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Packing Chromatic Numbers of Finite Super Subdivisions of Graphs

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Abstract. Given a graph G and a positive integer i, an *i-packing* in G is a subset W of the vertex set of G such that the distance between any two distinct vertices from W is greater than i. The *packing chromatic number* of a graph G, $\chi_\rho(G)$, is the smallest integer k such that the vertex set of G can be partitioned into sets V_i, $i \in \{1, \dots, k\}$, where each V_i is an i-packing. In this paper, we present some general properties of packing chromatic numbers of *finite super subdivisions* of graphs. We determine the packing chromatic numbers of the finite super subdivisions of complete graphs, cycles and some *neighborhood corona graphs*.

1. Introduction

In this paper, we consider only finite, simple graphs. We will use the following definitions and notations. For a given graph G, the vertex set of G is denoted by $V(G)$ and the edge set by $E(G)$. The *(open) neighborhood* of a vertex $u \in V(G)$ is the set of all vertices adjacent to u: $N_G(u) = \{v \in V(G) | uv \in E(G)\}$ (we often drop the subscript if the graph G is clear from context). The *degree* of u, denoted by $\deg_G(u)$ (or shorter $\deg(u)$), is $|N_G(u)|$. Further, the *minimum degree* of G, denoted by $\delta(G)$, is defined to be $\delta(G) = \min\{\deg(v) | v \in V(G)\}$.

The *distance* between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$ (or $d(u, v)$ in the case when a graph G is clear from context), is the length of a shortest u, v-path. The maximum of $\{d_G(x, y) | x, y \in V(G)\}$ is called the *diameter* of G and is denoted by $\mathrm{diam}(G)$. Given a positive integer i, an *i-packing* in G is a subset W of the vertex set of G with the property that the distance between any two distinct vertices from this set is greater than i. Note that this concept generalizes the notion of an independent set, which is equivalent to a 1-packing. Further, the *packing chromatic number* of a given graph G, $\chi_\rho(G)$, is defined as the smallest integer k such that the vertex set of G can be partitioned into sets $V_1, V_2, \dots, V_k$, where V_i is an i-packing for each $i \in \{1, 2, \dots, k\}$. The corresponding mapping $c : V(G) \to [k]$, satisfying the property that $c(u) = c(v) = i$ implies $d_G(u, v) > i$, is called a *k-packing coloring*. In the case when $k = \chi_\rho(G)$, we say that k-packing coloring is *optimal*.

The concept of the packing chromatic number was introduced in 2008 by Goddard et al. [15]. First, it was presented under the name broadcast chromatic number, and the current name was given in [6]. The

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concept arose from the area of frequency assignment in wireless networks [12, 27] and also has several additional applications, such as in resource replacement and biological diversity [6].

The packing chromatic number has been investigated in a number of papers, for example, there exist more than 10 papers, which were published only in the last two years (see [1, 2, 4, 5, 9, 16, 19, 20, 22, 23, 26]). This confirms a wide interest given to this concept. One of the main areas of investigation has been to determine the packing chromatic numbers of infinite graphs such as infinite grids, lattices, distance graphs, etc. [3, 6, 10, 11, 13, 21]. For instance, in the last paper in a series Barnaby et al. [3] prove that the packing chromatic number of the infinite square lattice is between 13 and 15. A lot of attention has been also given to the question of boundedness of the packing chromatic numbers in the class of cubic graphs. The question was answered in the negative by Balogh, Kostochka and Liu [1] and there is also known an explicit construction of an infinite family of subcubic graphs with unbounded packing chromatic number (see [5]). Note that the problem of determining the packing chromatic number is computationally (very) hard [12] as its decision version is NP-complete even when restricted to trees (see also a more recent investigation [19]).

It is known that the packing chromatic number satisfies the hereditary property in the sense that a graph cannot have smaller packing chromatic number that its subgraphs. The behaviour of the invariant under some local operations, as edge-contraction, vertex-deletion, edge-deletion, and edge subdivision was investigated in [7], while the packing chromatic number of subdivision of a given graph (a graph obtained from a given graph G by subdividing every edge of G, denoted by $S(G)$) was considered, for example in [6]. In particular, it was proven that for any connected graph G with at least three vertices, we have: $\omega(G) + 1 \leq \chi_\rho(S(G)) \leq \chi_\rho(G) + 1$ (recall that $\omega(G)$ denotes the *clique number* of G, i.e. the number of the vertices in a maximum clique of G).

Further, given a graph G and a positive integer m, the *finite super subdivision* of G, denoted by $\mathrm{FSSD}_m(G)$, is the graph obtained from G by replacing each of its edges with a complete bipartite graph $K_{2,m}$. In other words, $\mathrm{FSSD}_m(G)$ is obtained from G by first multiplying each of its edges m times and then making the subdivision graph from the resulting multigraph. Note that the subdivision of a given graph G, $S(G)$, is equivalent to $\mathrm{FSSD}_m(G)$, when $m = 1$, and hence finite super subdivision graphs are in some way a generalization of subdivision graphs.

A (finite) super subdivision of a graph was considered in several papers (see e.g. [18, 24, 25, 28]), but it is known only a little about the packing chromatic numbers of super subdivision graphs. Actually, we have found only one paper considering the packing chromatic number of such graphs (written by William and Roy [28]). Moreover, the authors of the mentioned paper determined only the packing chromatic number of finite super subdivision of star graphs, and hence, we consider this topic.

In the first part of the paper, we present some general properties of packing chromatic numbers of finite super subdivisions of graphs. More precisely, we give a lower and an upper bound for $\chi_\rho(\mathrm{FSSD}_m(G))$, where G is an arbitrary connected graph with at least three vertices and m is any positive integer. Namely, we prove that $\omega(G)+1 \leq \chi_\rho(\mathrm{FSSD}_m(G)) \leq \chi_\rho(G)+1$, which generalizes the result for subdivision of graph from [6]. Further, we prove that the packing chromatic numbers of graphs $\mathrm{FSSD}_m(G)$, $\mathrm{FSSD}_{m+1}(G)$, $\mathrm{FSSD}_{m+2}(G)$, $\ldots$ are equal, when m is large enough, i.e. it is greater than $\frac{\chi_\rho(G)}{\delta(G)}$. In addition, we determine also the packing chromatic numbers of super subdivision graphs G in the case when G is a bipartite graph, a complete graph or a cycle.

In the second part of the paper, we consider the packing chromatic numbers of finite super subdivision graphs of neighborhood corona graphs. Recall that the *neighborhood corona graph* of graphs G (with $|V(G)| = n_1$) and H is the graph, obtained by one copy of G and n_1 copies of H, such that each vertex of i-th copy of H is adjacent to all neighbors of i-th vertex of G. This graph is denoted by $G \star H$. In particular, when H is isomorphic to K_1, $G \star H$ is called a *splitting graph* and is denoted by $S'(G)$. Recall that neighborhood corona graphs have some nice properties about their structure (see e.g. [17]), which motivated us to consider their finite super subdivision graphs. In this paper, we provide the exact values for $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p))$, when m is a positive arbitrary integer, and the upper bound for $\chi_\rho(\mathrm{FSSD}_m(C_n \star P_p))$. At the end, we provide some remarks and open questions.

