H}$-symmetric manifolds $(P\mathcal{H}S)_n$ $(n > 3)$ which recovers some known structures on Lorentzian manifolds. Also, we provide several interesting results. Among others, we prove that if an Einstein $(P\mathcal{H}S)_n$ is a pseudosymmetric $(PS)_n$, then the scalar curvature of the manifold vanishes and conversely. Moreover, we deal with pseudo $\mathcal{H}$-symmetric perfect fluid spacetimes and obtain several interesting results. Also, we present some results of the spacetime satisfying divergence free $\mathcal{H}$-curvature tensor. Finally, we construct a non-trivial Lorentzian metric of $(P\mathcal{H}S)_4$.

1. Introduction

Curvature plays a crucial role in the development of differential geometry and Physics. According to Chern [17] ”a fundamental notion is curvature in its different forms”. Therefore, finding of curvature tensors are very interesting topics for the researchers. In this sense, we introduce a new curvature tensor to find applications in the theory of relativity and cosmology.
Investigating conformally flat Riemannian manifolds of dimension n of class one, R. N. Sen and M. C. Chaki [42] found that the curvature tensor R of type $(0,4)$ satisfies

$$R_{hijk,l} = 2a_l R_{hijk} + a_h R_{lijk} + a_i R_{hljk} + a_j R_{hilk} + a_k R_{hijl},$$

where “comma” denotes covariant derivative with respect to the metric and R_{hijk} are components of the curvature tensor R of type $(0,4)$. Hereafter, M. C. Chaki [8] and M. C. Chaki and U. C. De [10] examined the Riemannian manifolds with the above condition. The first author called such manifolds pseudosymmetric,

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since locally symmetric manifold satisfies the above condition with $a_l = 0$.
The above expression in index free notation can be written as

$$(\nabla_U R)(V,Z,W,X) = 2A(U)R(V,Z,W,X) + A(V)R(U,Z,W,X)$$
$$+A(Z)R(V,U,W,X) + A(W)R(V,Z,U,X) + A(X)R(V,Z,W,U)$$

for all vector fields X, U, V, Z, W, where A is a non-zero 1-form, ρ is a vector field defined by $g(U,\rho) = A(U)$ for all U. Here ∇ represents the covariant derivative operator with respect to the metric tensor g, A is an associated 1-form, and $R(V,Z,W,X) = g(\mathcal{R}(V,Z)W,X)$, where $\mathcal{R}$ is a curvature tensor of type $(1,3)$. If $A = 0$, then the manifold becomes locally symmetric manifold in the sense of Cartan. An n-dimensional pseudosymmetric manifold is denoted by $(PS)_n$. Pseudosymmetric manifolds have been studied by several authors such as ([9]-[11], [29], [30], [51], [52]) and many others.
The notion of pseudosymmetric manifolds was generalized by Tamássy and Binh [47] in 1989 and called weakly symmetric manifolds.

The idea of the generalized Robertson-Walker (GRW) spacetime was introduced by Alías et al. [1] in 1995. A GRW spacetime is an n-dimensional Lorentzian manifold M, that is, $M = -I \times f^2 M^*$, where I is an open interval of the real line $\Re$, M^*, a Riemannian submanifold of dimension $(n-1)$ and $f(>0)$, a smooth warping function (or scale factor). In [41], it is observed that the GRW spacetimes have applications in inhomogeneous spacetimes admitting an isotropic radiation. An n-dimensional Lorentzian manifold M with the metric (in local shape)

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(dt)^2 + f(t)^2 g^*_{\alpha\beta} dx^\alpha dx^\beta,$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \ldots, n)$ and f, the warping function of t only, is known as GRW spacetime. In particular, if $g^*_{\alpha\beta}$ has dimension 3 and constant curvature, then the spacetime converts into the RW spacetime. For instance, we refer ([2], [16], [19], [32], [33] and [41]).

Lorentzian manifolds with a non-vanishing Ricci tensor S are known as the perfect fluid spacetimes if

$$S = \mathfrak{a} g + \mathfrak{b} A \otimes A, \tag{1}$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are scalar fields and $g(\rho,\rho) = -1$. O'Neill [40] in his book listed that a Robertson-Walker spacetime is a perfect fluid spacetime. It is also noticed that a GRW spacetime (for $n = 4$) is a perfect fluid if and only if it is RW spacetime. If the energy-matter content of spacetime is a perfect fluid with fluid velocity ρ, then the Einstein's field equations reflect that the Ricci tensor assumes the form (1) and the scalars $\mathfrak{a}$ and $\mathfrak{b}$ are linearly related to the pressure p and the energy density μ measured in the locally comoving inertial frame [33].

