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Bézier Variant of the Szász-Durrmeyer Type Operators Based on the Poisson-Charlier Polynomials

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Abstract. In the present paper we introduce the Bézier variant of the Szász-Durrmeyer type operators, involving the Poisson-Charlier polynomials. Our study focuses on a direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness and the rate of convergence for differential functions whose derivatives are of bounded variation.

1. Introduction

In 1912 Bernstein [5] presented for any real-valued function $f : [0,1] \to \mathbb{R}$ the well known linear positive operators

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

as a useful and interesting tool for the proof of the Weierstrass approximation theorem. Thanks to some important properties as uniform approximation, shape preservation and variation diminishing, Bernstein polynomials have opened a new era in the approximation theory. These polynomials together with Bézier curves play an important role in computer aided geometric design, as well as in other areas of computer science. Powerful algorithms for their construction and visualization are available in the literature. They are used for the design of curves and could be taken as starting point for several generalizations. Among other polynomials with influence in applied mathematics, we recall here the class of orthogonal polynomials. Although the orthogonal polynomials are of particular importance in applied mathematics, they appear quite rarely in the approximation process by linear and positive operators. As we can see in [20], one example in this sense could be the operators constructed by Cheney and Sharma for a fixed $t \leq 0$, $f \in C[0,1]$ and $x \in [0,1)$

$$P_n(f;x) = (1-x)^{n+1} \cdot e^{\frac{tx}{1-x}} \sum_{k=0}^{\infty} L_k^{(n)}(t) x^k f\left(\tfrac{k}{k+n}\right),$$

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where $L_k^{(n)}(t)$ denotes the Laguerre polynomials. It is know that (see [18])

$$L_k^{(n)}(t) = \sum_{j=0}^{k} \binom{n+k}{k-j} \frac{(-t)^j}{j!}.$$

Also in [20], Varma and Taşdelen proposed for research a problem which requested to find a generalization of the Szász operators [19], involving the orthogonal polynomials. As a solution of the recalled problem, they considered the linear positive operators based on the Poisson-Charlier polynomials [18], with the generating function of the form

$$e^{-t}\left(1+\frac{t}{a}\right)^{u} = \sum_{k=0}^{\infty} C_k^{(a)}(u) \cdot \frac{t^k}{k!}, \quad |t| < a, \tag{1}$$

and the explicit representation

$$C_k^{(a)}(u) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} a^{-\nu} (u)_\nu,$$

where $(m)_r$ denote the rising factorial given by $(m)_0 = 1$, $(m)_r = m(m+1)\cdots(m+r-1)$, for $r \geq 1$. We note that for $a > 0$ and $u \geq 0$ the Poisson-Charlier polynomials are positive. However, depending on the context, the generating function of the Poisson-Charlier polynomials could also be

$$e^{t}\left(1-\frac{t}{a}\right)^{u} = \sum_{k=0}^{\infty} C_k^{(a)}(u) \cdot \frac{t^k}{k!}, \quad |t| < a. \tag{2}$$

The generating function written at (2) can be obtained from the relation (1) by changing the variable $t := -t$. Taking into account the definitions of the Laguerre, as well as of the Poisson-Charlier polynomials, in [12] is presented an interesting relationship between these two ortoghonal polynomials, given by

$$C_n^{(a)}(u) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} a^{-\nu} (u)_\nu = a^{-n} \cdot n! \cdot L_n^{(u-n)}(a), \tag{3}$$

which means that the Poisson-Charlier polynomials are a parametric reshuffling of the classical Laguerre polynomials. More detalis about these orthogonal polynomials can be found in [18] and [12]. Based on this relationship (3), the unknown reviewer of the present paper suggested to present some advantages (if there exists), concerning the use of the Poisson-Charlier polynomials instead of the classical Laguerre polynomials. At this moment, in literature we may find many articles that have the Poisson-Charlier polynomials as a central study issue at the expense of the classical Laguerre polynomials, but we cannot present some advantages of their use. If there exists, we let an open door for other researchers to bring them to the light. We return to our problem, so the explicit solution of the problem proposed by Varma and Taşdelen was to introduce the following positive linear operators