2. Finite super subdivisions

In this section, we study general properties of finite super subdivision graphs.

We will use the following notations for the vertices of $\mathrm{FSSD}_m(G)$. The vertices corresponding to the vertices of G, are denoted by $u_1, u_2, \ldots, u_n$, and for any pair of vertices u_i, u_j, $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, we denote the common neighbors of them by $u_{i,j}^k$ (these vertices will be called subdivided vertices), where $k \in \{1, \ldots, m\}$. Further, recall two well known results of packing chromatic number, which will be used several times in the sequel of this paper. While the first proposition states that the packing chromatic number satisfies the hereditary property, the second provides the values of packing chromatic numbers for cycles.

Proposition 2.1. *[15] For any subgraph H of a given graph G,*

$$\chi_\rho(H) \leq \chi_\rho(G).$$

Proposition 2.2. *[15] If C_n is any cycle of order n, then*

$$\chi_\rho(C_n) = \begin{cases} 3; & n = 4k, k \geq 1, \text{ or } n = 3, \\ 4; & \text{otherwise.} \end{cases}$$

While the next proposition have been already proven for subdivisions of graph (i.e. graphs $\mathrm{FSSD}_m(G)$, when $m = 1$), we present its generalization (considering the number m). Namely, we provide the lower and the upper bound for $\chi_\rho(\mathrm{FSSD}_m(G))$.

Proposition 2.3. *If $m \geq 1$ and G is a connected graph with at least three vertices, then*

$$\omega(G) + 1 \leq \chi_\rho(\mathrm{FSSD}_m(G)) \leq \chi_\rho(G) + 1.$$

Proof. It is known that the result holds for $m = 1$ [6]. Then, since for any $m \geq 1$ $\mathrm{FSSD}_m(G)$ contains a subgraph isomorphic to $\mathrm{FSSD}_1(G)$, Proposition 2.1 implies that $\omega(G) + 1 \leq \chi_\rho(\mathrm{FSSD}_1(G)) \leq \chi_\rho(\mathrm{FSSD}_m(G))$ for each $m \geq 1$. In order to provide the upper bound, denote by c any optimal packing coloring of G and define a coloring c' of $\mathrm{FSSD}_m(G)$ as follows: $c'(u_{i,j}^k) = 1$ and $c'(u_i) = c(u_i) + 1$ for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, and $k \in \{1, \ldots, m\}$. It is easy to check that c' is a packing coloring. Namely, suppose that $c'(u_i) = c'(u_j) = l \geq 2$ for some i, j and color l. Then $c(u_i) = c(u_j) = l - 1$, which implies that $d_G(u_i, u_j) \geq l$ and thus $d_{\mathrm{FSSD}_m(G)}(u_i, u_j) \geq 2l \geq l + 1$ for any $l \geq 2$. Additionally, note that the distance between any two vertices both colored with color 1 is at least 2. Therefore, c' is a packing coloring of $\mathrm{FSSD}_m(G)$ and since it uses color 1 and exactly $\chi_\rho(G)$ other colors, $\chi_\rho(\mathrm{FSSD}_m(G)) \leq \chi_\rho(G) + 1$, what completes the proof. □

Note that in the case of graphs G with $\chi_\rho(G) = \omega(G)$, the written bounds provide the exact values for $\chi_\rho(\mathrm{FSSD}_m(G))$, $m \geq 1$. For example, applying the written proposition, we derive the following corollary.

Corollary 2.4. *For any $n \geq 3$ and $m \geq 1$, $\chi_\rho(\mathrm{FSSD}_m(K_n)) = n + 1$.*

In addition, complete graphs (and other graphs G with $\chi_\rho(G) = \omega(G)$) prove that the above written bounds are sharp.

Since by Proposition 2.3, the packing chromatic number of the family of graphs $\{\mathrm{FSSD}_1(G), \mathrm{FSSD}_2(G), \mathrm{FSSD}_3(G), \ldots\}$ is bounded from above, and by Proposition 2.1, $\chi_\rho(\mathrm{FSSD}_m(G)) \leq \chi_\rho(\mathrm{FSSD}_{m+1}(G))$ for any graph G and any $m \geq 1$, we infer that the packing chromatic numbers of graphs $\mathrm{FSSD}_m(G), \mathrm{FSSD}_{m+1}(G), \mathrm{FSSD}_{m+2}(G), \ldots$ are equal, when m is large enough. Hence, we are interested in question, for which m is $\chi_\rho(\mathrm{FSSD}_m(G)) = \chi_\rho(\mathrm{FSSD}_{m+1}(G)) = \chi_\rho(\mathrm{FSSD}_{m+2}(G)) = \ldots$. With the following two propositions, we prove that the written equalities hold for any m greater than $\frac{\chi_\rho(G)}{\delta(G)}$. In addition, we prove that in the case of complete graphs of order at least 3, bipartite graphs of order at least 3 and cycles, the equality actually holds for any $m \geq 1$.

Proposition 2.5. *Let G be a graph and $m \geq 1$ a positive integer. If there exists an optimal packing coloring c of $\mathrm{FSSD}_m(G)$ which assigns to all subdivided vertices a color 1, then $\chi_\rho(\mathrm{FSSD}_m(G)) = \chi_\rho(\mathrm{FSSD}_{m+1}(G))$.*

Proof. Let G be a graph, $m \geq 1$ a positive integer and c an optimal packing coloring of $\mathrm{FSSD}_m(G)$ which assigns to all subdivided vertices a color 1. By setting $c'(u_i) = c(u_i)$ for any u_i, and assigning a color 1 to all other vertices of $\mathrm{FSSD}_{m+1}(G)$, we get a $\chi_\rho(FSSD_m(G))$-packing coloring of $\mathrm{FSSD}_{m+1}(G)$ and hence, $\chi_\rho(\mathrm{FSSD}_m(G)) = \chi_\rho(\mathrm{FSSD}_{m+1}(G))$. □

Proposition 2.6. *If G is any graph and $m > \frac{\chi_\rho(G)}{\delta(G)}$ a positive integer, then*

$$\chi_\rho(\mathrm{FSSD}_m(G)) = \chi_\rho(\mathrm{FSSD}_{m+1}(G)).$$

Proof. Let $m > \frac{\chi_\rho(G)}{\delta(G)}$ be an arbitrary positive integer. Then $m > \frac{\chi_\rho(G)}{\deg_G(u_i)}$ for each $u_i, i \in \{1, \ldots, n\}$, and thus $m \cdot \deg_G(u_i) > \chi_\rho(G)$. Clearly, $\deg_{\mathrm{FSSD}_m(G)}(u_i) = \deg_G(u_i) \cdot m$ for any u_i, thus $1 + \deg_{\mathrm{FSSD}_m(G)}(u_i) > 1 + \chi_\rho(G)$. By Proposition 2.3, it follows that $1 + \deg_{\mathrm{FSSD}_m(G)}(u_i) > \chi_\rho(\mathrm{FSSD}_m(G))$.