It is well known that the conformal curvature tensor is invariant under conformal transformation and projective curvature tensor is invariant under projective transformation. In a Riemannian or Lorentzian manifold of dimension n, the conformal curvature tensor C and the projective curvature tensor $\mathcal{P}$ are defined as follows:

$$\begin{aligned} C(U,V)Z &= \mathcal{R}(U,V)Z - \frac{1}{n-2}[S(V,Z)U - S(U,Z)V + g(V,Z)QU \\ &\quad -g(U,Z)QV] + \frac{r}{(n-1)(n-2)}[g(V,Z)U - g(U,Z)V] \end{aligned} \tag{2}$$

and $$\mathcal{P}(U,V)Z = \mathcal{R}(U,V)Z - \frac{1}{n-1}[S(V,Z)U - S(U,Z)V], \tag{3}$$

where $\mathcal{R}$ is the Riemannian curvature tensor of type $(1,3)$, the Ricci operator Q is defined by $g(QU,V) = S(U,V)$, and r denotes the scalar curvature.
Conformal curvature tensor and projective curvature tensor play an important role in the differential

geometry as well as in the theory of relativity. In this paper, we introduce a new tensor named $\mathcal{H}$-tensor of type $(1,3)$ which is a linear combination of conformal and projective curvature tensors and defined by

$$\mathcal{H}(U,V)Z = a\,\mathcal{C}(U,V)Z + [a + (n-2)b]\mathcal{P}(U,V)Z, \tag{4}$$

where a and b are real numbers (not simultaneously zero). If $a = 1$ and $b = -\frac{1}{n-2}$, then $\mathcal{H} \equiv \mathcal{C}$, also if $a = 0$ and $b = \frac{1}{n-2}$, then $\mathcal{H} \equiv \mathcal{P}$. Since the conformal curvature tensor vanishes for $n = 3$, we consider the dimension of the manifold $n > 3$.

A non-flat Lorentzian manifold $(M^n, g)\,(n > 3)$ is said to be pseudo $\mathcal{H}$-symmetric if the H-tensor of type $(0,4)$ satisfies the condition

$$\begin{aligned}&(\nabla_U H)(V,Z,W,X) = 2A(U)H(V,Z,W,X) + A(V)H(U,Z,W,X)\\&+A(Z)H(V,U,W,X) + A(W)H(V,Z,U,X) + A(X)H(V,Z,W,U),\end{aligned} \tag{5}$$

where A is a non-zero 1-form, ρ is a vector field defined by $g(U,\rho) = A(U)$ for all U, and $H(V,Z,W,X) = g(\mathcal{H}(V,Z)W,X)$. The 1-form A is called the associated 1-form of the manifold. Such a manifold is denoted by $(P\mathcal{H}S)_n$. This manifold includes pseudo-conformally symmetric manifolds [20] or conformally quasi-recurrent manifolds ([34], [43]) and pseudo projective symmetric manifolds [13].

The notion of Codazzi type of Ricci tensor S was introduced by Gray [26]. A Lorentzian manifold is said to satisfy Codazzi type of Ricci tensor if S satisfies the condition

$$(\nabla_U S)(V,Z) = (\nabla_V S)(U,Z),$$

which implies that $div\,\mathcal{R} = 0$, that is, the manifold possesses the harmonic curvature, where 'div' denotes divergence. A Riemannian connection is a solution of the Yang-Mills equations on the tangent bundle if and only if the Riemannian manifold possesses a harmonic curvature [6]. The another reason for this study on the metric with harmonic curvature, that is, $div\,R = 0$ is the fact that a Riemannian manifold has harmonic curvature if and only if the Riemannian connection is a solution of the Yang-Mills equations on the tangent bundle [6]. Let M be an n-dimensional differentiable manifold and T_pM is the tangent space at each point p of M. If there exists an endomorphism F at each point of the tangent space T_pM such that

$$F^2 = I,$$

then we say that the tensor F of type $(1,1)$ gives an almost product structure to the manifold M and we call the manifold an almost product manifold [49].

