Filomat 34:10 (2020), 3251–3264
https://doi.org/10.2298/FIL2010251K

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost Complete Convergence for the Sequence of Approximate Solutions in Linear Calibration Problem With α−Mixing Random Data

Samia Khalfoune[a], Halima Zerouati[a]

[a]*Laboratoire de Mathématiques Appliquées, Faculté des Sciences Exactes, Université de Bejaia, Bejaia, Algérie*

Abstract. In this work, we propose a stochastic method which gives an estimated solution for a linear calibration problem with α−mixing random data. We establish exponential inequalities of Fuk Nagaev type, for the probability of the distance between the approximate solutions and the exact one. Furthermore, we build a confidence domain for the so mentioned exact solution. To check the validity of our results, a numerical example is proposed.

1. Introduction

Calibration is a classical problem which appears often in a regression setup under fixed design. In calibration, also called inverse regression, interest centers on estimating an unknown value of an exogenous variable x, which corresponds to an observed value of an endogenous variable y. Some influential and relevant works on calibration have been published. In 1991, a review of statistical calibration is given by Osborne [17]. In 1992, Sarndal [20] and Deville [6] published respectively a book (which became a reference for the survey statistician) and the article entitled "Calibration Estimators in Survey Sampling". Calibration problems arise often in the experimental sciences (Martens [15] and Frank[11]), in the field of chemometrics, (Garcia-Dorado [12]) in experimental cardiology. Experimental data are always noisy. A natural way to deal with the treatement of the errors is to consider the probabilistic framework (Aiane [2]) and random variables are used to represent noisy data. In a precedent publication, Zerouati [23] proposed a stochastic method for linear calibration problem with independent random data. One notes that, independent random data failed for modeling some phenomena. Indeed, dependent random data are more adjusted to reality. In this context, (Doukhan [7] and [8]), studied stochastic algorithms with a weakly dependence noise. In this paper, one considers a non restrictive mixing condition to characterize the dependence between the data random errors. We suppose them to be strong mixing or α−mixing (Aiane [1]). The notion of α−mixing was firstly introduced by Rosenblatt [19] and the central limit theorem has been established. The strong mixing random variables have many interests in linear processes and found many applications.

2. Preliminaries

Definition 2.1. *Let* $(\Omega, \mathrm{F}, \mathbb{P})$ *be a probability space and we note* $\mathbb{F}_{-\infty}^{k} \subset \mathbb{F}$ *(respectively,* $\mathbb{F}_{n+k}^{+\infty} \subset \mathbb{F}$*) the* σ*- algebra generated by* $e_i, i \leq k$ *(respectively by* $e_i, i \geq n+k$*).*

2010 *Mathematics Subject Classification*. 47A52, 53C38, 62L20.
Keywords. Calibration; α−mixing random data; Ill-posed problem; Almost complete convergence
Received: 21 October 2019; Revised: 20 February 2020; Accepted: 17 April 2020
Communicated by Calegro Vetro
Email addresses: khalfounesamia93@gmail.com (Samia Khalfoune), h_zerouati@yahoo.fr (Halima Zerouati)

The sequence $(e_n)_{n\in\mathbb{N}}$ ($\mathbb{N}$ is the set of nonnegative natural numbers) is said to be $\alpha-$mixing (or strongly mixing), if

$$\alpha(n) = \sup_k \sup_{A\in F_{-\infty}^k} \sup_{B\in F_{n+k}^{+\infty}} |P(A\cap B) - P(A)P(B)| \underset{n\to\infty}{\to} 0.$$

The sequence $(e_n)_{n\in\mathbb{N}^*}$ ($\mathbb{N}^* = \mathbb{N}\backslash\{0\}$) is said to be algebraically $\alpha-$mixing with rate $b > 1$ if

$$\exists\, C > 0,\ \alpha(n) \leq Cn^{-b}.$$

We say that the sequence of random variables $(z_n)_n$ converges almost completely (*a.co.*) if

$$\forall\varepsilon > 0, \sum_{n=1}^{+\infty} P\{\|z_n\| > \varepsilon\} < +\infty.$$

The aim of this work is to propose a stochastic method which gives an estimated solution for a linear calibration problem with $\alpha-$mixing random data. We establish exponential inequalities of Fuk- Nagaev type, for the probability of the distance between the approximate solutions and the exact one. These inequalities yield almost complete convergence (*a.co*) with convergence rate of approximate solution sequence. Furthermore, we build a confidence domain for the so mentioned exact solution.

The paper is organized as follows: In section 3, the statement of the problem is described. In section 4, some known results concerning the estimation of the operator of linear calibration problem are recalled . In section 5, some new results were established by using stochastic methods. Therefore a stable solution is obtained. In section 6 the validity of our approach is illustrated by a numerical example. We finish by a conclusion.

3. Statement of the problem

A large variety of problems arising from various domains of applied sciences can be often regarded mathematically as an integral equation

$$\begin{array}{lll} A: & L^2[a,b] & \longrightarrow L^2[a,b] \\ & x & \mapsto Ax = u,\ \ u \text{ is given.} \end{array}$$

where

$$Ax(t) = \int_a^b K(t,s)\,x(s)\,ds. \tag{1}$$

The kernel K satisfies

$$\int_a^b \int_a^b K(t,s)^2\, dsdt < +\infty.$$

The operator A is compact, hence its inverse exists but it is not continuous [3].