Next, let c be any optimal packing coloring of $\mathrm{FSSD}_m(G)$. Prove that, if c assigns a color 1 to a vertex u_i, $i \in \{1, \ldots, n\}$, then it uses at least $1 + \deg_{\mathrm{FSSD}_m(G)}(u_i)$ colors. Suppose that $c(u_i) = 1$ for some $i \in \{1, \ldots, n\}$. Then, the neighbors of u_i get pairwise different colors (since they are pairwise at distance 2, but cannot be colored with color 1) and hence c uses at least $1 + \deg_{\mathrm{FSSD}_m(G)}(u_i)$ colors (actually these colors are already required for a packing coloring of vertices from $N_{\mathrm{FSSD}_m(G)}[u_i]$). But since $1 + \deg_{\mathrm{FSSD}_m(G)}(u_i) > \chi_\rho(\mathrm{FSSD}_m(G))$, c is not optimal and therefore, $c(u_i) \neq 1$ for any $i \in \{1, \ldots, n\}$. Without loss of generality, we may assume that all other (subdivided) vertices get color 1 (since they are pairwise at distance at least 2). Then, Proposition 2.5 implies the result. □

Therefore, if $m > \frac{\chi_\rho(G)}{\delta(G)}$, then $\chi_\rho(\mathrm{FSSD}_m(G)) = \chi_\rho(\mathrm{FSSD}_{m+1}(G))$, but the case when $m \leq \frac{\chi_\rho(G)}{\delta(G)}$ is still opened. Clearly, also in this case for any graph G and any $m \geq 1$ the following inequality holds: $\chi_\rho(\mathrm{FSSD}_m(G)) \leq \chi_\rho(\mathrm{FSSD}_{m+1}(G))$. As we will see, there exist graphs, for example cycles of order $2k + 1$, $k \geq 2$ (see Propositions 2.2 and 2.8), which satisfy the property that $\chi_\rho(G) = \chi_\rho(\mathrm{FSSD}_m(G))$ for any $m \geq 1$. But for the others, there is a question of when there appear a change of the value of the packing chromatic number. More precisely, for which m we have $\chi_\rho(\mathrm{FSSD}_m(G)) < \chi_\rho(\mathrm{FSSD}_{m+1}(G)) = \chi_\rho(\mathrm{FSSD}_{m+i}(G))$ for any $i \geq 2$? Based on the Proposition 2.5, we derive that a necessary condition for the inequality of $\chi_\rho(\mathrm{FSSD}_m(G))$ and $\chi_\rho(\mathrm{FSSD}_{m+1}(G))$ is that any optimal packing coloring of $\mathrm{FSSD}_m(G)$ assigns to at least one of the subdivided vertices a color greater than 1. For example, any optimal packing coloring of $\mathrm{FSSD}_1(K_2)$ assigns to a subdivided vertex a color 2 and we have: $2 = \chi_\rho(\mathrm{FSSD}_1(K_2)) < \chi_\rho(\mathrm{FSSD}_m(K_2)) = 3$, $m \geq 2$. Also for the Petersen graph, $\chi_\rho(\mathrm{FSSD}_1(P)) = 5$ (see [8]) and it is easy to observe that there exists an optimal packing coloring $\mathrm{FSSD}_m(P)$, $m \geq 2$, which assigns to all subdivided vertices a color 1, which implies that $\chi_\rho(\mathrm{FSSD}_m(P)) = \chi_\rho(\mathrm{FSSD}_{m+1}(P)) \geq 6$ for any $m \geq 2$. We have to mention that we were not able to find any other graph G such that $\chi_\rho(\mathrm{FSSD}_1(G)) < \chi_\rho(\mathrm{FSSD}_2(G))$, and in addition, we have not found any graphs G with the property that $\chi_\rho(\mathrm{FSSD}_m(G)) < \chi_\rho(\mathrm{FSSD}_{m+1}(G))$ for any $m \geq 2$. Hence, there arises an open question of whether there exists a graph G with the written property.

We continue with the consideration of packing chromatic numbers of finite super subdivision graphs of bipartite graphs and (other) cycles. While the result has been already known for all bipartite graphs (it follows from Proposition 3.3. in [15]), we give the exact values also for cycles.

Proposition 2.7. *For any bipartite graph G of order at least 3 and any $m \geq 1$,*

$$\chi_\rho(\mathrm{FSSD}_m(G)) = 3.$$

Proposition 2.8. *If $n \geq 3$ and $m \geq 1$, then*

$$\chi_\rho(\mathrm{FSSD}_m(C_n)) = \begin{cases} 3; & n \text{ is even}, \\ 4; & n \text{ is odd}. \end{cases}$$

Proof. Let $n \geq 3$ and $m \geq 1$ be arbitrary integers. If n is even, then Proposition 2.7 implies the result. Otherwise, note that $\mathrm{FSSD}_m(C_n)$ contains a subgraph, which is isomorphic to a cycle C_{2n}. Since in this case $2n$ is not a multiple of 4 and $2n \neq 3$, the results from Proposition 2.1 and Proposition 2.2 imply that $\chi_\rho(\mathrm{FSSD}_m(C_n)) \geq 4$. To show that $\chi_\rho(\mathrm{FSSD}_m(C_n)) \leq 4$, color all vertices $u^k_{i,j}$, $i, j \in \{1, \dots, n\}$, $i \neq j$, $k \in \{1, \dots, m\}$, with color 1 and the consecutive vertices $u_1, u_2, \dots, u_n$ one after another using the following pattern of colors: $2, 3, 2, 3, \dots, 2, 3, 4$. Clearly, this is a 4-packing coloring of $\mathrm{FSSD}_m(C_n)$, thus $\chi_\rho(\mathrm{FSSD}_m(C_n)) = 4$. □

3. Finite super subdivision graphs of neighborhood corona graphs

We continue with determining the packing chromatic numbers of finite super subdivision graphs of neighborhood corona graphs. Note that neighborhood corona graphs were defined in Section 1.

In the sequel of this paper we consider finite super subdivision graphs of the following neighborhood corona graphs: $K_n \star P_p$ and $C_n \star P_p$, where $n \geq 3$ and $p \geq 1$. We use the following notations of the vertices of $\mathrm{FSSD}_m(K_n \star P_p)$ respectively $\mathrm{FSSD}_m(C_n \star P_p)$. The vertices of $K_n \star P_p$ respectively $C_n \star P_p$, which are corresponding to the vertices from $V(K_n)$ respectively $V(C_n)$, are denoted by $u_1, u_2, \dots, u_n$. For any $u_i, i \in \{1, \dots, n\}$, denote the corresponding copy of P_p by $P_{i,p}$ and the vertices of $P_{i,p}$ by $v_{i,g}$, where $g \in \{1, 2, \dots, p\}$ (in particular, when $p = 1$, these vertices are denoted by v_i). The common neighbors of u_i and u_j, where $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, are denoted by $u^k_{i,j}$, where $k \in \{1, 2, \dots, m\}$. Analogously, the vertices which connect $v_{i,g}$ and $v_{i,h}$, where $i \in \{1, 2, \dots, n\}$ and $g, h \in \{1, 2, \dots, p\}$, $g \neq h$, are denoted by $v^k_{i,g,h}$, where $k \in \{1, 2, \dots, m\}$. Finally, the vertices connecting u_j and $v_{i,g}$, where $i, j \in \{1, 2, \dots, n\}$, $i \neq j$ and $g \in \{1, 2, \dots, p\}$, are labeled by $s^k_{j,i,g}$, where $k \in \{1, 2, \dots, m\}$. In particular, if $p = 1$, then the common neighbors of u_i and v_j are denoted by $s^k_{i,j}$ for any $i, j \in \{1, 2, \dots, n\}$ $i \neq j$, and $k \in \{1, 2, \dots, m\}$.