General relativity governs by the Einstein equations and the energy-momentum tensor T is of vanishing divergence. This shows that the energy-momentum tensor to be covariantly constant, that is, $\nabla T = 0$. To find the nature of spacetimes, the energy-momentum tensor plays an important role. In the general theory of relativity, the energy-momentum tensor plays an important role and the condition on energy-momentum tensor for a perfect fluid spacetime changes the nature of spacetime [46]. In [12], Chaki and Ray studied general relativistic spacetime with covariant constant energy-momentum tensor. Recently, De and Velimirović [22] studied spacetimes with semisymmetric energy momentum tensor.

The spacetime of general relativity and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) equipped with a Lorentzian metric g of signature $(-,+,+,+)$. The geometry of Lorentzian manifold starts with the study of causal character of vectors of the manifold. For this causality, the Lorentzian manifold becomes a convenient choice for the study of general theory of relativity. Indeed, by basing its study on Lorentzian manifold the general theory of relativity opens the way to the study of global questions about it ([5], [14], [15], [18], [25], [27], [28]). Also, several authors studied spacetimes in different way such as ([21]-[24], [38], [50]) and many others.

Einstein's field equation without cosmological constant is given by

$$S(U,V) - \frac{r}{2}g(U,V) = \kappa T(U,V), \tag{6}$$

where κ is the gravitational constant. From (6), we can say that the geometry of spacetimes is determined by matter and conversely, and the motion of the matter is evaluated by the metric tensor of non-flat space. In general relativity, the matter content of the spacetime is described by the energy momentum tensor. The matter content is assumed to be fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion.
The energy momentum tensor T of type $(0, 2)$, in a perfect fluid spacetime, takes the form [40]:

$$T(U, V) = pg(U, V) + (\sigma + p)A(U)A(V), \tag{7}$$

where p and σ are the isotropic pressure and the energy density, respectively. Here ρ denotes the velocity vector field, which is metrically equivalent to the non-zero 1-form A such that $g(\rho, \rho) = -1$, that is, ρ is a unit time-like vector field. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [27]. A perfect fluid spacetime is called isentropic if $p = \sigma$ [27].

Recently, De et al. [22] studied spacetimes with semisymmetric energy momentum tensor. Also in [30], Mallick, Suh and De studied spacetimes with pseudo-projective curvature tensor. Moreover in [35], Mantica and Molinari studied weakly Z-symmetric manifolds. Also, several authors studied spacetimes in different way such as ([21], [22], [36]-[38], [50]) and many others. In [12], Chaki and Ray studied spacetimes with covariant constant energy momentum tensor. Motivated by the above studies in the present paper we study pseudo $\mathcal{H}$-symmetric manifolds and its applications to spacetimes.
The present paper is organized as follows:
After introduction, in Section 2 we study some basic geometric properties of $\mathcal{H}$-curvature tensor. Section 3 is devoted to study $\mathcal{H}$-flat Lorentzian manifolds. We prove that a $\mathcal{H}$-flat Lorentzian manifold is an almost product manifold. In Section 4, we study Lorentzian manifolds satisfying $div\mathcal{H} = 0$. Next, we obtain a necessary and sufficient condition for the vanishing scalar curvature in a $(P\mathcal{H}S)_n$. Section 5 deals with the study of scalar curvature. In section 6, we study Einstein $(P\mathcal{H}S)_n$ and prove that if an Einstein $(P\mathcal{H}S)_n$ is a $(PS)_n$, then the scalar curvature of the manifold vanishes and conversely. Moreover in Section 7, we characterize pseudo $\mathcal{H}$-symmetric spacetimes and spacetimes with divergence free $\mathcal{H}$-tensor and obtain several interesting results. Finally, we construct an example of $(PHS)_4$.

2. Preliminaries

We derive some basic formulas of $(P\mathcal{H}S)_n$, which will be useful to find our results. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \le i \le n$. We define the Ricci tensor and scalar curvature in a Lorentzian manifold as $S(U, V) = \sum_{i=1}^{n} \epsilon_i g(QU, V)$ and $r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i)$, where $\epsilon_i = g(e_i, e_i) = \pm 1$.