Moreover if

$$K(t,s) = K(s,t) \tag{2}$$

A is self-adjoint

In the integral equation

$$Ax = u \tag{3}$$

A satisfies the assumption:

(H_1) A is compact, self-adjoint (and the sequence of normalized eigenfunctions ϕ_i of A corresponding to the eigenvalues λ_i different from zero form an orthonormal basis in $L^2[a,b]$).

$\langle .,. \rangle$ will denote the inner product of $L^2[a,b]$ and $\|.\|$ will be the associated norm.

Denote by u_{ex} the exact value of the unknown second member of equation (3) and by x_{ex} the element of $L^2[a,b]$ for which the existence and uniqueness is known and which satisfies

$$Ax_{ex} = u_{ex}.$$

However, in some categories of applied problems we find a class for which A is known only for some points, furthermore the second member is only known approximately. A natural way to deal with this situation is to consider the probabilistic framework.

The integral operator equation becomes a calibration problem which involves data collected in two stages. In order to solve it, we proceed in two steps. The first one consists in estimating the operator (from the first stage of data collection). Using Cuevas et al. results [4], an estimator $\widehat{A_N}$ for the underlying linear operator A, is built. Then, we estimate the second member of the equation (3). Indeed, the data from the second stage of experimentation consist of several observations of the response variable which corresponds to an unknown value of the regressor variable. The strong law of large numbers gives the empirical mean $\overline{u}$ as a natural exhaustive estimate.

Thus, the problem becomes to solve the following equation

$$\widehat{A_N}x = \overline{u} \tag{4}$$

The operator $\widehat{A_N}$ is non injective (because of finite rank), thus the equation (4) is an ill-posed problem and can not be solved by using classical methods ([9]; [22] and [21]).

In the second step, we establish the almost complete convergence of the sequence of approximate solutions obtained when solving (4). Furthermore, we specify its convergence rate and we build a confidence domain for the exact solution. These results are obtained after the establishment of our aim result, i.e. the exponential inequalities for the probability of the distance between the approximate solution and the exact one.

4. Estimation of the operator and the second member

4.1. *Construction of the estimator*

Based on the works of Cuevas et al. [4], we will use its estimator corresponding to our functional setup. The functional framework involves some difficulties for the construction of a consistent estimator.

Statistical calibration involves data collected in two stages. In the first stage, several values of an endogenous variable are observed, each corresponding to a known value of an exogenous variable. In the second stage, one or more values of the endogenous variable are observed which correspond to an unknown value of the exogenous variable.

In the first stage, we suppose that when carrying out N independent experiments, we obtain functional data which consist of the pairs

$$(x_{iN}(t), u_{iN}(t)) \in L^2[a,b] \times L^2[a,b] \ , \ i = 1, \cdots, N$$

observed according to the model

$$u_{iN}(t) = Ax_{iN}(t) + e_{iN}(t), \qquad i = 1, \cdots, N. \tag{5}$$

The data e_{iN} are independent and identically distributed functional random variables.

For each i, x_{iN} is a known fixed value of independent variable and u_{iN} is the corresponding observed response.

Moreover, we suppose that

For $i = 1, \cdots, N$ we have

$$\mathbb{E}\, e_{iN}(t) = 0 \quad , \quad \mathbb{E}\, \|e_{iN}\|^2 = \sigma^2$$

and there is $C > 0$ such that the random variables e_{iN} fulfill Cramer's assumption

$$\mathbb{E}\, \|e_{iN}\|^p \leq \frac{p!}{2}\sigma^2 C^{p-2} \quad , \quad p \geq 2 \tag{6}$$

where $\mathbb{E}$ represents the mathematical expectation.

An estimator $\widehat{A_N}(x_{1N}, x_{2N}, \cdots, x_{NN}, u_{1N}, u_{2N}, \cdots, u_{NN})$ which verifies the strong consistency to A. is defined (see[4]) as follows:

It is assumed (see [4]) that for every N, the set of indices $\{1, ..., N\}$ can be partitioned into $m = m_N$ subsets $J_{1N}, J_{2N}, ..., J_{mN}$ with card $J_{iN} = k_{iN}$ and $k_N = \min_i k_{iN} \to \infty$, $m = m_N \to \infty$ such that for $i = 1, ..., m$ the averages

$$\overline{x}i = \overline{x}_{iN} = \frac{1}{k_{iN}} \sum_{j \in \mathbf{J}_{in}} x_{jN}$$

are linearly independent. We define

$$\overline{u}_i = \overline{u}_{iN} = \frac{1}{k_{iN}} \sum_{j \in \mathbf{J}_{in}} u_{jN}$$

$$\overline{e}_i = \overline{e}_{iN} = \frac{1}{k_{iN}} \sum_{j \in \mathbf{J}_{iN}} e_{jN}.$$

The closed linear spans

$$\mathbb{H}_N = \mathcal{L}(x_{1N}, x_{2N}, ..., x_{mN})$$

fulfill

$$(H_2) \qquad \mathbb{H}_N \subset \mathbb{H}_{N+1} \text{ and } \mathcal{L}\left(\underset{N}{\cup}\mathbb{H}_N\right) = L^2[a, b]$$

(H_3) The smallest eigenvalue of the matrix Q with elements $\left\langle \overline{x}_{iN}, \overline{x}_{jN} \right\rangle$,
$1 \leq i, j \leq m$ is bounded below by a constant $c_0 > 0$ for all N.