Before determining $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p))$ we need the following lemma.

Lemma 3.1. *Let $n \geq 3$, $p \geq 2$ and $m \geq 1$ be arbitrary integers. Then, for any a-packing coloring c of the graph $\mathrm{FSSD}_m(K_n \star P_p)$, where $a \leq n + 3$, the following holds:*

1. $c(u_i) \neq 1$ *for all* $i \in \{1, 2, \dots, n\}$;
2. $c(v_{i,g}) \neq 1$ *for all* $i \in \{1, 2, \dots, n\}$ *and* $g \in \{1, 2, \dots, p\}$.

Proof. Let $n \geq 3$, $p \geq 2$, $m \geq 1$ be arbitrary integers, and let c be an arbitrary a-packing coloring of $\mathrm{FSSD}_m(K_n \star P_p)$, where $a \leq n + 3$.

First we prove that $c(u_i) \neq 1$ for all $i \in \{1, 2, \dots, n\}$. Suppose to the contrary that $c(u_i) = 1$ for some $i \in \{1, \dots, n\}$ and without loss of generality, assume that $i = 1$. Note that u_1 has $m(n-1) + m(n-1)p$ neighbors. Since $m \geq 1$ and $p \geq 2$, we derive that u_1 has actually at least $3n - 3$ neighbors, which are pairwise at distance 2 and hence c assigns them pairwise different colors from $\{2, 3, \dots\}$. Therefore, c uses at least $3n - 3 + 1 = 3n - 2$ colors, which is more than a, a contradiction to c being an a-packing coloring. Hence, $c(u_i) \neq 1$ for all $i \in \{1, \dots, n\}$ and without loss of generality, we may assume that $c(u^k_{i,j}) = 1$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$, and $k \in \{1, \dots, m\}$.

Next, we prove that $c(v_{i,g}) \neq 1$ for all $i \in \{1, 2, \dots, n\}$ and $g \in \{1, 2, \dots, p\}$. Again, suppose to the contrary that $c(v_{i,g}) = 1$ for some i and g, say $c(v_{2,1}) = 1$. Note that $v_{2,1}$ has at least mn neighbors. If $m \geq 2$, then $\deg(v_{2,1}) \geq 2n$ and hence c uses at least $2n + 1$ different colors, which is more than a, a contradiction. Therefore, in the remainder of the proof, we only need to consider the case when $m = 1$. We distinguish three cases with respect to n.

Case 1. $n \geq 5$.
In this case, c assigns to (at least) four vertices $s^1_{i,2,1}$, $i \in \{1, 3, 4, \ldots, n\}$ pairwise different colors from $\{2, 3, 4, \ldots\}$, and note that at least three of the mentioned vertices receive colors from $\{3, 4, \ldots\}$ by c. Since these colors cannot be used for packing coloring of the vertices from $\{u_1, u_2, \ldots, u_n\}$, Corollary 2.4 implies that c uses at least $n + 4$ colors, a contradiction.

Case 2. $n = 4$.
Vertices $s^1_{1,2,1}$, $s^1_{3,2,1}$, $s^1_{4,2,1}$ and $v^1_{2,1,2}$ receive four different colors by c. Note that, if $c(s^1_{1,2,1})$, $c(s^1_{3,2,1})$, $c(s^1_{4,2,1}) \in \{3, 4, \ldots\}$, then by the same consideration as above follows a contradiction. Therefore, $c(s^1_{i,2,1}) = 2$ for some $i \in \{1, 3, 4\}$, say $i = 1$. If $c(v^1_{2,1,2}) = 3$, then by using Corollary 2.4, we infer that c uses at least $n + 3$ colors for a packing coloring of the subgraph of $\mathrm{FSSD}_1(K_n \star P_p)$, which is induced by the set of vertices $V(FFSD_1(K_4)) \cup \{s^1_{1,2,1}, s^1_{3,2,1}, s^1_{4,2,1}, v_{2,1}, v^1_{2,1,2}\}$. Since $a \leq n + 3$, the colors 2 and 3 must be used for a packing coloring of the vertices from $\{u_1, u_2, u_3, u_4\}$, which implies $c(u_2) = 3$. But then it is easy to check that there is no available colors for at least one of the vertices $v_{2,2}$, $s^1_{1,2,2}$, $s^1_{3,2,2}$ or $s^1_{4,2,2}$ (note that at least one of them cannot receive a color 1). The same result follows if $c(v^1_{2,1,2}) = 4$. In the case when $c(v^1_{2,1,2}) \geq 5$, colors $c(v^1_{2,1,2})$, $c(s^1_{3,2,1})$ and $c(s^1_{4,2,1})$ cannot be used by c for packing coloring of the vertices of subgraph isomorphic to $\mathrm{FSSD}_1(K_4)$, thus by Corollary 2.4, c uses more than a colors, a contradiction.

Case 3. $n = 3$.
Recall that c is an a-packing coloring of $\mathrm{FSSD}_1(K_3 \star P_p)$ and in this case $a \leq 6$. Since $c(u_i) \neq 1$ for all $i \in \{1, 2, 3\}$, let be $c(u_1) = b$, $c(u_2) = c$ and $c(u_3) = d$, where b, c, d are three pairwise distinct colors from $\{2, 3, \ldots a\}$. We consider five sub-cases with respect to the colors b, c, d.

Case 3.1. $b = 2$.
Vertex $s^1_{1,2,1}$ can be colored only with color $e \in \{3, 4, \ldots, a\}$ by c, where $e \neq c, d$, and $s^1_{3,2,1}$ can receive one of the colors from $\{2, f\}$ by c, where $f \in \{3, 4, \ldots, a\}$, $f \neq c, d, e$.

If $c(s^1_{3,2,1}) = 2$, then $c(v^1_{2,1,2}) \in \{f, c\}$. The first case implies that $c(v_{2,2}) = c = 3$. The second case yields that $c \leq 4$ and $c(v_{2,2}) \in \{1, f\}$; if 1 is used, then $c(s^1_{1,2,2}) = f$ and $c(s^1_{3,2,2}) = e = 3$.

If $c(s^1_{3,2,1}) = f$, then $c(v_{2,2}) \in \{1, c\}$; the first case implies that $f = 3$ ($c(s^1_{1,2,2}) = f = 3$), and the second that $c = 3$.