Then from (2) and (3) we have the following:

$$\sum_{i=1}^{n} \epsilon_i C(U, e_i, e_i, V) = 0 = \sum_{i=1}^{n} \epsilon_i C(U, V, e_i, e_i). \tag{8}$$

$$\sum_{i=1}^{n} \epsilon_i C(e_i, U, V, e_i) = \sum_{i=1}^{n} \epsilon_i C(e_i, e_i, U, V) = 0, \tag{9}$$

$$\sum_{i=1}^{n} \epsilon_i P(e_i, U, V, e_i) = 0, \tag{10}$$

and

$$\sum_{i=1}^{n} \epsilon_i P(U, e_i, e_i, V) = \frac{n}{n-1}[S(U, V) - \frac{r}{n} g(U, V)] = \bar{P}(U, V)(\text{say}). \tag{11}$$

Using the curvature properties of the conformal curvature tensor and projective curvature tensor from (4) we can easily verify that the tensor $\mathcal{H}$ satisfies the following properties:

$$i)\ \mathcal{H}(U,V)W = -\mathcal{H}(V,U)W,$$
$$ii)\ \mathcal{H}(U,V)W + \mathcal{H}(V,W)U + \mathcal{H}(W,U)V = 0. \tag{12}$$

Also using (8)-(11) from (4) we have

$$\sum_{i=1}^{n} \epsilon_i H(U,V,e_i,e_i) = 0 = \sum_{i=1}^{n} \epsilon_i H(e_i,e_i,U,V),$$

$$\sum_{i=1}^{n} \epsilon_i H(e_i,V,W,e_i) = 0,$$

$$\begin{aligned}\sum_{i=1}^{n} \epsilon_i H(U,e_i,e_i,X) &= \frac{n}{n-1}\{a+(n-2)b\}[S(U,X) - \frac{r}{n}g(U,X)] \\ &= \{a+(n-2)b\}\bar{P}(U,X).\end{aligned}$$

From (4) and (12) it follows that

$$(i)\ H(U,V,W,X) = -H(V,U,W,X),$$
$$(ii)\ H(U,V,W,X) \neq H(U,V,X,W),$$
$$(iii)\ H(U,V,W,X) \neq H(W,X,U,V),$$
$$(iv)\ H(U,V,W,X) + H(V,W,U,X) + H(W,U,V,X) = 0$$

for any vector fields U, V, W and X.

3. $\mathcal{H}$-flat Lorentzian manifolds

A Riemannian or a Lorentzian manifold of dimension $n(>2)$ is said to be an Einstein manifold if

$$S(U,V) = \lambda g(U,V),$$

where λ is a constant, from which it follows that $QU = \lambda U$. We first suppose that the manifold is an Einstein manifold. Then

$$Q(QU) = \lambda Q(U) = \lambda^2 U. \tag{13}$$

Now, let us consider an endomorphism F at each point of the tangent space T_pM such that

$$F(U) = \frac{1}{\lambda}Q(U). \tag{14}$$

So, we have

$$F(F(U)) = F(\frac{1}{\lambda}Q(U)). \tag{15}$$

Using (14) in (15) we have

$$F(F(U)) = \frac{1}{\lambda^2}Q^2U.$$

Now using (13) we get $F^2(U) = U$, which is an almost product structure. So we conclude that an Einstein manifold is an almost product manifold. Thus we can state the following:

Proposition 3.1. *Every Einstein manifold of dimension $n(> 2)$ is an almost product manifold.*

Let us consider a $\mathcal{H}$-flat Lorentzian manifold. Then from (4) we have

$$a\,\mathcal{C}(U,V)Z + [a + (n-2)b]\mathcal{P}(U,V)Z = 0,$$

which implies that the conformal curvature tensor $\mathcal{C}$ and projective curvature tensor $\mathcal{P}$ are equivalent. This forces $\mathcal{P}$ to have the algebraic symmetries of the tensor $\mathcal{C}$. However, in general it does not, but if it does then the manifold must be an Einstein. This result was mentioned (without proof) by Barnes [4]. Thus with the help of Proposition 3.1, we can state the following:

Theorem 3.2. *A $\mathcal{H}$-flat Lorentzian manifold (M^n, g) $(n > 3)$ is an almost product manifold.*

4. Lorentzian manifolds with divergence free $\mathcal{H}$-tensor

We know that

$$\mathcal{H}(U,V)Z = a\mathcal{C}(U,V)Z + [a + (n-2)b]\mathcal{P}(U,V)Z,$$

where $a, b \in \Re$ (set of real numbers). Therefore

$$(div\,\mathcal{H})(U,V)Z = a(div\,\mathcal{C})(U,V)Z + [a + (n-2)b](div\,\mathcal{P})(U,V)Z. \tag{16}$$