Let us consider

$$\overline{u}_i = A\overline{x}_i + \overline{e}_i$$

We can choose (see [4]) a conjugate system of functions $x_1^*, ..., x_m^*$ such that

$$\langle x_r^*, \overline{x}_i \rangle = \delta_{ir}$$

(where δ_{ir} is Kronecker's symbol) i.e.,

$$x_r^* = \sum_{j=1}^{m} \left(Q^{-1}\right)_{rj} \overline{x}_j$$

then $\widehat{A_N}$ the linear operator

$$\widehat{A_N}x(t) = \int_a^b K_N(s,t)\, x(s)\, ds$$

associated with the kernel

$$K_N(s,t) = \sum_{i=1}^{m} x_i^*(s)\ \overline{u_i}(t).$$

Assume (H_2), (H_3) together with:

$$\lim_{N\to\infty} \min_{1\le i\le m} \frac{k_i}{m_N \log m_N} = \infty.$$

Then, the estimator $\widehat{A_N}$ satisfies (see [4], theorem 1, page 290)

$$\lim_{N\to\infty} \left\| \widehat{A_N} - A \right\| = 0 \text{ completely.}$$

4.2. Estimation of the second member

The second member of $Ax = u$ is observed with $\alpha-$mixing random errors and the data from the second stage of experimentation consist of n observations of the response variable, $u_{N+1}, ..., u_{N+n}$, assumed to be related to a single unknown regressor value x_{ex}. To simulate the process of the error mesurements on u_{ex} we set

$$\begin{aligned} u_{N+j} &= u_{ex} + e_{N+j} \quad j = 1, ..., n \\ &= Ax_{ex} + e_{N+j} \quad j = 1, ..., n \end{aligned}$$

where $\left(e_{N+j}\right)_j$ is a sequence of identically distributed and algebraically $\alpha-$mixing with rate $b > 1$ functional random variables, defined on $(\Omega, \mathcal{F})$ with values in $L^2[a,b]$.

We substitute to u_{ex} the averages $\overline{u}_n$ of the sample (given by the strong law of large numbers) defined by:

$$\overline{u}_n = \frac{1}{n}\sum_{j=1}^{n} u_{N+j}$$

where $\overline{u}_n$ is an approximate value of u_{ex} since $\overline{u}_n$ converges to u_{ex}, as $n \to \infty$.

The problem becomes solving the following equation

$$\widehat{A_N}x = \overline{u}_n. \tag{7}$$

This is a typical feature of an ill-posed problem. For this, Lavrentiev [13] has replaced equation (7) by another one, precisely

$$\left(\widehat{A_N} + \gamma I\right) x = \overline{u}_n \tag{8}$$

where I is the operator identity and γ a positive real. The solution $\overline{X}_n^{\gamma}$ of (8) is defined by

$$\overline{X}_n^{\gamma} := \left(\widehat{A_N} + \gamma I\right)^{-1} \left(\overline{u}_n\right).$$

Under some assumptions on the parameter γ and on the class of possible solutions, the previous problem is stable for small variations of $\overline{u}_n$ and can be solved for all $\overline{u}_n$ of $L^2[a,b]$.

The invertibility of the operator $\left(\widehat{A_N}+\gamma I\right)$ is deduced from the following result (see [16], Proposition 3.3)

Assume that $\widetilde{A},\widetilde{B}\in\mathcal{L}(X,Y)$, are linear operators between two Banach spaces X,Y. If $\widetilde{A}$ is invertible and

$$\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{A}-\widetilde{B}\right\|<1$$

then $\widetilde{B}$ is also invertible and

$$\left\|\widetilde{B}^{-1}\right\|\le\frac{\left\|\widetilde{A}^{-1}\right\|}{1-\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{A}-\widetilde{B}\right\|},\left\|\widetilde{B}^{-1}-\widetilde{A}^{-1}\right\|\le\frac{\left(\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{A}-\widetilde{B}\right\|\right)\left\|\widetilde{A}^{-1}\right\|}{1-\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{A}-\widetilde{B}\right\|}.$$

Taking $\widetilde{A}=A+\gamma I$ and $\widetilde{B}=\widehat{A_N}+\gamma I$, we prove that $\widehat{A_N}+\gamma I$ is invertible since the assumptions of the proposition are fulfilled. Indeed, the positivity of A ensures the strict positivity of its eigenvalues [18] and hence the operator $A+\gamma I$ is invertible and its inverse is defined by

$$(A+\gamma I)^{-1}\left(\phi_i\right)=\frac{1}{\gamma+\lambda_i}\phi_i$$

where λ_i are the eigenvalues of the operator A corresponding to the eigenfunctions ϕ_i. Hence,

$$\left\|(A+\gamma I)^{-1}\right\|\le\frac{1}{\gamma}\tag{9}$$

and

$$\left\|\left(\widehat{A_N}+\gamma I\right)^{-1}-(A+\gamma I)^{-1}\right\|\le\frac{\left\|\widehat{A_N}-A\right\|}{\gamma^2\left(1-\left\|(A+\gamma I)^{-1}\right\|\left\|\widehat{A_N}-A\right\|\right)}.\tag{10}$$

