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Pseudoanalytic Extension on $F(p, p-2, s)$ Spaces and Applications

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Abstract. In this paper, we generalized the main results in [9]. As an applications, we give a characterization of the closure of $F(p, p-2, s)$ spaces in Lipschitz-type spaces $\mathcal{A}_\omega$ by pseudoanalytic extension.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. For $0 < p < \infty$, H^p denotes the Hardy space, which consisting of all functions $f \in H(\mathbb{D})$ satisfied (see [13])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ is the set of bounded analytic functions in $\mathbb{D}$ and $\mathcal{A}$ denotes the disc algebra.

Let $0 < p < \infty$, $-2 < q < \infty$ $s \geq 0$. The $F(p, q, s)$ ([27]) space is the set of all $f \in H(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q \left(1-|\varphi_a(z)|^2\right)^s dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ and $dA(z) = \frac{1}{\pi}dxdy$. When $q = p-2$, $F(p, p-2, s)$ is Möbius invariant Besov-type spaces. When $0 < s < 1$, $F(2, 0, s) = Q_s$ ([24, 25]); If $s = 1$, $F(2, 0, 1) = BMOA$, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation. When $s > 1$, $F(2, 0, s) = \mathcal{B}$ (the Bloch space).

Let $\omega : [0, \infty) \to \mathbf{R}$ be a right-continuous with $\omega(0) = 0$. If ω is increasing and $\frac{\omega(t)}{t}$ is nonincreasing for $t > 0$, there exists constant $C(\omega)$ such that

$$\int_0^\delta \frac{\omega(t)}{t} dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \leq C(\omega) \cdot \omega(\delta),$$

then we say that ω is a regular majorant, where $0 < \delta < 1$.

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Given a regular majorant ω and a compact set $E \subset \mathbb{C}$, the Lipschitz-type spaces $\Lambda_\omega(E)$ consists of those functions $f : E \to \mathbb{C}$, such that

$$\|f\|_{\Lambda_\omega} = \sup\left\{\frac{|f(z)-f(w)|}{\omega(|z-w|)} : z, w \in E, z \neq w\right\} < \infty.$$

In this paper, we shall be concerned with the space $\mathcal{A}_\omega =: \mathcal{A} \cap \Lambda_\omega(\overline{\mathbb{D}})$. When $\omega(t) = t^\alpha$, $0 < \alpha < 1$, it give the classical Lipschitz space Λ_α. For more informations on $\mathcal{A}_\omega$, we refer to [4] and the paper referin there.

Pseudoanalytic extension, as explained in [10], an analytic function in $\mathbb{D}$ can be extended to $\mathbb{D}_e = \{z : |z| > 1\}$ as a C^1 function whose Cauchy-Riemann $\overline{\partial}$-derivative becomes appropriately small. There are many applications for pseudoanalytic extension, for example: K-property ([9]); inner-outer factorization ([10]); Bernstein-type inequality related to kernel of H^p spaces ([6]) and so on.

In this paper, we generalize the main results in [9] to $F(p, p-2, s)$ spaces. Moreover, we also give an application on our result to studying the closure of $F(p, p-2, s)$ spaces in Lipschitz-type spaces $\mathcal{A}_\omega$ (denoted by $C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p-2, s))$) by pseudoanalytic extension.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2. Auxiliary results

If Q is a measurable subset of $\mathbb{C}$ and Q varies over all discs in $\mathbb{C}$, $|Q|$ will denote the measure (area) of Q. Let ω be a positive measurable function on $\mathbb{C}$. We say that ω is an A_t-weight ($t > 1$) if (see [22])

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(z)\, dA(z)\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{\omega^{\frac{1}{t-1}}(z)} dA(z)\right)^{t-1} < \infty.$$

Remark 1. Let $t > 1$ and ω be an A_t-weight and T be a Calderon-Zygmund operator. It is well know that (see [22])

$$\int_{\mathbb{C}} |Tf(z)|^t \omega(z) dA(z) \lesssim \int_{\mathbb{C}} |f(z)|^t \omega(z) dA(z), \quad for\ \ all\ \ f \in L^t(\omega).$$

Here $L^t(\omega)$ denote the space of functions $f \in L^t$ which satisfy

$$\int_{\mathbb{C}} |f(z)|^t \omega(z) dA(z) < \infty.$$

The following lemma generalized [9, Proposition 1].