$$L_n(f;x,a) = \sum_{k=0}^{\infty} W_{n,k}^{a}(x) f\left(\frac{k}{n}\right) = e^{-1}\left(1-\tfrac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} f\left(\frac{k}{n}\right), \tag{4}$$

for $a > 1$ and $x \in [0,\infty)$. In the case when $a \to \infty$ and taking $x - \frac{1}{n}$ instead of x, these operators (4) become the well-known Szász operators [19]. Based on the operators (4), for $\gamma > 0$ and $f \in C_\gamma[0,\infty) = \{f \in C[0,\infty) \,|\, f(t) = O(t^\gamma) \text{ as } t \to \infty\}$ Kajla and Agrawal [11] introduced the following Durrmeyer type modification defined as follows

$$S_{n,a}(f;x) = \sum_{k=0}^{\infty} W_{n,k}^{a}(x) \frac{1}{B(k+1,n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} f(t)dt, \tag{5}$$

where $n > \gamma$, $a > 1$ and $B(k+1, n)$ is the beta function given by

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}, \text{ for } x, y > 0.$$

Also in [11], some interesting and nice properties for the presented operators (5) are established. The Bézier variant of Bernstein-Durrmeyer operators was introduced by Zeng and Chen in [23]. Srivastava and Gupta [14] got the rate of convergence for the Bézier variant of the Bleimann Butzer and Hahn operators for functions with bounded variation. In 2007, Guo et al. [9] studied Baskakov-Bézier operators and established direct, inverse and equivalence approximation theorems with the help of Ditzian-Totik modulus of smoothness. Very recently, Agrawal et al. [3] introduced mixed hybrid operators for which they got direct results and the rate of convergence for differential functions whose derivatives are of bounded variation. Many other interesting Bézier type operators were studied by several researchers, and the reader is invited to see [1], [2], [4], [6], [8], [10], [13], [15], [16], [17], [21], [22]. For $\theta \geq 1$ we consider the Bézier variant of the operators $S_{n,a}f$, defined by

$$S^a_{n,\theta}(f;x) = \sum_{k=0}^{\infty} Q^{(\theta)}_{n,k,a}(x)\frac{1}{B(k+1,n)}\int_0^\infty \frac{t^k}{(1+t)^{n+k+1}}f(t)dt, \tag{6}$$

where $Q^{(\theta)}_{n,k,a}(x) = \left(J^a_{n,k}(x)\right)^\theta - \left(J^a_{n,k+1}(x)\right)^\theta$, with $J^a_{n,k}(x) = \sum_{v=k}^{\infty} W^a_{n,v}(x)$. Alternatively, we may rewrite the operators (6) as

$$S^a_{n,\theta}(f;x) = \int_0^\infty P_{n,\theta,a}(x,t)f(t)dt, \quad x \in [0,\infty), \tag{7}$$

where

$$P_{n,\theta,a}(x,t) = \sum_{k=0}^{\infty} Q^{(\theta)}_{n,k,a}(x)\frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}}.$$

For $\theta = 1$, the operators $S^a_{n,\theta}f$ reduce to the operators $S_{n,a}f$.

The aim of this paper is to introduce the Bézier variant (6) of the Szász-Durrmeyer type operators, involving the Poisson-Charlier polynomials. Our further study focuses on a direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness and the rate of convergence for differential functions whose derivatives are of bounded variation on every finite subinterval of $(0,\infty)$, for the presented operators (6).

2. Auxiliary results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this paper, C denotes a positive constant independent of n and x, not necessarily the same at each occurrence. For these new operators (6) we establish some auxiliary results. The monomials $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ called test functions play an important role in uniform approximation by linear positive operators. We recall some results established in [11].