5. Results

Theorem 5.1. *For the model defined by (1), and (5) assume that* (H_1) *holds and that* A *is positive. The operator,* $\widehat{A_N}$ *being a strongly consistent estimator of* A *(i.e.* $\left\|\widehat{A_N}-A\right\|\to 0$*, a.s.), assume that the following assumptions are fulfilled.*

(H_4) *The error trajectories* $e_{N+1},...,e_{N+n}$ *are identically distributed and algebraically* $\alpha-$*mixing, with rate* $b>1$ *and if*

$$\begin{aligned}\exists\, p &> 2, \exists\,\theta>2 \text{ and } M>0 \text{ such that}\\ \forall\, t &> M,\ P\left(\|e_1\|>t\right)\le t^{-p},\ \text{and } s_n^{-\frac{(b+1)p}{b+p}}=o\left(n^{-\theta}\right)\\ \text{with } s_n^{-2} &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|cov\left(\|e_i\|,\left\|e_j\right\|\right)\right|\end{aligned}$$

(H_5) *let* $\gamma=\gamma(N,n)$ *a sequence of parameters be chosen such that, as* $n,N\to\infty$,

$$\gamma\longrightarrow 0,\ \gamma^2\ln n\to+\infty,\ s_n^{-\frac{(b+1)p}{b+p}}(\ln n)^{\frac{3}{2}\frac{(b+1)p}{b+p}-2}n\gamma^{-\frac{(b+1)p}{b+p}}\longrightarrow 0\ .$$

$$\text{and} \quad \frac{\left\|\widehat{A_N} - A\right\|}{\gamma^2} \longrightarrow 0, \ a.s$$

Then, for all $\varepsilon > 0$,

$$P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \varepsilon\right\} \leq C \exp\left(-\frac{n\varepsilon_0^2\gamma^2\delta^2}{8} + \frac{\varepsilon_0^4\gamma^4\delta^4}{64}\right)$$

$$+ 2^{\frac{(b+1)p}{b+p}} n (\ln n)^{\frac{3\frac{(b+1)p}{b+p}}{2} - 2} (\varepsilon_0\delta)^{-\frac{(b+1)p}{b+p}} s_n^{-\frac{(b+1)p}{b+p}} \gamma^{-\frac{(b+1)p}{b+p}} \tag{11}$$

with

$$\delta = 1 - \left\|(A + \gamma I)^{-1}\right\| \left\|\widehat{A_N} - A\right\|$$

Proof. The following identity

$$\begin{aligned} & P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \varepsilon\right\} \\ = \ & P\left\{\left\|\left(\widehat{A_N} + \gamma I\right)^{-1} A x_{ex} + \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{A_N} + \gamma I\right)^{-1} e_{N+i} - x_{ex}\right\| > \varepsilon\right\}\end{aligned}$$

gives

$$P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \varepsilon\right\}$$

$$\leq P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{A_N} + \gamma I\right)^{-1} e_{N+i}\right\| > \varepsilon - \left\|\left(\widehat{A_N} + \gamma I\right)^{-1} A x_{ex} - x_{ex}\right\|\right\} \tag{12}$$

We first prove [4] that:

$$\left\|\left(\widehat{A_N} + \gamma I\right)^{-1} A x_{ex} - x_{ex}\right\| \longrightarrow 0, \quad a.s.$$

According to (10) we get

$$\begin{aligned}\left\|\left(\widehat{A_N} + \gamma I\right)^{-1} A x_{ex} - x_{ex}\right\| &\leq \left\|\left(\widehat{A_N} + \gamma I\right)^{-1} A x_{ex} - (A + \gamma I)^{-1} A x_{ex}\right\| \\ &\quad + \left\|(A + \gamma I)^{-1} A x_{ex} - x_{ex}\right\| \\ &\leq \frac{\left\|\widehat{A_N} - A\right\|}{\gamma^2\delta} \|A x_{ex}\| + \left\|(A + \gamma I)^{-1} A x_{ex} - x_{ex}\right\|.\end{aligned}$$

From the assumption (H_5) we have

$$\frac{\left\|\widehat{A_N} - A\right\|}{\gamma^2\delta} \longrightarrow 0, \ a.s., \text{ as } N, \ n \to \infty.$$

If

$$x_{ex} = \sum_i \mu_i \phi_i$$

then (H_1) involves

$$A x_{ex} = \sum_i \lambda_i \mu_i \phi_i$$

and

$$(A+\gamma I)^{-1} A x_{ex} - x_{ex} = \sum_i \mu_i \left(\frac{\lambda_i}{\lambda_i + \gamma} - 1 \right) \phi_i .$$