Next, consider the vertices $s^1_{1,3,1}$, $s^1_{1,3,2}$, $v_{3,1}$ and $v_{3,2}$. It is easy to observe that in each case there is no available colors by c for at least one of the mentioned vertices, a contradiction to c being an a-packing coloring.

Case 3.2. $c = 2$
In this case $\{c(s^1_{1,2,1}), c(s^1_{3,2,1}), c(v^1_{2,1,2})\} = \{2, e, f\}$, $e, f \in \{3, \ldots, a\}$, $e \neq f$, $e \neq b, d$, $f \neq b, d$.

First, suppose that $c(v^1_{2,1,2}) = 2$ (and $c(s^1_{1,2,1}) = e$, $c(s^1_{3,2,1}) = f$). This implies that $c(v_{2,2}) = 1$ and $c(s^1_{1,2,2}) = f = 3$, but then there is no available color for the vertex $s^1_{3,2,2}$, which yields a contradiction to c being an a-packing coloring. Therefore $c(v^1_{2,1,2}) \neq 2$ and without loss of generality we may assume that $c(s^1_{1,2,1}) = 2$ (and $c(s^1_{3,2,1}) = e$, $c(v^1_{2,1,2}) = f$). Next distinguish three posibilities with respect to the colors $c(v_{1,1})$ and $c(v_{1,2})$.

Case 3.2.1. $c(v_{1,1}), c(v_{1,2}) \neq 1$.
The only possibility is that the vertices $v_{1,1}$ and $v_{1,2}$ receive colors $3 = b$ and $4 = f$ by c, which implies $c(v_{3,1}) = c(v_{3,2}) = 1$. But then there is no available colors for $s^1_{2,3,2}$, a contradiction to c being an a-packing coloring.

Case 3.2.2. $c(v_{1,1}) = 1, c(v_{1,2}) \neq 1$ (the proof in the case when $c(v_{1,1}) \neq 1, c(v_{1,2}) = 1$ is analogous).
Vertex $s^1_{2,1,1}$ can receive color e, if $e = 3$, or color f, if $f \in \{3, 4, 5\}$, by c.

If $c(s^1_{2,1,1}) = 3$ (i.e. either $e = 3$ or $f = 3$), then $c(v_{1,2}) = f = 4$ (and therefore $c(s^1_{2,1,1}) = e = 3$), so $v_{3,1}$ and $v_{3,2}$ can receive only color 1 by c, but then we have the same contradiction as in the situation 3.2.1. The case when $c(s^1_{2,1,1}) = 4 = f$ yields $c(v_{1,2}) = 3 = b$ and again, the vertices $v_{3,1}$ and $v_{3,2}$ can receive only color 1 by c (note that $d, e \geq 5$), a contradiction. Therefore, $c(s^1_{2,1,1}) = 5 = f$ and then $c(v_{1,2}) = 3 = b$, which yields that the vertices $v_{3,1}$ and $v_{3,2}$ can receive only colors 1 and $e = 4$ by c. In each case it is impossible to color all of the vertices from $N(v_{3,1}) \cup N(v_{3,2})$ by c, a contradiction to c being an a-packing coloring.

Case 3.2.3. $c(v_{1,1}) = c(v_{1,2}) = 1$.

This implies that $c(s^1_{3,1,1})$ is 2 or $f = 3$. If $c(s^1_{3,1,1}) = f = 3$, then there is no available colors for $s^1_{2,1,1}$. Thus $c(s^1_{3,1,1}) = 2$, which implies $c(s^1_{3,1,2}) = f = 3$, but then there is no available colors for $s^1_{2,1,2}$, a contradiction to c being an a-packing coloring.

Case 3.3. $b = 3$ and $c, d \neq 2$ (i.e. $c, d \in \{4, \dots, a\}$).
In this case $\{c(s^1_{1,2,1}), c(s^1_{3,2,1})\} = \{2, e\}$, where $e \in \{4, \dots, a\}, e \neq c, d$. First, consider the case when $c(s^1_{1,2,1}) = 2$ and $c(s^1_{3,2,1}) = e$. We derive that $c(s^1_{1,3,1}) = c(s^1_{1,3,2}) = 1$, hence the vertices $v_{3,1}$ and $v_{3,2}$ receive color 2 and color $e = 4$ (thus, $c \neq 4$). But then, there is no available color for $v^1_{2,1,2}$, a contradiction. If $c(s^1_{1,2,1}) = e$ and $c(s^1_{3,2,1}) = 2$, then the vertices $s^1_{1,3,1}, s^1_{1,3,2}, s^1_{2,3,1}, s^1_{2,3,2}, v_{3,1}$ and $v_{3,2}$ receive the colors from $\{1, 2\}$. But since it is impossible to color all of the mentioned vertices with these two colors, we have a contradiction to c being a packing coloring.

Case 3.4. $c = 3$ and $b, d \neq 2$ (i.e. $b, d \in \{4, \dots, a\}$).
This yields that $\{c(s^1_{1,2,1}), c(s^1_{3,2,1})\} = \{2, e\}$, where $e \in \{4, \dots, a\}, e \neq b, d$. First assume that $c(s^1_{1,2,1}) = 2$ and $c(s^1_{3,2,1}) = e$. The vertices $s^1_{3,1,1}, s^1_{2,1,1}, s^1_{2,1,2}, s^1_{3,1,2}, v_{1,1}$ and $v_{1,2}$ receive colors 1 and 2. Again, it is impossible to color all of the mentioned vertices only with two colors, thus we have a contradiction. If $c(s^1_{1,2,1}) = e$ and $c(s^1_{3,2,1}) = 2$, then by considering the vertices $s^1_{2,3,1}, s^1_{1,3,1}, s^1_{2,3,2}, s^1_{1,3,2}, v_{3,1}$ and $v_{3,2}$ by analogous consideration as above follows a contradiction.

Case 3.5. $\{b, c, d\} = \{4, 5, 6\}$.
Vertices $s^1_{1,2,1}$ and $s^1_{3,2,1}$ can receive only colors 2 and 3, thus $c(v^1_{2,1,2}) = c = 4$. Without loss of generality, we may assume that $c(s^1_{1,2,1}) = 2$ and $c(s^1_{3,2,1}) = 3$. Then the vertices $s^1_{3,1,1}$ and $s^1_{3,1,2}$ can get only colors 1 or 2, so at least one of them is colored by 1, say $s^1_{3,1,1}$. This yields that $c(v_{1,1}) = 2$, $c(v_{1,2}) = 1$ and $c(v^1_{1,1,2}) = 3$, but then there is no available colors for $s^1_{2,1,2}$ and $s^1_{3,1,2}$ (only one of them can get color 2), a contradiction to c being an a-packing coloring.