Lemma 2.1. [11] *For any $n \in \mathbb{N}$, $n > 2$, the images of test functions by Durrmeyer type operators* (5) *are given by*

$$S_{n,a}(e_0;x) = 1, \quad S_{n,a}(e_1;x) = \frac{nx+2}{n-1}, \quad S_{n,a}(e_2;x) = \frac{1}{(n-2)(n-1)}\left(n^2x^2 + nx\left(6 + \frac{1}{a-1}\right) + 7\right).$$

Lemma 2.2. [11] *For any $n \in \mathbb{N}$, $n > 2$, the computation of the central moments up to the second order for Durrmeyer type operators* (5) *is given by*

$$S_{n,a}(e_1 - x; x) = \frac{x+2}{n-1}, \quad S_{n,a}\left((e_2 - x)^2; x\right) = \frac{1}{(n-2)(n-1)}\left(nx\left(2 + \frac{1}{a-1}\right) + x^2(n+2) + 8x + 7\right).$$

Lemma 2.3. *Let f be a real-valued function continuous and bounded on $[0, \infty)$, with $\|f\| = \sup_{x\in[0,+\infty)} |f(x)|$, then $|S_{n,a}(f)| \leq \|f\|$.*

Lemma 2.4. *Let f be a real-valued function continuous and bounded on $[0, \infty)$, then $|S^a_{n,\theta}(f)| \leq \theta\|f\|$.*

Proof. Applying the well known property $|a^\alpha - b^\alpha| \leq \alpha|a - b|$, with $0 \leq a, b \leq 1$, $\alpha \geq 1$ and the definition of $Q^{(\theta)}_{n,k,a}(x)$, we have

$$0 < \left(J^a_{n,k}(x)\right)^\theta - \left(J^a_{n,k+1}(x)\right)^\theta \leq \theta(J^a_{n,k}(x) - J^a_{n,k+1}(x)) = \theta W^a_{n,k}(x). \tag{8}$$

Hence, from the definition of $S^a_{n,\theta}(f)$ operators and Lemma 2.3, we get

$$|S^a_{n,\theta}(f)| \leq \theta|S_{n,a}(f)| \leq \theta\|f\|.$$

□

Lemma 2.5. *Let $x \in (0, \infty)$, then for $\theta \geq 1$ and sufficiently large n, we have*

i) $\zeta_{n,\theta,a}(x, y) = \int_0^y P_{n,\theta,a}(x, t)dt \leq \frac{\theta\lambda(a)}{n}\frac{\phi^2(x)}{(x-y)^2}, \quad 0 \leq y < x,$

ii) $1 - \zeta_{n,\theta,a}(x, z) = \int_z^\infty P_{n,\theta,a}(x, t)dt \leq \frac{\theta\lambda(a)}{n}\frac{\phi^2(x)}{(z-x)^2}, \quad x < z < \infty,$

where $\lambda(a)$ is a positive constant depending on a.

Proof. *i*) Using Lemma 2.4 and (Eq. (2.2) from [11]), we get

$$\zeta_{n,\theta,a}(x, y) = \int_0^y P_{n,\theta,a}(x, t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 P_{n,\theta,a}(x, t)dt$$
$$\leq S^a_{n,\theta}((t-x)^2; x)\,(x-y)^{-2} \leq \theta S_{n,a}((t-x)^2; x)(x-y)^{-2} \leq \frac{\theta\lambda(a)}{n}\frac{\phi^2(x)}{(x-y)^2}, \quad 0 \leq y < x.$$

ii) Analogously could be proved the second relation. □

In order to present our further results, we recall from [7] the definitions of the Ditizian-Totik modulus of smoothness. Let $\phi(x) = \sqrt{x(1+x)}$, and $0 \leq \eta \leq 1$, then

$$\omega_{\phi^\eta}(f, t) = \sup_{0<h\leq t}\ \sup_{x\pm h\phi^\eta(x)/2\geq 0} \left\{\left|f\left(x + \frac{h\phi^\eta(x)}{2}\right) - f\left(x - \frac{h\phi^\eta(x)}{2}\right)\right|\right\},$$

and the appropriate Peetre's K-functional is defined by

$$\overline{K}_{\phi^\eta}(f, t) = \inf_{g\in V_\eta}\{\|f - g\| + t\|\phi^\eta g'\|\}, \quad t > 0,$$

where $V_\eta = \{g \in C[0, +\infty) \,|\, g \in AC_{\text{loc}}[0, +\infty), \|\phi^\eta g'\| < \infty\}$. Based on the theory from [7], it is well known that $\overline{K}_{\phi^\eta}(f, t) \sim \omega_{\phi^\eta}(f, t)$, which means that there exists a constant $M > 0$, such that

$$M^{-1}\omega_{\phi^\eta}(f, t) \leq \overline{K}_{\phi^\eta}(f, t) \leq M\omega_{\phi^\eta}(f, t). \tag{9}$$

3. Direct Theorem

Now we are able to prove the following direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness.