Since $\left(\phi_i\right)_i$ is an Hilbertian basis, we obtain

$$\left\| (A+\gamma I)^{-1} A x_{ex} - x_{ex} \right\|^2 = \sum_i \mu_i^2 \left(\frac{\lambda_i}{\lambda_i + \gamma} - 1 \right)^2 .$$

The dominated convergence theorem implies that the right-hand side tends to zero as $\gamma \to 0$, since $\sum\limits_i \mu_i^2 < \infty$. Consequently, for N, n large enough, we have

$$\left\| \left(\widehat{A}_N + \gamma I\right)^{-1} A x_{ex} - x_{ex} \right\| \leq \frac{\varepsilon}{2}.$$

Hence, the inequality (12) becomes

$$P\left\{ \left\| \overline{X}_n^{\gamma} - x_{ex} \right\| > \varepsilon \right\} \leq P\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \left(\widehat{A}_N + \gamma I\right)^{-1} e_{N+i} \right\| > \frac{\varepsilon}{2} \right\}. \tag{13}$$

We define

$$\Delta = \left\| (A+\gamma I)^{-1} \right\| \left\| \widehat{A}_N - A \right\| ;$$

it holds

$$\Delta \leq \frac{1}{\gamma} \left\| \widehat{A}_N - A \right\| .$$

We obtain

$$\left\| \left(\widehat{A}_N + \gamma I\right)^{-1} e_{N+i} \right\| \leq \left\| \left(\widehat{A}_N + \gamma I\right)^{-1} \right\| \left\| e_{N+i} \right\| .$$

And since

$$\left\| \left(\widehat{A}_N + \gamma I\right)^{-1} \right\| \leq \frac{\left\| (A+\gamma I)^{-1} \right\|}{1-\Delta},$$

so, from (9), we have

$$\left\| \left(\widehat{A}_N + \gamma I\right)^{-1} \right\| \leq \frac{1}{\gamma\,(1-\Delta)}.$$

From (H_5) we deduce that $\frac{1}{\gamma} \left\| \widehat{A}_N - A \right\| \leq 1/4$ almost surely and consequently

$$\left\| (A+\gamma I)^{-1} \right\| \left\| \widehat{A}_N - A \right\| < 1.$$

Hence

$$P\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \left(\widehat{A}_N + \gamma I\right)^{-1} e_{N+i} \right\| > \frac{\varepsilon}{2} \right\}$$

becomes

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon\delta\gamma}{2}\right\} \text{ with } \delta = 1-\Delta.$$

Now, we show that

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon\delta\gamma}{2}\right\} < \infty .$$

Applying directly proposition A.11 (Ferraty and Vieu [10]), we obtain

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon\delta\gamma}{2}\right\} \leq C\left(1+\frac{n^2\varepsilon^2\gamma^2\delta^2}{4rs_n^2}\right)^{-\frac{r}{2}} + nr^{-1}\left(\frac{2r}{n\varepsilon\delta\gamma}\right)^{q}; \tag{14}$$

where

$$q = \frac{(b+1)\,p}{b+p}.$$

We replace in (14) r and ε by

$$r = (\ln n)^2 \text{ and } \varepsilon = \varepsilon_0\sqrt{n^{-2}s_n^2\ln n};$$

we get

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon_0\delta\gamma\sqrt{n^{-2}s_n^2\ln n}}{2}\right\} \leq C\left(1+\frac{n^2\varepsilon_0^2\gamma^2\delta^2u_n}{4(\ln n)^2 s_n^2}\right)^{-\frac{(\ln n)^2}{2}} + n(\ln n)^{-2}\left(\frac{2(\ln n)^2}{n\varepsilon_0\delta\gamma\sqrt{u_n}}\right)^{q}.$$

We set $u_n = n^{-2}s_n^2\ln n$ $\left(\lim\limits_{n\to\infty} u_n = 0\right)$, then

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon_0\delta\gamma\sqrt{u_n}}{2}\right\} \leq C\left(1+\frac{n^{-2}\varepsilon_0^2 s_n^2\ln n\gamma^2\delta^2n^2}{4(\ln n)^2 s_n^2}\right)^{-\frac{(\ln n)^2}{2}} + n(\ln n)^{-2}\left(\frac{2(\ln n)^2}{n^{-1}s_n(\ln n)^{\frac{1}{2}}\varepsilon_0\delta n\gamma}\right)^{q},$$

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon_0\delta\gamma\sqrt{u_n}}{2}\right\} \leq C\left(1+\frac{\varepsilon_0^2\gamma^2\delta^2}{4(\ln n)}\right)^{-\frac{(\ln n)^2}{2}} + n(\ln n)^{-2}\left(\frac{2(\ln n)^{\frac{3}{2}}}{\varepsilon_0\delta s_n\gamma}\right)^{q}.$$

Using the fact $\ln(1+x) = x - \frac{x^2}{2} + o\left(x^2\right)$ when x close to 0, we get