By symmetry, we can switch the role of the colors b and d, and derive that our claim holds. □

Theorem 3.2. *If $n \geq 3$, $p \geq 2$ and $m \geq 1$, then*

$$\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p)) = n + 3.$$

Proof. In order to show that $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p)) \leq n + 3$ holds for any $n \geq 3$, $p \geq 2$ and $m \geq 1$, we form a $(n+3)$-packing coloring of $\mathrm{FSSD}_m(K_n \star P_p)$. First, color the vertices $u^k_{i,j}$, $v^k_{i,g,h}$ and $s^k_{i,j,g}$ for all $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, $g, h \in \{1, 2, \dots, p\}$, $g \neq h$ and $k \in \{1, \dots, m\}$, with color 1. Then color the uncolored vertices, which correspond to the vertices from $V(P_{i,p})$, $i \in \{1, \dots, n\}$, one after another with the following pattern of colors: $2, 3, 2, 3, \dots$ Finally, color the vertices $u_1, u_2, \dots, u_n$ with n new colors. It is easy to check that the described coloring is $(n+3)$-packing coloring of a given graph, thus $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p)) \leq n + 3$.

Next, prove that $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p)) \geq n+3$ holds for any $n \geq 3$, $p \geq 2$ and $m \geq 1$. First, consider a graph $\mathrm{FSSD}_m(K_n \star P_2)$, where $n \geq 3$ and $m \geq 1$ are arbitrary integers. Denote by c an optimal packing coloring of this graph. Note that c is an a-packing coloring, where $a \leq n + 3$. Using Lemma 3.1 we infer that $c(u_i) \neq 1$ and $c(v_{i,g}) \neq 1$ for all $i \in \{1, 2, \dots, n\}$ and $g \in \{1, 2\}$. Let be $A = \{u_i; 1 \leq i \leq n\} \cup \{v_{i,g}; 1 \leq i \leq n, 1 \leq g \leq 2\}$. Note that all vertices from $V(\mathrm{FSSD}_m(K_n \star P_2)) \setminus A$ (can) receive color 1 by c and all vertices from A receive colors from $\{2, 3, \dots, a\}$ by c. Since the vertices from A are pairwise at distance at most 4, we have: $|c^{-1}(i) \cap A| = 1$ for all $i \in \{4, 5, \dots, a\}$. In other words, for each color $i \in \{4, 5, \dots, a\}$, there is only one vertex from A, colored by i. Then, since c is an optimal packing coloring of $\mathrm{FSSD}_m(K_n \star P_2)$, it assigns colors 2 and 3 to at most as possible vertices. If there exists $i \in \{1, \dots, n\}$ such that $c(u_i) = 2$ (respectively $c(u_i) = 3$), then $|c^{-1}(2) \cap A| \leq 2$ (respectively $|c^{-1}(3) \cap A| \leq 2$). Otherwise, c can assign a color 2 (respectively 3) to at most n vertices (one vertex in each P_2 can be colored with color 2), which is more than 2. Therefore, $c(u_i) \neq 2$ and $c(u_i) \neq 3$ (and recall that $c(u_i) \neq 1$) for all $i \in \{1, 2, \dots, n\}$, which implies that c uses at least $n + 3$ colors. Hence, $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_2)) \geq n + 3$. Furthermore, since $\mathrm{FSSD}_m(K_n \star P_p)$, $m \geq 1, n \geq 3, p \geq 2$, contains a subgraph isomorphic to $\mathrm{FSSD}_m(K_n \star P_2)$, $\chi_\rho(\mathrm{FSSD}_m(K_n \star P_p)) \geq \chi_\rho(\mathrm{FSSD}_m(K_n \star P_2)) \geq n + 3$, what completes the proof. □

In Fig. 1 is shown a graph $\mathrm{FSSD}_m(K_3 \star P_2)$ and its packing coloring, as is described in the proof of the previous theorem. Note that all unlabeled vertices of presented graph receive a color 1.

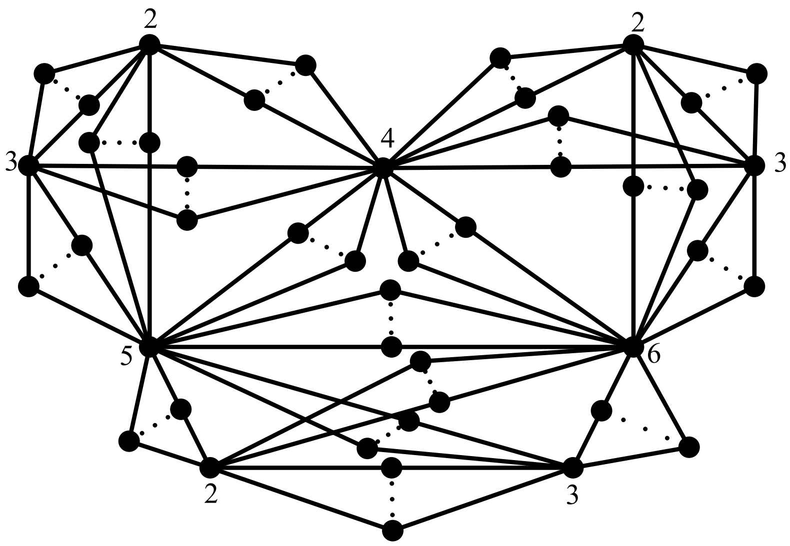

Figure 1: $FSSD_m(K_3 \star P_2)$ and a packing coloring of this graph

We continue this section with determining the packing chromatic numbers of graphs $\mathrm{FSSD}_m(C_n \star P_p)$, $n \geq 3, p \geq 2$. While in the case of graphs $\mathrm{FSSD}_m(K_n \star P_p)$, we provided the exact values of their packing chromatic numbers, in the case of cycles the task gets much harder for us, hence we present only the upper bound for $\chi_\rho(\mathrm{FSSD}_m(C_n \star P_p))$.

Theorem 3.3. *If $n \geq 3$, $m \geq 1$ and $p \geq 2$, then*

$$\chi_\rho(\mathrm{FSSD}_m(C_n \star P_p)) \leq \begin{cases} 6; & n = 3, \\ 7; & n \geq 4, n \notin \{5,7,11\}, \\ 8; & n \in \{5,7,11\}. \end{cases}$$

Proof. In the case when $n = 3$ the result clearly holds, since C_3 is isomorphic to K_3 and thus $\mathrm{FSSD}_m(K_3 \star P_p)$ is isomorphic to $\mathrm{FSSD}_m(C_3 \star P_p)$ for any $m \geq 1$ and $p \geq 2$. Thus, Theorem 3.2 yields the result.

Next, prove the desired bounds in the case when $n \geq 4$. Hence, form a packing coloring c of a given graph $\mathrm{FSSD}_m(C_n \star P_p)$. First, let be $c(u^k_{i,j}) = c(v^k_{i,g,h}) = c(s^k_{j,i,g}) = 1$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, $g, h \in \{1, \ldots, p\}$, $g \neq h$, $k \in \{1, \ldots, m\}$. Then, for each $i \in [n]$ color the vertices from $\{v_{i,g}; 1 \leq g \leq p\}$ one after another with the following sequence of colors: $2, 3, 2, 3, \ldots$. The remaining vertices of G (i.e. the vertices u_i) are colored one after another using the following pattern of colors.

Case 1. $n = 4, n = 5$.
In the case when $n = 4$ use the pattern 4567, and in the case when $n = 5$, color the vertices u_i with color pattern 45678.