Theorem 3.1. *Let $f \in C_B[0,\infty)$, then for any $x \in [0,+\infty)$ yields*

$$\left|S^a_{n,\theta}(f;x) - f(x)\right| \leq C\omega_{\phi^\eta}\left(f, \frac{\phi^{1-\eta}(x)}{\sqrt{n}}\right), \tag{10}$$

where C is an absolute constant.

Proof. Using the definition of $\overline{K}_{\phi^\eta}(f,t)$, for fixed n, x, η we can choose $g = g_{n,x,\eta} \in V_\eta$, such that

$$\|f-g\| + \frac{\phi^{1-\eta}(x)}{\sqrt{n}}\|\phi^\eta g'\| \leq 2\overline{K}_{\phi^\eta}\left(f, \frac{\phi^{1-\eta}(x)}{\sqrt{n}}\right). \tag{11}$$

Since $S^a_{n,\theta}(e_0;x) = 1$, we may write

$$\left|S^a_{n,\theta}(f;x) - f(x)\right| \leq 2\|f-g\| + |S^a_{n,\theta}(g;x) - g(x)|. \tag{12}$$

Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we get

$$\left|S^a_{n,\theta}(g;x) - g(x)\right| = \left|S^a_{n,\theta}\left(\int_x^t g'(u)du;x\right)\right| \leq S^a_{n,\theta}\left|\left(\int_x^t g'(u)du;x\right)\right| \leq \left\|\phi^\eta g'\right\| S^a_{n,\theta}\left(\left|\int_x^t \frac{du}{\phi^\eta(u)}\right|;x\right). \tag{13}$$

Applying the Hölder's inequality and the relation $|a+b|^p \leq |a|^p + |b|^p$, for $a, b \in \mathbb{R}$, $0 \leq p \leq 1$, we have

$$\begin{aligned} \left|\int_x^t \frac{du}{\phi^\eta(u)}\right| &\leq |t-x|^{1-\eta}\left|\int_x^t \frac{du}{\phi(u)}\right|^\eta \leq |t-x|^{1-\eta}\left|\int_x^t \frac{du}{\phi(u)}\right|^\eta\left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+t}}\right)^\eta \\ &\leq |t-x|\frac{2^\eta}{x^{\eta/2}}\left((1+t)^{-\eta/2} + (1+x)^{-\eta/2}\right). \end{aligned} \tag{14}$$

Combining (13) and (14), we get

$$\left|S^a_{n,\theta}(g;x) - g(x)\right| \leq 2^\eta\|\phi^\eta g'\|\left\{\phi^{-\eta}(x)S^a_{n,\theta}(|t-x|;x) + x^{-\eta/2}S^a_{n,\theta}\left(\frac{|t-x|}{(1+t)^{\eta/2}};x\right)\right\}. \tag{15}$$

In order to estimate the second term on the right hand side of (15), we note that for any $m \geq 0$, the inequality

$$S^a_{n,\theta}((1+t)^{-m};x) \leq C_{m,a}(1+x)^{-m} \tag{16}$$

holds, where $C_{m,a}$ is a constant that depends on m and a. To prove (16), we remark that the presented inequality holds for $m = 0$. For $m > 0$, we may write

$$\begin{aligned} S^a_{n,\theta}((1+t)^{-m};x) &= \sum_{k=0}^\infty \frac{Q^{(\theta)}_{n,k}(x)}{B(k+1,n)}\int_0^\infty \frac{t^k}{(1+t)^{n+k+m+1}}dt = \sum_{k=0}^\infty \frac{Q^{(\theta)}_{n,k}(x)\Gamma(n+k+1)\Gamma(n+m)}{\Gamma(n+m+k+1)\Gamma(n)} \\ &\leq \theta(1+x)^{-m}\sum_{k=0}^\infty e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{C^{(a)}_k(-(a-1)nx)}{k!}\cdot\frac{x^k}{(1+x)^{n+k-m}}\frac{\Gamma(n+k+1)\Gamma(n+m)}{\Gamma(n+m+k+1)\Gamma(n)}. \end{aligned} \tag{17}$$