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon_0\delta\gamma\sqrt{u_n}}{2}\right\} \leq C\left(e^{-\frac{\varepsilon_0^2\ln n\,\gamma^2\delta^2}{8}+\frac{\varepsilon_0^4\gamma^4\delta^4}{64}}\right) + 2^q n(\ln n)^{\frac{3q}{2}-2}(\varepsilon_0\delta)^{-q}s_n^{-q}\gamma^{-q}$$

to ensure

$$P\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} e_{N+i}\right\| > \frac{\varepsilon\delta\gamma}{2}\right\} \leq C\exp\left(-\frac{n\varepsilon_0^2\gamma^2\delta^2}{8} + \frac{\varepsilon_0^4\gamma^4\delta^4}{64}\right)$$
$$+ 2^q n (\ln n)^{\frac{3q}{2}-2} (\varepsilon_0\delta)^{-q} s_n^{-q} \gamma^{-q}. \tag{15}$$

Finally (11) follows from (13) and (15). □

Corollary 5.2. *Under the assumptions of Theorem 2, the sequence $\left(\overline{X}_n^{\gamma}\right)$ converges almost completely to the exact solution x_{ex} of equation (3).*

Proof. Indeed, it is easy to find $\gamma = \gamma(n, N)$ satisfying the assumption (H_5) such that

$$P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \varepsilon\right\} \leq Cn^{-\gamma^2-\nu_0}; \tag{16}$$

hence, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \varepsilon\right\} < +\infty. \tag{17}$$

Then the result follows. □

Corollary 5.3. *Under the assumptions of Theorem 2,*

$$\overline{X}_n^{\gamma} - x_{ex} = O(\gamma\sqrt{n^{-2}s_n^2\ln n}) \quad a.co.. \tag{18}$$

Proof. We recall that $\overline{X}_n^{\gamma} - x_{ex} = O(u_n)$ *a.co.*, where $(u_n)_n$ is a sequence of real positive numbers, if there exist $\epsilon_0 > 0$ such that

$$\sum_{n=1}^{+\infty} P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \epsilon_0 u_n\right\} < +\infty.$$

If we apply the result (16) with $\varepsilon = \epsilon_0\gamma\sqrt{u_n}$, we obtain, for all ϵ_0,

$$P\left\{\left\|\overline{X}_n^{\gamma} - x_{ex}\right\| > \epsilon_0\gamma\sqrt{u_n}\right\} \leq Cn^{-\gamma^2-\nu_0}. \tag{19}$$

The right-hand side of the precedent inequality is the term of convergent series. Thus we deduce (18). □

Corollary 5.4. *Under the assumptions of Theorem 2, for a given level β, there is a natural integer n_β for which:*

$$P\left\{\left\|\overline{X}_{n_\beta}^{\gamma} - x_{ex}\right\| \leq \varepsilon\right\} \geq 1 - \beta, \tag{20}$$

i.e., the exact solution x_{ex} of equation (3) belongs to the closed ball of center $X_{n_\beta}^{\gamma}$ and radius ε with a probability greater than or equal to $1 - \beta$.

Proof. Indeed, we have

$$\lim_{n\to+\infty} Cn^{-\gamma^2-\nu_0} = 0, \tag{21}$$

which implies the existence of a natural integer n_β such that

$$n \geq n_\beta \Longrightarrow Cn^{-\gamma^2-\nu_0} \leq \beta; \tag{22}$$

thus, (20) arises from (16) and (22). □

6. Numerical example

In this section, a simulation is proposed to illustrate the validity of the approximate solution $\overline{X}_n^{\gamma}$.

Example 6.1. *Let us consider the model*

$$u_i = Ax_i + e_i$$

where

$$\begin{array}{llll} A: & L^2[0,\pi] & \longrightarrow L^2[0,\pi] \\ & x & \mapsto Ax \end{array}$$

is defined by the kernel

$$K(s,t) = \begin{cases} \frac{s(\pi-t)}{\pi} & , 0 \le s \le t, \\ \\ \frac{t(\pi-s)}{\pi} & , t \le s \le \pi, \end{cases}$$

and

$$u_i(t) = \frac{1}{2}\sin 2t.$$

The design of repeated observations is used by taking the orthonormal basis $\overline{x}_i(t) = \sqrt{\frac{2}{\pi}}\sin it$*, hence the estimated kernel given in the construction is*

$$K_N(s,t) = \frac{2}{\pi}\sum_{i=1}^{m}\frac{\sin(is)\ \sin(it)}{i^2}.$$

The approximate solutions are given by

$$\overline{X}_n^{\gamma} := \left(\widehat{A_N} + \gamma I\right)^{-1}(\overline{u}_n).$$

To characterize the strong mixing random errors (e_i)*, we consider an autoregressive model* $(e_i)_i$ *of order 1 (see ([5])) described as follows:*

$$\begin{array}{rcl} e_0 & = & 0 \\ e_{i+1} & = & \varphi e_i + g_i \end{array}$$

where g_i *is a Gaussian white noise process,* φ *is a constant such that* $|\varphi| < 1$*. For the simulation of Gaussian random variables* $(g_i)_i$*, we use the method of Box-Muller:*

$$\begin{array}{rcl} g_1 & = & \sqrt{-2\ln(u_1)}\cos(2\pi u_2) \\ g_2 & = & \sqrt{-2\ln(u_1)}\sin(2\pi u_2) \end{array}$$