Lemma 1. *Suppose that* $1 < p < \infty$, $0 < s < 1$, $p + s > 2$, $z \in \mathbb{C}$ *and* $a \in \mathbb{D}$. *Then* $\left|1-|z|^2\right|^{p-2}\left|\frac{1}{|\varphi_a(z)|^2} - 1\right|^s$ *is an* A_p*-weight.*

Proof. Since

$$\left|1-|z|^2\right|^{p-2}\left|\frac{1}{|\varphi_a(z)|^2} - 1\right|^s = \frac{(1-|a|^2)^s \left||z|^2-1\right|^{p-2+s}}{|z-a|^{2s}}.$$

Let

$$M_a(z) = \frac{(1-|a|^2)^s \left||z|^2-1\right|^{p-2+s}}{|z-a|^{2s}}$$

and

$$N_a(z) = \frac{\left||z|^2-1\right|^{p-2+s}}{|z-a|^{2s}}.$$

It is easily to see that $M_a(z)$ is an A_p-weight if and only if $N_a(z)$ is an A_p-weight. Now, we adopt and modify the method in [9, Proposition 1]. Suppose that $N_a(z) = J(z)K_a(z)$, where

$$J(z) = \left||z|^2 - 1\right|^{p-2+s}, \quad K_a(z) = \frac{1}{|z-a|^{2s}}, \quad K_0(z) = \frac{1}{|z|^{2s}}.$$

From [22, page 218], we known that $K_0(z)$ is an A_t-weight ($t > 1$). Since $K_a(z)$ are translates of $K_0(z)$, we have $K_a(z)$ is also an A_t-weight, that is,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q K_a(z)dA(z)\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{K_a^{\frac{1}{t-1}}(z)}dA(z)\right)^{t-1} < \infty. \tag{$*$}$$

Let $r \in (1, \frac{p-1}{p-2+s})$ and Q be any disc. Let $\frac{1}{r} + \frac{1}{r'} = 1$. Then, for any $a \in \mathbb{D}$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q N_a(z)\,dA(z)\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{N_a^{\frac{1}{p-1}}(z)}dA(z)\right)^{p-1}\\
=&\left(\frac{1}{|Q|}\int_Q J(z)\,K_a(z)\,dA(z)\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{J^{\frac{1}{p-1}}(z)\,K_a^{\frac{1}{p-1}}(z)}dA(z)\right)^{p-1}\\
\lesssim&\left[\sup_{z\in Q} J(z)\right]\times\left(\frac{1}{|Q|}\int_Q K_a(z)\,dA(z)\right)\\
&\quad\times\left(\frac{1}{|Q|}\int_Q \frac{1}{J^{\frac{r}{p-1}}(z)}dA(z)\right)^{\frac{p-1}{r}}\times\left(\frac{1}{|Q|}\int_Q \frac{1}{K_a^{\frac{r'}{p-1}}(z)}dA(z)\right)^{\frac{p-1}{r'}}.
\end{aligned}$$

By direct calculation (or see [9, page 484]), we obtain

$$\left[\sup_{z\in Q} J(z)\right]\times\left(\frac{1}{|Q|}\int_Q \frac{1}{J^{\frac{r}{p-1}}(z)}dA(z)\right)^{\frac{p-1}{r}} < \infty.$$

Thus,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q N_a(z)\,dA(z)\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{N_a^{\frac{1}{p-1}}(z)}dA(z)\right)^{p-1}\\
\lesssim&\left(\frac{1}{|Q|}\int_Q K_a(z)\,dA(z)\right)\times\left(\frac{1}{|Q|}\int_Q \frac{1}{K_a^{\frac{r'}{p-1}}(z)}dA(z)\right)^{\frac{p-1}{r'}}.
\end{aligned}$$