By using the ratio test, it follows that for each $x > 0$, the series on the right hand side (17) is convergent. Lemma 2.4, the Cauchy-Schwarz inequality and the estimation presented in [11] (equation (2.2)), lead us to

$$S^a_{n,\theta}(|t-x|;x) \leq \left(S^a_{n,\theta}((e_1-x)^2;x)\right)^{1/2} \leq \frac{\sqrt{\theta\lambda(a)}\phi(x)}{\sqrt{n}}. \tag{18}$$

Using again the same estimation presented in [11] (equation (2.2)), form the relation (16) we get

$$S^a_{n,\theta}\left(\frac{|t-x|}{(1+t)^{\eta/2}};x\right) \le \theta S_{n,a}\left(\frac{|t-x|}{(1+t)^{\eta/2}};x\right) \le \theta\left(S_{n,a}((e_1-x)^2;x)\right)^{1/2}\left(S_{n,a}((1+t)^{-\eta};x)\right)^{1/2} \le C_1\frac{\sqrt{\theta\lambda(a)}\phi(x)}{\sqrt{n}}(1+x)^{-\eta/2}. \tag{19}$$

Replacing the inequalities (18) and (19) in the relation (15), it follows

$$\left|S^a_{n,\theta}(g;x)-g(x)\right| \le C\|\phi^\eta g'\|\frac{\phi^{1-\eta}(x)}{\sqrt{n}}. \tag{20}$$

Combining the relations (9), (11), (12) and (20), we get the desired result (10). □

4. Rate of Convergence

Let $f\in DBV_\gamma(0,\infty)$, $\gamma\ge 0$, be the class of differentiable functions defined on $(0,\infty)$, whose derivatives f' are of bounded variation on every finite subinterval of $(0,\infty)$ and $|f(t)|\le Mt^\gamma$, for all $t>0$ and some $M>0$. The functions $f\in DBV_\gamma(0,\infty)$ could be represented

$$f(x)=\int_0^x g(t)dt+f(0),$$

where g is a function of bounded variation on each finite subinterval of $(0,\infty)$.

Theorem 4.1. *Let $f\in DBV_\gamma(0,\infty)$, $\theta\ge 1$ and $\bigvee_a^b(f'_x)$ be the total variation of f'_x on $[a,b]\subset(0,\infty)$. Then, for every $x\in(0,\infty)$ and sufficiently large n, we have*

$$\left|S^a_{n,\theta}(f;x)-f(x)\right| \le \frac{\sqrt{\theta}}{\theta+1}\left|f'(x+)+\theta f'(x-)\right|\sqrt{\frac{\lambda(a)}{n}}\phi(x)+\sqrt{\frac{\lambda(a)}{n}}\,\phi(x)\frac{\theta^{3/2}}{\theta+1}\left|f'(x+)-f'(x-)\right| + \frac{\theta\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}\,]}\bigvee_{x-x/k}^{x}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}(f'_x)+\frac{\theta\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}\,]}\bigvee_{x}^{x+x/k}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}}(f'_x),$$

where $\lambda(a)$ is a positive constant depending on a and the auxiliary function f'_x is defined by

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\le t<x\\ 0, & t=x\\ f'(t)-f'(x+) & x<t\le 1.\end{cases}$$

Proof. Since $\int_0^\infty P_{n,\theta,a}(x,t)dt=S^a_{n,\theta}(e_0;x)=1$, we can write

$$S^a_{n,\theta}(f;x)-f(x)=\int_0^\infty (f(t)-f(x))\,P_{n,\theta,a}(x,t)dt=\int_0^\infty\left(\int_x^t f'(u)du\right)P_{n,\theta,a}(x,t)dt. \tag{21}$$

Using the definition of the function f'_x, for any $f\in DBV_\gamma(0,\infty)$, it follows

$$f'(t)=\frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)+f'_x(t) + \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\left(\mathrm{sgn}(t-x)+\frac{\theta-1}{\theta+1}\right)+\delta_x(t)\Big(f'(x)-\frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\Big), \tag{22}$$

where

$$\delta_x(t) = \begin{cases} 1 \ , \ x = t \\ 0 \ , \ x \neq t. \end{cases}$$

It is clear that

$$\int_0^{\infty} P_{n,\theta,a}(x,t) \int_x^t \Big(f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\Big)\delta_x(u)du dt = 0.$$