Also it is known that [48]

$$(div\,\mathcal{C})(U,V)Z \quad = \quad \frac{n-3}{n-2}[\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\} - \frac{1}{2(n-1)}\{dr(U)g(V,Z) - dr(V)g(U,Z)\}], \tag{17}$$

and

$$(div\,\mathcal{P})(U,V)Z = \frac{n-2}{n-1}\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\}. \tag{18}$$

Using (17) and (18) in (16) we have

$$\begin{aligned}(div\,\mathcal{H})(U,V)Z = {} & a\frac{n-3}{n-2}[\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\} - \frac{1}{2(n-1)}\{dr(U)g(V,Z) - dr(V)g(U,Z)\}] \\ & +[a + (n-2)b]\frac{n-2}{n-1}\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\}.\end{aligned}$$

It follows that

$$\begin{aligned}(div\,\mathcal{H})(U,V)Z = {} & \{a\frac{n-3}{n-2} + [a + (n-2)b]\frac{n-2}{n-1}\}\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\} \\ & -\frac{a}{2(n-1)}\frac{n-3}{n-2}\{dr(U)g(V,Z) - dr(V)g(U,Z)\}.\end{aligned} \tag{19}$$

Suppose that $(div\,\mathcal{H})(U,V)Z = 0$. Then from the above equation we infer that

$$\begin{aligned}& \{a\frac{n-3}{n-2} + [a + (n-2)b]\frac{n-2}{n-1}\}\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\} \\ & -\frac{a}{2(n-1)}\frac{n-3}{n-2}\{dr(U)g(V,Z) - dr(V)g(U,Z)\} = 0.\end{aligned} \tag{20}$$

Taking a frame field and contracting V and Z in (20) we get $\frac{n-2}{n-1}[a + (n-2)b]dr(U) = 0$. Therefore $dr(U) = 0$ for all vector field U provided $a + (n-2)b \neq 0$. Thus we can conclude the following:

Theorem 4.1. *In a Lorentzian manifold (M^n, g) $(n > 3)$ with divergence free $\mathcal{H}$-curvature tensor, the scalar curvature r is constant, provided $a + (n-2)b \neq 0$.*

Using $r = constant$ and $(div\,\mathcal{H})(U,V)Z = 0$ in (19) yields $[a(2n^2-8n+7)+b(n-2)^3]\{(\nabla_U S)(V,Z)-(\nabla_V S)(U,Z)\} = 0$, that is, either

$$a(2n^2 - 8n + 7) + b(n - 2)^3 = 0 \tag{21}$$

or the Ricci tensor is of Codazzi type. Thus we are in position to state the following:

Corollary 4.2. *If a Lorentzian manifold (M^n, g) $(n > 3)$ possesses the divergence free $\mathcal{H}$-curvature tensor, then either the Ricci tensor S of M^n is of Codazzi type or a and b are linearly dependent.*

Now, let us suppose that a Lorentzian manifold (M^n, g) $(n > 3)$ possesses the constant scalar curvature and a, b satisfy the equation (21). Thus we have $b = -\frac{a(2n^2-8n+7)}{(n-2)^2}$. Substituting this value in (19), we conclude that $div\,\mathcal{H} = 0$. Thus we can state the following:

Corollary 4.3. *If the scalar curvature of a Lorentzian manifold (M^n, g) $(n > 3)$ is constant and $a(2n^2 - 8n + 7) + b(n - 2)^3 = 0$, then M^n possesses the divergence free $\mathcal{H}$-curvature tensor.*

Since a Lorentzian manifold with divergence free $\mathcal{H}$-curvature tensor and $a(2n^2 - 8n + 7) + b(n - 2)^3 \neq 0$ possesses the Codazzi type of Ricci tensor, therefore from (17) we infer $div\,C = 0$. Hence we can state:

Corollary 4.4. *If an n-dimensional Lorentzian manifold with $n > 3$ possesses a divergence free $\mathcal{H}$-curvature tensor, then $div\,C = 0$ provided $a(2n^2 - 8n + 7) + b(n - 2)^3 \neq 0$.*