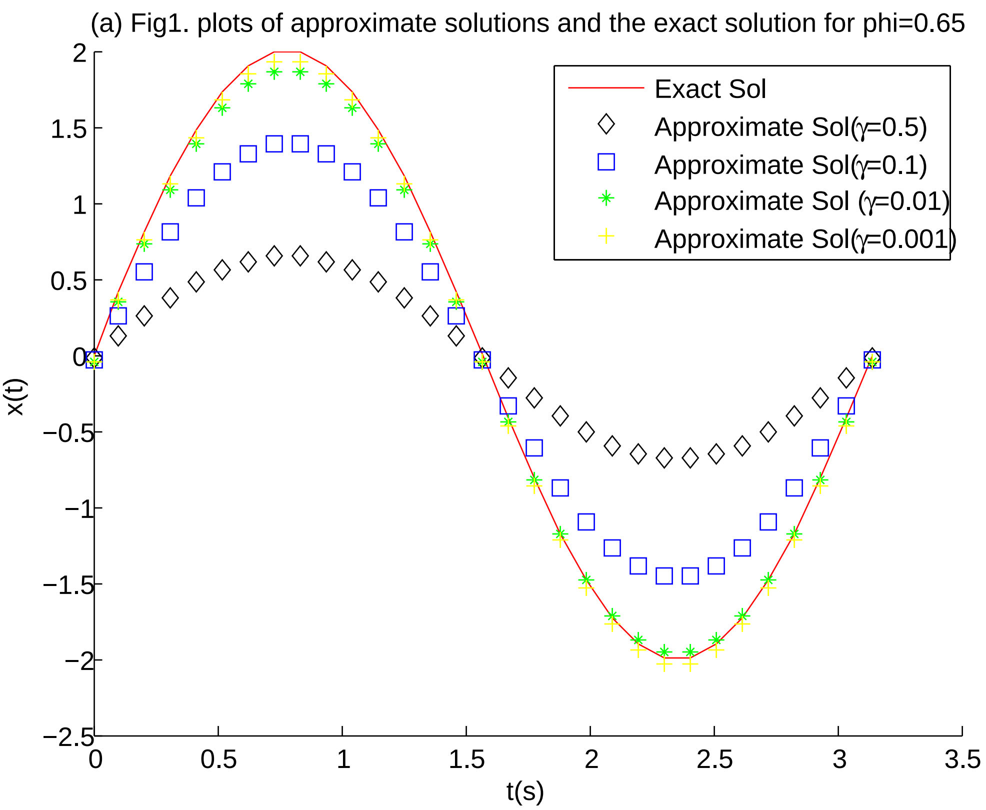
where u_1 *and* u_2 *are uniform distributed random numbers.*

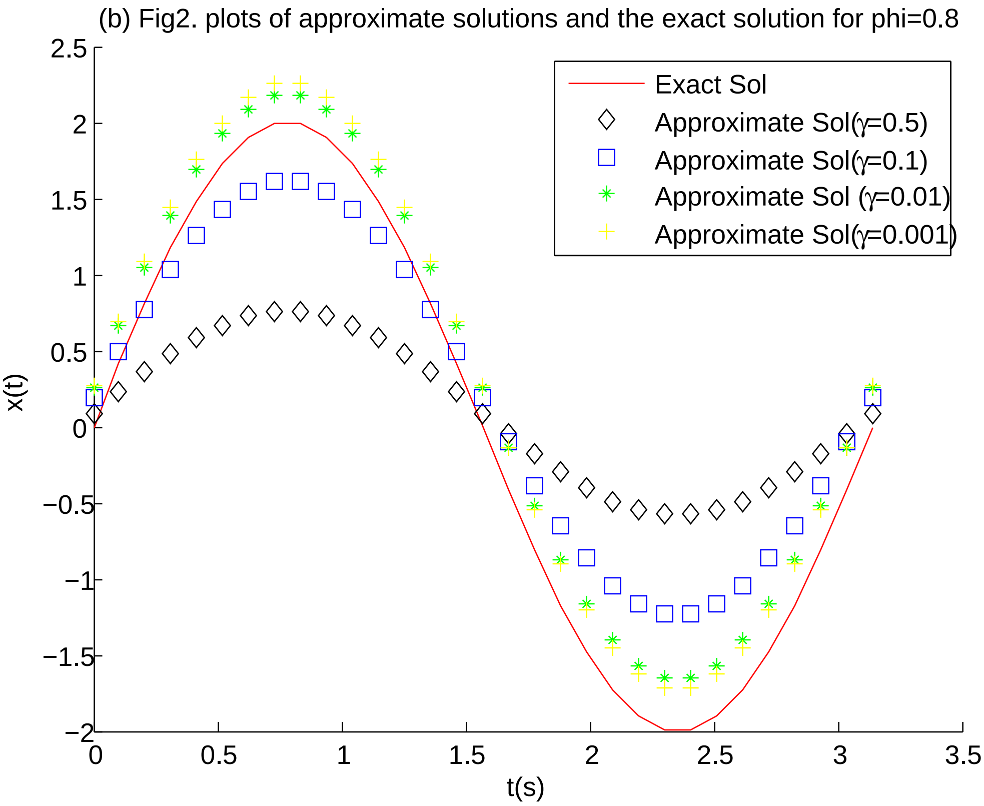
The plots of the exact solution and results of simulation are shown on **(a) Fig1** and **(b) Fig2**.