Using the definition of the operators (7), then simple computations lead us to

$$\begin{aligned} E_1 &= \int_0^{\infty}\Big(\int_x^t \frac{1}{\theta+1}\Big(f'(x+) + \theta f'(x-)\Big)du\Big)P_{n,\theta,a}(x,t)dt \qquad (23) \\ &= \frac{1}{\theta+1}\Big|f'(x+) + \theta f'(x-)\Big| \int_0^{\infty} |t-x| P_{n,\theta,a}(x,t)dt \leq \frac{1}{\theta+1}\Big(f'(x+) + \theta f'(x-)\Big)\Big(S^a_{n,\theta}((e_1-x)^2;x)\Big)^{1/2} \\ &\leq \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+) + \theta f'(x-)\Big|\sqrt{\frac{\lambda(a)}{n}}\phi(x) \end{aligned}$$

and

$$\begin{aligned} E_2 &= \int_0^{\infty}\Big(\int_x^t \frac{1}{2}\Big(f'(x+) - f'(x-)\Big)\Big(\mathrm{sgn}(u-x) + \frac{\theta-1}{\theta+1}\Big)du\Big)P_{n,\theta,a}(x,t)dt \qquad (24) \\ &\leq \frac{\theta}{\theta+1}\Big|f'(x+) - f'(x-)\Big| \int_0^{\infty} |t-x|\, P_{n,\theta,a}(x,t)dt = \frac{\theta}{\theta+1}\Big|f'(x+) - f'(x-)\Big| S^a_{n,\theta}\,(|t-x|\,;x) \\ &\leq \frac{\theta}{\theta+1}\Big|f'(x+) - f'(x-)\Big|\Big(S^a_{n,\theta}\Big((e_1-x)^2;x\Big)\Big)^{1/2} \leq \frac{\theta^{3/2}}{\theta+1}\Big|f'(x+) - f'(x-)\Big|\sqrt{\frac{\lambda(a)}{n}}\phi(x). \end{aligned}$$

Involving the relations (21)–(24), we obtain the following estimate

$$\begin{aligned} \Big|S^a_{n,\theta}(f;x) - f(x)\Big| &\leq \Big|A_{n,\theta,a}(f'_x,x) + B_{n,\theta,a}(f'_x,x)\Big| + \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+) + \theta f'(x-)\Big|\sqrt{\frac{\lambda(a)}{n}}\phi(x) \qquad (25) \\ &\quad + \frac{\theta^{3/2}}{\theta+1}\Big|f'(x+) - f'(x-)\Big|\sqrt{\frac{\lambda(a)}{n}}\,\phi(x), \end{aligned}$$

where

$$A_{n,\theta,a}(f'_x,x) = \int_0^x \Big(\int_x^t f'_x(u)du\Big)P_{n,\theta,a}(x,t)dt \quad \text{and} \quad B_{n,\theta,a}(f'_x,x) = \int_x^{\infty}\Big(\int_x^t f'_x(u)du\Big)P_{n,\theta,a}(x,t)dt.$$

For a complete proof of the theorem, it remains to estimate the terms $A_{n,\theta,a}(f'_x,x)$ and $B_{n,\theta,a}(f'_x,x)$. Since $\int_a^b d_t\zeta_{n,\theta,a}(x,t) \leq 1$, for all $[a,b] \subseteq (0,\infty)$, using integration by parts and applying Lemma 2.5 with $y = x - (x/\sqrt{n})$, it follows

$$\begin{aligned} \Big|A_{n,\theta,a}(f'_x,x)\Big| &= \Big|\int_0^x \Big(\int_x^t f'_x(u)du\Big)d_t\zeta_{n,\theta,a}(x,t)\Big| = \Big|\int_0^x \zeta_{n,\theta,a}(x,t)f'_x(t)dt\Big| \leq \Big(\int_0^y + \int_y^x\Big)\Big|f'_x(t)\Big|\Big|\zeta_{n,\theta,a}(x,t)\Big|dt \\ &\leq \theta\frac{\lambda(a)\phi^2(x)}{n}\int_0^y \bigvee_t^x (f'_x)(x-t)^{-2}dt + \int_y^x \bigvee_t^x (f'_x)dt \\ &\leq \theta\frac{\lambda(a)\phi^2(x)}{n}\int_0^y \bigvee_t^x (f'_x)(x-t)^{-2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x (f'_x). \end{aligned}$$