5. Nature of the scalar curvature of a $(P\mathcal{H}S)_n$ $(n > 3)$

In this section we consider pseudo $\mathcal{H}$-symmetric manifolds $(P\mathcal{H}S)_n$. Then the H-tensor satisfies the equation (5). Taking a frame field and contracting W and Z in (5) we yield

$$\begin{aligned}[a + (n-2)b](\nabla_U \bar{P})(V,X) = {} & 2[a + (n-2)b]A(U)\bar{P}(V,X) \\ & +[a + (n-2)b]A(V)\bar{P}(U,X) + H(V,U,\rho,X) \\ & +[a + (n-2)b]A(X)\bar{P}(V,U) + H(V,\rho,U,X).\end{aligned} \tag{22}$$

Again contracting V and X in (22) we get $[a + (n-2)b]\bar{P}(U,\rho) = 0$. It follows that

$$\bar{P}(U,\rho) = 0, \tag{23}$$

provided $[a + (n-2)b] \neq 0$. Using (11) in (23) we get $nS(U,\rho) - rg(U,\rho) = 0$. Therefore

$$S(U,\rho) = \frac{r}{n} g(U,\rho). \tag{24}$$

Thus we can state the following:

Theorem 5.1. *In a $(P\mathcal{H}S)_n (n > 3)$ with $a + (n-2)b \neq 0$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.*

6. Einstein $(P\mathcal{H}S)_n$ $(n > 3)$

This section deals with an Einstein $(P\mathcal{H}S)_n$. Then the Ricci tensor satisfies

$$S(U,V) = \frac{r}{n} g(U,V), \tag{25}$$

from which it follows that

$$dr(U) = 0 \quad \text{and} \quad (\nabla_Z S)(U,V) = 0 \tag{26}$$

for all vector fields U, V, Z. Again from (4) we have

$$(\nabla_U H)(V,Z,W,X) = a(\nabla_U C)(V,Z,W,X) + [a + (n-2)b](\nabla_U P)(V,Z,W,X). \tag{27}$$

Making use of (2) and (3) in (27) and in view of (26) and (27) we have

$$(\nabla_U H)(V,Z,W,X) = [2a + (n-2)b](\nabla_U R)(V,Z,W,X). \tag{28}$$

Again from (25) we get

$$C(V,Z,W,X) = P(V,Z,W,X) \tag{29}$$

for all V, Z, W and X. Thus using (28) and (29) in (5) we obtain

$$\begin{aligned}&[2a + (n-2)b](\nabla_U R)(V,Z,W,X) = [2a + (n-2)b][2A(U)P(V,Z,W,X) + A(V)P(U,Z,W,X)\\ &+A(Z)P(V,U,W,X) + A(W)P(V,Z,U,X) + A(X)P(V,Z,W,U)].\end{aligned}$$

This shows that either $2a + (n-2)b = 0$ or $(\nabla_U R)(V,Z,W,X) = 2A(U)P(V,Z,W,X) + A(V)P(U,Z,W,X) + A(Z)P(V,U,W,X) + A(W)P(V,Z,U,X) + A(X)P(V,Z,W,U)$. If possible, we suppose that $2a + (n-2)b = 0 \implies a + (n-2)b = -a$. Using this result and (29) in (4) , we obtain $\mathcal{H} = 0$. Thus $2a + (n-2)b \neq 0$ and

$$\begin{aligned}&(\nabla_U R)(V,Z,W,X) = 2A(U)P(V,Z,W,X) + A(V)P(U,Z,W,X)\\ &+A(Z)P(V,U,W,X) + A(W)P(V,Z,U,X) + A(X)P(V,Z,W,U).\end{aligned} \tag{30}$$

Using (3) in (30) we obtain

$$\begin{aligned}&(\nabla_U R)(V,Z,W,X) = 2A(U)[R(V,Z,W,X) - \frac{r}{n-1}\{g(Z,W)g(V,X) - g(V,W)g(Z,X)\}]\\ &+A(V)[R(U,Z,W,X) - \frac{r}{n-1}\{g(Z,W)g(U,X) - g(U,W)g(Z,X)\}] + A(Z)[R(V,U,W,X)\\ &-\frac{r}{n-1}\{g(U,W)g(V,X) - g(V,W)g(U,X)\}] + A(W)[R(V,Z,U,X) - \frac{r}{n-1}\{g(Z,U)g(V,X)\\ &-g(V,U)g(Z,X)\}] + A(X)[R(V,Z,W,U) - \frac{r}{n-1}\{g(Z,W)g(V,U) - g(V,W)g(Z,U)\}].\end{aligned} \tag{31}$$