In this example, the exact solution is $(x(t) = 2\sin(2t))$ and the graphical results obtained for $\varphi = 0,65$ and $\varphi = 0,80$ are in good agreement with the exact solution.

The algorithm of the approximate solutions $\overline{X}_n^{\gamma}$ is implemented using MATLAB.

Conclusion 6.2. *In this work, we have established exponential inequalities for the probability of the distance between approximate solutions and the exact one for a linear calibration problem with* $\alpha-$*mixing random data. A numerical example was proposed as illustration. The numerical results show that the approximate solutions are in good agreement with the exact one. The proposed method is applicable to linear problems with* $\alpha-$*mixing random data. In a future work, it is interesting to consider quasi associated random errors.*


(a) Fig1. plots of approximate solutions and the exact solution for phi=0.65
Exact Sol
Approximate Sol(γ=0.5)
Approximate Sol(γ=0.1)
Approximate Sol (γ=0.01)
Approximate Sol(γ=0.001)
x(t)
t(s)
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
0.5
1
1.5
2
2.5
3
3.5


(b) Fig2. plots of approximate solutions and the exact solution for phi=0.8
Exact Sol
Approximate Sol(γ=0.5)
Approximate Sol(γ=0.1)
Approximate Sol (γ=0.01)
Approximate Sol(γ=0.001)
x(t)
t(s)
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
0.5
1
1.5
2
2.5
3
3.5

References

[1] Aiane N, Dahmani A. On the rate of convergence of the Robbins-Monro's algorithm in a linear stochastic ill-posed problem with $\alpha-$mixing data. Communications in Statistics-Theory and Methods. (2017); 46(13): 6694-6703.

[2] Aiane N, Dahmani. A Consistency of Tikhonov's regularization in an ill-posed problem with $\alpha-$mixing data. Communications in Statistics-Theory and Methods. (2017); 46(12): 5633-5642.

[3] Brezis H. Analyse fonctionnelle. Théorie et applications. Dunod. Paris. (1999).

[4] Cuevas A, Febrero M, Fraiman R. Linear functional regression: the case of fixed design and functional response. The Canadian Journal of Statistics. (2002); 30(2): 285-300.

[5] Dahmani A, Ait Saidi A. Consistency of Robbins Monro's algorithm within a mixing framework, Bulletin de la Société Royale des Sciences de Liège. (2010); 79: 131-140.

[6] Deville JC, Sarndal CE. Calibration Estimators in Survey Sampling. Journal of American Statistical Association.(1992); 87(418): 376-382.

[7] Doukhan P. Mixing. Properties and examples. Lecture notes in Statistics 85,Springer Verlag, New York. (1994); 182pp.

[8] Doukhan P, Bandriere O. Dependence Noise for stochastic algorithms, C.R. Math. Acad. Sci. Paris. (2003); 337,no.7,473-476.

[9] Engl HW, Hanke M, Neubauer A. Regularization of Inverse Problems. Kluwer. Dordrecht. (1996).

[10] Ferraty F, Vieu P. Non parametric moddelling for functional data. Springer: monograph. (2005).

[11] Frank I E, Friedman J H. A statistical view of some chemometrics regression tools. Technometrics. (1993); 35, 109-135.

[12] Garcio-Dorado D, Inserte J ,Ruiz-Meana M , Gonzàlez M A ,Solares J, Julia M, Barrabés J A &Soler-Soler J. Gap junction uncoupler heptanol prevents cell-to-cell progression of hypercontracure and limits necrosis during myocardial reperfusion. Circulation. (1997); 96, 3579-3586.

[13] Lavrentiev MM, Avdeev AV, Priimenko VI. Inverse Problems Of Mathematical Physics. Walter de Gruyter. (2003).

[14] Lavrentiev MM, Romanov VG, Shishatskii SP. Ill-posed problems of mathematical physics and analysis. American Mathematical Society. Providence. (1986).

[15] Martens H, Naes T. Multivariate calibration. John Wiley and Sons. Chichester. (1989).

[16] Nashed MZ. Perturbations and approximations for generalized inverses and linear operator equations. Generalized Inverses and Applications. (Academic Press). New York. (1976); 325-396.

[17] Osborne C. Statistical calibration: a review. International Statistical Review. (1991); 59: 309-336.

[18] Riesz F, Nagy BS. Functional analysis. Frederick Ungar. New York. (1955).

[19] Rosenblatt M. A central limit theorem and a strong mixing condition, Proc.Nat. Acad.Sci. (1956); 42: 43-47, USA.

[20] Sarndal CE, Swensson B, Wretman J. Model Assisted Survey Sampling. Springer Verlag. New York. (1992).

[21] Tarantola A. Inverse Problem Theory and Methods for Model Parameter Estimation. SIAM. Philadelphia. (2005).

[22] Tikhonov AN, Arsenin VA. Solutions of Ill-posed Problems. Winston & Sons. Washington. (1977).

[23] Zerouati H, Dahmani A. Exponential inequalities in linear calibration problem. Communications in Statistics-Theory and Methods. (2016); 46(12): 5633-5642.