Case 2. $n \cong 0 \pmod 6$.
Use the color pattern 456457 for the packing coloring of the vertices u_i.

Case 3. $n \cong 1 \pmod 6$.
In this case start the coloring of the consecutive vertices with the colors 7546574567456 and repeat the pattern 457456. If $n = 13$, then the repeated block is omitted, and if $n = 7$, then assign to the vertices $u_i, i \in [7]$, the colors of the following pattern: 4564578.

Case 4. $n \cong 2 \pmod 6$.
Color the vertices one after another using this pattern of colors 75465745674564 and repeat the pattern 754654, if necessary. If $n = 8$, then use the color pattern: 75467456.

Case 5. $n \cong 3 \pmod 6$.
When $n \cong 3 (\mathrm{mod}\ 6)$, start the coloring of the vertices u_i with the colors 4657, repeat the block 456457 and end by 45675. Note that, the repeated block is omitted in the case when $n = 9$ (see Fig. 2).

Case 6. $n \cong 4 \pmod 6$.
In this case repeat the sequence of colors 456457 and end by 4567.

Case 7. $n \cong 5 \pmod 6$.
Start the coloring with the patten 754657456, repeat the block 457456 and end by 75467546. Note that the repeated block can be omitted. In the case when $n = 11$, use the colors 75465745648.

Since in each case the described coloring is a packing coloring of a given graph, the proof is completed. □

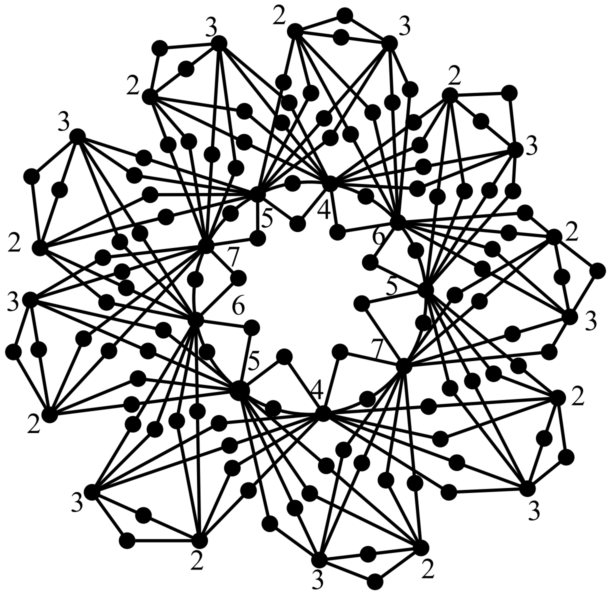

Figure 2: $FSSD_2(C_9 \star P_2)$ and a packing coloring of this graph

In the sequel of this section, we determine the packing chromatic numbers of graphs $\mathrm{FSSD}_m(K_n \star P_p)$ and $\mathrm{FSSD}_m(C_n \star P_p)$, when $p = 1$. In other words, we consider so called splitting graph.

Proposition 3.4. *If $n \geq 3$ and $m \geq 1$, then*

$$\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) = \begin{cases} 3; & \textit{if n is even,} \\ 5; & \textit{if n is odd.} \end{cases}$$

Proof. **Case 1.** n is even.
Since $\mathrm{FSSD}_m(S'(C_n))$ contains a subgraph, which is isomorphic to $\mathrm{FSSD}_m(C_n)$, by Propositions 2.1 and 2.8 follows that $\chi_\rho(\mathrm{FSSD}_m(S'(C_n))) \geq 3$ for any $m \geq 1$ and $n \geq 3$.

In order to prove that $\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) \leq 3$ holds for any $m \geq 1$ and any even $n \geq 3$, we form a 3-packing coloring of considered graph. First, color all vertices u_i, $i \in \{1,\ldots,n\}$, one after another using the following pattern of colors: $2,3,2,3,\ldots,2,3$. Then, color each vertex v_i, $1 \leq i \leq n$, with the color, which is assigned to u_i (for example, see Fig. 3). The remaining vertices are colored with color 1. Clearly, this is a 3-packing coloring of considered graph and therefore $\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) = 3$ for any $m \geq 1$ and any even $n \geq 3$.

Case 2. n is odd.
A graph $\mathrm{FSSD}_m(S^{'}(C_n))$ contains a subgraph, which is isomorphic to $\mathrm{FSSD}_m(C_n)$, and analogically as above by Propositions 2.1 and 2.8 follows that $\chi_\rho(\mathrm{FSSD}_m(S'(C_n))) \geq 4$ for any $m \geq 1$ and any odd $n \geq 3$.

Next, suppose that $\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) = 4$ and let c be any 4-packing coloring of considered graph. If $c(u_i) = 1$ for some $i \in \{1,\ldots,n\}$, then all neighbors of u_i get pairwise different colors. Since $\deg(u_i) \geq 4$, c uses at least 5 colors, a contradiction. Therefore, $c(u_i) \in \{2,3,4\}$ for all $i \in \{1,\ldots,n\}$. If there exists $i \in \{1,\ldots,n\}$, such that $c(u_i) = 4$ and $c(u_{i-1}) = 2$, $c(u_{i+1}) = 3$ (resp., $c(u_{i-1}) = 3$, $c(u_{i+1}) = 2$), then there is no available color for v_i or its neighbor. Therefore, $c(u_i) = 4$ implies that either $c(u_{i-1}) = 2$ and $c(u_{i+1}) = 2$ or $c(u_{i-1}) = 3$ and $c(u_{i+1}) = 3$. But then, by replacing color 4 with color 3 (for each u_i, when $c(u_i) = 4$ and $c(u_{i-1}) = c(u_{i+1}) = 2$) or color 2 (for each u_i, when $c(u_i) = 4$ and $c(u_{i-1}) = c(u_{i+1}) = 3$), we infer that $\chi_\rho(\mathrm{FSSD}_m(C_n)) \leq 3$, a contradiction to Proposition 2.8. Thus, $\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) \geq 5$.

In order to show that $\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) \leq 5$, we form a 5-packing coloring of considered graph. First, color all vertices u_i, $1 \leq i \leq n-1$, using the following pattern of colors: $2,3,2,3,\ldots,2,3$. Then color each vertex v_i, $1 \leq i \leq n-1$, with the color assigned to u_i, vertex u_n with color 4 and vertex v_n with color 5. The remaining vertices color by 1. This is a 5-packing coloring of a given graph and hence $\chi_\rho(\mathrm{FSSD}_m(S^{'}(C_n))) = 5$. □

Fig. 3 shows a graph $\mathrm{FSSD}_m(S'(C_4))$ and its packing coloring, described in the previous proof.