Now if the Einstein $(P\mathcal{H}S)_n$ is a $(PS)_n$ with the same associated 1-form A, then from the above equation we get either $r = 0$ or,

$$\begin{aligned}&2A(U)\{g(Z,W)g(V,X) - g(V,W)g(Z,X)\} + A(V)\{g(Z,W)g(U,X) - g(U,W)g(Z,X)\}\\ &+A(Z)\{g(U,W)g(V,X) - g(V,W)g(U,X)\} + A(W)\{g(Z,U)g(V,X) - g(V,U)g(Z,X)\}\\ &+A(X)\{g(Z,W)g(V,U) - g(V,W)g(Z,U)\} = 0.\end{aligned} \tag{32}$$

Contracting V and X in (32) we get

$$2(n+1)A(U)g(Z,W) + (n-2)A(Z)g(U,W) + (n-2)A(W)g(U,Z) = 0. \tag{33}$$

Again contracting U and W in (33) we have $n(n+1)A(Z) = 0$. Then it follows that $A(Z) = 0$ for all U, which is a contradiction. Therefore the only possibility is $r = 0$. Thus we conclude the following:

Theorem 6.1. *If an Einstein $(P\mathcal{H}S)_n$ $(n > 3)$ is a $(PS)_n$, then the scalar curvature of the manifold vanishes.*

Again in an Einstein $(P\mathcal{H}S)_n$ if $r = 0$, then from (31) it follows that $(\nabla_U R)(V,Z,W,X) = 2A(U)R(V,Z,W,X) + A(V)R(U,Z,W,X) + A(Z)R(V,U,W,X) + A(W)R(V,Z,U,X) + A(X)R(V,Z,W,U)$. Hence we can state the following:

Theorem 6.2. *If in an Einstein $(P\mathcal{H}S)_n (n > 3)$ with $2a + (n-2)b \neq 0$ the scalar curvature vanishes, then the manifold is a $(PS)_n$.*

7. Pseudo $\mathcal{H}$-symmetric perfect fluid spacetimes

Now we suppose that the matter distribution is a perfect fluid. Therefore, the energy momentum tensor T of type $(0,2)$ satisfies the equation (7). Hence from the Einstein's field equation and (7), we get

$$S(U,V)-\frac{r}{2}g(U,V)=\kappa[pg(U,V)+(\sigma+p)A(U)A(V)]. \tag{34}$$

Contracting U and V in the above equation we have

$$\frac{(2-n)r}{2}=\kappa[(n-1)p-\sigma]. \tag{35}$$

Replacing V by ρ in (34) and using (24) we obtain

$$\frac{(2-n)r}{2}=-\kappa\,\sigma. \tag{36}$$

Equations (35) and (36) yield

$$\sigma+p=0, \tag{37}$$

which is of the form $p=p(\sigma)$. In [45] Shepley and Taub studied a perfect fluid spacetime in dimension $n=4$, with equation of state $p=p(\sigma)$ and the additional condition that the conformal curvature tensor has null divergence. They proved that the metric is Robertson-Walker, the flow is irrotational, shear-free, and geodesic. We assume that the pseudo $\mathcal{H}$-symmetric spacetimes satisfies $div\,\mathcal{H}=0$. Then from Corollary 4.4 we get $div\,C=0$. Thus from (37) and the above results of Shepley and Taub we can state the following:

Theorem 7.1. *A 4-dimensional pseudo $\mathcal{H}$-symmetric perfect fluid spacetime with divergence free $\mathcal{H}$-curvature tensor obeying Einstein's field equations is a Robertson-Walker spacetime. Also the flow is irrotational, shear free, and geodesic, provided* $a(2n^2-8n+7)+b(n-2)^3\neq 0$.