Taking $u = x/(x-t)$ into account, we get

$$\theta\frac{\lambda(a)\phi^2(x)}{n}\int_0^{x-x/\sqrt{n}}(x-t)^{-2}\bigvee_t^x(f'_x)dt = \theta\frac{\lambda(a)(1+x)}{n}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^x(f'_x)du$$
$$\leq \theta\frac{\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-x/u}^x(f'_x)du \leq \theta\frac{\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^x(f'_x).$$

Hence, we reach the following estimation

$$\left|A_{n,\theta,a}(f'_x,x)\right| \leq \theta\frac{\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^x(f'_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x(f'_x). \tag{26}$$

Using again the integration by parts and applying Lemma 2.5 with $z = x + x/\sqrt{n}$, it follows

$$\begin{aligned}
|B_{n,\theta,a}(f'_x,x)| &= \left|\int_x^\infty\left(\int_x^t f'_x(u)du\right)P_{n,\theta,a}(x,t)dt\right| \qquad (27)\\
&= \left|\int_x^z\left(\int_x^t f'_x(u)du\right)d_t(1-\zeta_{n,\theta,a}(x,t)) + \int_z^\infty\left(\int_x^t f'_x(u)du\right)d_t(1-\zeta_{n,\theta,a}(x,t))\right|\\
&= \Bigg|\left[\left(\int_x^t f'_x(u)du\right)(1-\zeta_{n,\theta,a}(x,t))\right]_x^z - \int_x^z f'_x(t)(1-\zeta_{n,\theta,a}(x,t))dt\\
&\quad + \int_z^\infty\left(\int_x^t f'_x(u)du\right)d_t(1-\zeta_{n,\theta,a}(x,t))\Bigg|\\
&= \Bigg|\left(\int_x^z f'_x(u)du\right)(1-\zeta_{n,\theta,a}(x,z)) - \int_x^z f'_x(t)(1-\zeta_{n,\theta,a}(x,t))dt\\
&\quad + \left[\left(\int_x^t f'_x(u)du\right)(1-\zeta_{n,\theta,a}(x,t))\right]_z^\infty - \int_z^\infty f'_x(t)(1-\zeta_{n,\theta,a}(x,t))dt\Bigg|\\
&= \left|\int_x^z f'_x(t)(1-\zeta_{n,\theta,a}(x,t))dt + \int_z^\infty f'_x(t)(1-\zeta_{n,\theta,a}(x,t))dt\right|\\
&< \theta\frac{\lambda(a)\phi^2(x)}{n}\int_z^\infty\bigvee_x^t(f')_x(t-x)^{-2}dt + \int_x^z\bigvee_x^t(f'_x)dt\\
&\leq \theta\frac{\lambda(a)\phi^2(x)}{n}\int_{x+x/\sqrt{n}}^\infty\bigvee_x^t(f'_x)(t-x)^{-2}dt + \frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f'_x).
\end{aligned}$$

Taking $u = x/(t-x)$ into account, we get

$$\theta\frac{\lambda(a)\phi^2(x)}{n}\int_{x+x/\sqrt{n}}^\infty\bigvee_x^t(f'_x)(t-x)^{-2}dt = \theta\frac{\lambda(a)\phi^2(x)}{xn}\int_0^{\sqrt{n}}\bigvee_x^{x+x/u}(f'_x)du \tag{28}$$
$$\leq \theta\frac{\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_x^{x+x/u}(f'_x)du \leq \theta\frac{\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+x/k}(f'_x).$$

Using the relations (27)–(28), we get the following estimation

$$|B_{n,\theta,a}(f'_x,x)| \leq \theta\frac{\lambda(a)(1+x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+x/k}(f'_x) + \frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f'_x). \tag{29}$$

The relations (25), (26) and (29) lead us to the desired result. □

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