Proposition 3.5. *If $n \geq 3$ and $m \geq 1$, then*

$$\chi_\rho(\mathrm{FSSD}_m(S'(K_n))) = n+2.$$

Proof. First, show that $\chi_\rho(\mathrm{FSSD}_m(S'(K_n))) \leq n+2$ holds for any $n \geq 3$ and $m \geq 1$. Hence, form a $(n+2)$-packing coloring of $\mathrm{FSSD}_m(S'(K_n))$. First, color all vertices v_i, $1 \leq i \leq n$, with color 2, and all vertices u_i, $1 \leq i \leq n$, with pairwise different colors from $\{3,4,\ldots,n+2\}$. Finally, the remaining vertices of a given graph color with color 1. Clearly, such coloring is $(n+2)$-packing coloring of $\mathrm{FSSD}_m(S'(K_n))$ and thus $\chi_\rho(\mathrm{FSSD}_m(S'(K_n))) \leq n+2$ for all $n \geq 3$, $m \geq 1$.

Next, prove that $\chi_\rho(\mathrm{FSSD}_m(S'(K_n))) \geq n+2$ holds for any $n \geq 3$ and $m \geq 1$. If $n = 3$, then Proposition 3.4 implies that $\chi_\rho(\mathrm{FSSD}_m(S'(K_3))) = 5$ and we are done. Otherwise, let c be any optimal packing coloring of $\mathrm{FSSD}_m(S'(K_n))$, where $n \geq 4$. Note that c uses at most $n+2$ colors. Suppose that there exists $i \in \{1,\ldots,n\}$ such that $c(u_i) = 1$. Then, $c(u^k_{i,j}) \neq 1$ and $c(s^k_{i,j}) \neq 1$ for any $j \in \{1,\ldots,n\}$, $i \neq j$, and $k \in \{1,\ldots,m\}$. Since $m \geq 1$, c uses at least $(n-1)+(n-1)$ colors different from 1, hence $\chi_\rho(\mathrm{FSSD}_m(S'(K_n))) \geq 2n-1$. Since $2n-1 > n+2$ for any $n \geq 4$, we have a contradiction to c being an optimal packing coloring of a given graph. Therefore, $c(u_i) \neq 1$ for any $i \in \{1,\ldots,n\}$. Then, suppose that there exists $j \in \{1,\ldots,n\}$ such that $c(v_j) = 1$. Without loss of generality assume that $c(v_1) = 1$, which implies that $c(s^k_{i,1}) \neq 1$ for any $i \in \{2,3,\ldots,n\}$. Recall that also $c(u_i) \neq 1$ for any $i \in \{1,\ldots,n\}$. Since the vertices from $\{u_1, u_i, s^k_{i,1}; 2 \leq i \leq k\}$ are pairwise at distance at most 3, the only color which can be used at least twice for packing coloring of these vertices, is color 2. But since any two vertices $s^k_{i,1}$, $i \in \{2,\ldots,n\}$, and also any two vertices u_i, $i \in \{1,\ldots,n\}$ are pairwise at distance 2, c assigns a color 2 to at most two of the mentioned vertices. Therefore, c uses at least $(n-1)+(n-1)+1$ colors (at least $n-1$ colors for the neighbors of v_1, beside that also $n-1$ additional colors for the vertices u_i and color 1), what is more than $n+2$ for any $n \geq 4$, a contradiction. Hence, $c(v_i) \neq 1$ for any $i \in \{1,\ldots,n\}$. Without loss of generality we may assume that c assigns to all vertices from $V(\mathrm{FSSD}_m(S'(K_n))) \setminus \{u_i, v_i; 1 \leq i \leq n\}$ a

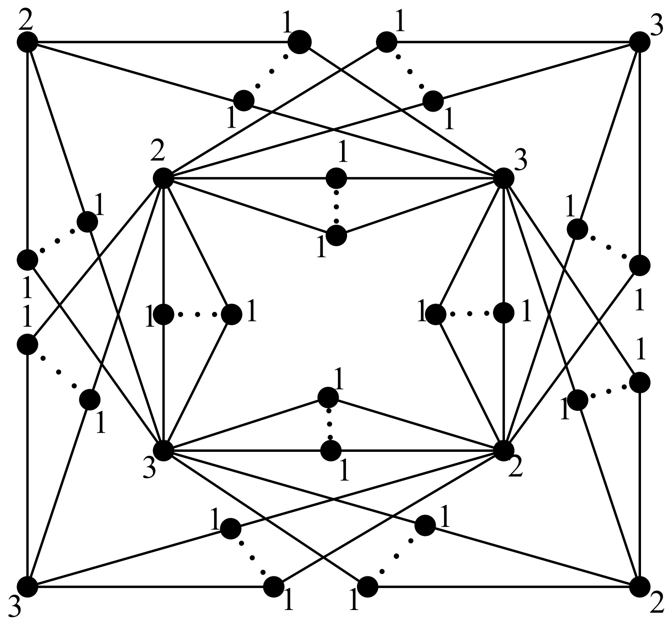

Figure 3: $FSSD_m(S'(C_4))$ and a packing coloring of this graph

color 1. Note that any two vertices from $A = \{u_i, v_i; i = 1, \ldots, n\}$ are at distance at most 4, which implies $|c^{-1}(i) \cap A| \leq 1$ for all $i \geq 4$. If c assigns colors 2 and 3 to two vertices from $\{u_i; 1, \ldots, n\}$, then at most one vertex from $\{v_i; 1, \ldots, n\}$ receive color 2 and at most one receive a color 3. Using Corollary 2.4 and the fact that $n \geq 4$ we infer that c uses at least $n+3$ colors, a contradiction. If $c(u_i) \neq 2$ or $c(u_i) \neq 3$ for all $i \in \{1, \ldots, n\}$, then all vertices v_i (can) receive a color 2 respectively 3, which yields that c uses at least $n+2$ colors ($n+1$ colors for the vertices $u_1, u_2, \ldots, u_n$ and a color 2 resp. 3). This completes the proof. □

4. Concluding remarks

It is well known that some operations or only local changes of a given graph, can change its packing chromatic number. While the packing chromatic number of a subdivision of a given graph has been studied in a number of papers, we consider the operation of finite super subdivisions. Since some authors considered subdivisions of graphs in relation to the concept of so called $S = (s_1, s_2, \ldots, s_k)$-packing coloring (see e.g. [14]), it would be interesting to consider such coloring and the influence of the operation of finite super subdivisions on it.

There are some additional open problems about finite super subdivisions of graphs that follows directly from our work. Namely, as we mentioned, we have found only two graphs G such that $\chi_\rho(\text{FSSD}_1(G)) < \chi_\rho(\text{FSSD}_2(G))$. Therefore, an open problem is to determine all graphs G such that $\chi_\rho(\text{FSSD}_1(G)) < \chi_\rho(\text{FSSD}_2(G))$. In addition, we have not found any graph G with the property that $\chi_\rho(\text{FSSD}_m(G)) < \chi_\rho(\text{FSSD}_{m+1}(G))$ for any $m \geq 2$, and it would be interesting to know whether there exists any such graph G.

Another natural problem that arises from Theorem 3.3, is to determine the exact values of packing chromatic numbers of graphs $\text{FSSD}_m(C_n \star P_p)$. We propose also a problem of providing the packing chromatic number of finite super subdivisions of some other classes of graphs (not necessarily neighborhood corona graphs).

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