Remark 7.2. *In a pseudo $\mathcal{H}$-symmetric perfect fluid spacetime, we get $\sigma=-p$, that is, $p=p(\sigma)$. Hence we conclude that the fluid is isentropic [27].*

Remark 7.3. *From (37), it is noticed that a 4-dimensional pseudo $\mathcal{H}$-symmetric perfect fluid spacetime satisfying Einstein's field equations represents dark matter.*

Remark 7.4. *The dark energy is usually described by an equation of state parameter $\omega=\frac{p}{\sigma}$. In a pseudo $\mathcal{H}$-symmetric perfect fluid spacetime, the equation of state parameter $\omega=\frac{p}{\sigma}=-1$. Therefore the model describes the equation in the Phantom barrier. In 2003, on the basis of the observation data, Caldwell et al. [7] noted that the equation of state parameter ω has very narrow range around $\omega=-1$ with more like hood to the side of $\omega<-1$. So, he argued that this possibility could not be ignored for the dark energy fluid.*

Now we consider spacetimes with $div\,\mathcal{H}=0$ and the additional condition that a and b are linearly dependent. Then from Corollary 4.2 it follows that the Ricci tensor is of Codazzi type. In [44] Ray proved the following:

Theorem 7.5. *If the Ricci tensor of the perfect fluid spacetime is of Codazzi type, then the velocity vector field U of the fluid is hypersurface orthogonal and energy density is constant over a hypersurface orthogonal to U. Further the fluid has vanishing vorticity and vanishing shear.*

It has been proved by Barnes [3], if a perfect fluid spacetime is vorticity free and shear-free and velocity vector field U of the fluid is hypersurface orthogonal and energy density is constant over a hypersurface orthogonal to U, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O (conformally flat). Thus we can state the following:

Theorem 7.6. *If the $\mathcal{H}$-curvature tensor is divergence free in a perfect fluid spacetime with a and b are linearly dependent, then the possible local cosmological structures of the spacetime are of Petrov type I, D or O (conformally flat).*

Remark 7.7. *Mantica et al. [39] proved that an n-dimensional GRW spacetime satisfying $div\,C=0$ if and only if the spacetime is a perfect fluid. This result with Corollary 4.4 tells us that a GRW spacetime with $div\,\mathcal{H}=0$ is a perfect fluid spacetime, provided* $a(2n^2-8n+7)+b(n-2)^3\neq 0$.

8. Example of a $(\mathcal{PHS})_4$

We construct an example of a pseudo $\mathcal{H}$-symmetric spacetime whose metric is taken from the paper [31].
We consider a Lorentzian manifold (M^4, g) equipped with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = -(dx^4)^2 + (x^2)^2(dx^3)^2 + (x^1)^2(dx^2)^2 + (dx^1)^2, \ \forall \ i, \ j = 1, 2, 3, 4.$$

With the help of above equation, we can easily find the non-zero components of the curvature tensor, Christoffel symbols, and the Ricci tensor as

$$\Gamma^1_{22} = -x^1, \ \ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \ \ \Gamma^2_{12} = \frac{1}{x^1}, \ \ \Gamma^3_{23} = \frac{1}{x^2}, \ R_{1332} = -\frac{x^2}{x^1}, \ \ S_{12} = -\frac{1}{x^1x^2}.$$

The non-vanishing components of $\mathcal{H}$-curvature tensor and its covariant derivatives are given by

$$H_{1332} = -[2a + (n-2)b]\frac{x^2}{x^1}, \ H_{1332,1} = [2a + (n-2)b]\frac{2x^2}{(x^1)^2}, \ H_{1332,2} = -[2a + (n-2)b]\frac{1}{x^1}.$$

Let

$$A_i(x) = \begin{cases} -\frac{2}{3x^1}, & \text{for i=1} \\ \frac{1}{3x^2}, & \text{for i=2} \\ 0, & \text{otherwise} \end{cases}$$

at any point $x \in M^4$ denote the 1-form. From (5) we have

$$H_{1332,1} = 2A_1H_{1332} + A_1H_{1332} + A_3H_{1132} + A_3H_{1312} + A_2H_{1331} \tag{38}$$

and

$$P_{1332,2} = 2A_2P_{1332} + A_1P_{2332} + A_3P_{1232} + A_3P_{1322} + A_2P_{1332} \tag{39}$$

We can observed that the equations (38) and (39) are true on M^4. Thus, (M^4, g) is a pseudo $\mathcal{H}$-symmetric spacetime, that is, $(\mathcal{PHS})_4$.
Hence we have

Theorem 8.1. *A 4-dimensional Lorentzian manifold $(\mathbb{R}^4, g)$ with a Lorentzian metric g defined by*

$$ds^2 = g_{ij}dx^i dx^j = -(dx^4)^2 + (x^2)^2(dx^3)^2 + (x^1)^2(dx^2)^2 + (dx^1)^2,$$

is a pseudo $\mathcal{H}$-symmetric spacetime.

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