If $2 - s < p \le 2$, it easily to see that $\frac{r'}{p-1} > 1$. If $p > 2$, noted that $r \in (1, \frac{p-1}{p-2+s})$ and $\frac{1}{r} + \frac{1}{r'} = 1$, we can also deduce that $\frac{r'}{p-1} > 1$. Let $t = \frac{p-1+r'}{r'} > 1$. Combined with ($*$), we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q K_a(z)\,dA(z)\right)\times\left(\frac{1}{|Q|}\int_Q \frac{1}{K_a^{\frac{r'}{p-1}}(z)}dA(z)\right)^{\frac{p-1}{r'}}.\\
=&\left(\frac{1}{|Q|}\int_Q K_a(z)\,dA(z)\right)\times\left(\frac{1}{|Q|}\int_Q \frac{1}{K_a^{\frac{1}{t-1}}(z)}dA(z)\right)^{t-1} < \infty.
\end{aligned}$$

Therefore,

$$\left(\frac{1}{|Q|}\int_Q N_a(z)\,dA(z)\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{N_a^{\frac{1}{p-1}}(z)}dA(z)\right)^{p-1} < \infty,$$

for any $a \in \mathbb{D}$. The proof is completed. □

3. Pseudoanalytic extension on $F(p, p-2, s)$

Now, let us consider the pseudoanalytic extension on $F(p, p-2, s)$.

Theorem 1. Suppose that $p > 1$, $0 < s < 1$, $p + s > 2$ and $f \in \bigcap_{0<p<\infty} H^p$. Then the following are equivalent:
(1) $f \in F(p, p-2, s)$;
(2)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}\left(\frac{1}{|\varphi_a(z)|^2}-1\right)^s dA(z) < \infty;$$

(3) There exists a function $F \in C^1(\mathbb{D}_e)$ satisfying

$$F(z) = O(1), \quad as \quad z \to \infty, \tag{a}$$

$$\lim_{r\to 1^+} F(re^{i\theta}) = f(e^{i\theta}), \;\; a.e \;\; and \;\; in \;\; L^q([-\pi,\pi]) \;\; for \;\; all \;\; q \in [1,\infty), \tag{b}$$

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}_e} |\bar{\partial}F(z)|^p(|z|^2-1)^{p-2}(|\varphi_a(z)|^2-1)^s dA(z) < \infty. \tag{c}$$

Proof. (1) ⇔ (2). Since $F(p, p-2, s)$ space is Möbius invariant, we only need to prove that (the case $a = 0$)

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z)\\
\asymp &\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}\left(\frac{1}{|z|^2}-1\right)^s dA(z)\\
= &\int_{\mathbb{D}} |f'(z)|^p\frac{(1-|z|^2)^{p-2+s}}{|z|^{2s}}dA(z).
\end{aligned}$$

On the one hand, it is obvious that

$$\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z) \le \int_{\mathbb{D}} |f'(z)|^p\frac{(1-|z|^2)^{p-2+s}}{|z|^{2s}}dA(z).$$

On the other hand, let

$$M_p(r, f')^p = \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^p d\theta.$$

Bearing in mind that $M_p(r, f')^p$ is an increasing function of r, we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^p\frac{(1-|z|^2)^{p-2+s}}{|z|^{2s}}dA(z) = \int_0^1 M_p(r, f')^p(1-r^2)^{p-2+s}r^{1-2s}dr\\
\le &M_p\left(\frac{1}{2}, f'\right)^p\int_0^{\frac{1}{2}}(1-r^2)^{p-2+s}r^{1-2s}dr + 4^s\int_{\frac{1}{2}}^1 M_p(r, f')^p(1-r^2)^{p-2+s}rdr\\
\le &(C(p,s)+4^s)\int_{\frac{1}{2}}^1 M_p(r, f')^p(1-r^2)^{p-2+s}rdr\\
\lesssim &\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z),
\end{aligned}$$

where

$$C(p,s)=\frac{\int_0^{\frac{1}{2}}(1-r^2)^{p-2+s}r^{1-2s}dr}{\int_{\frac{1}{2}}^{1}(1-r^2)^{p-2+s}rdr}<\infty.$$

We get the desired result.
(1) ⇒ (3). Suppose $f\in F(p,p-2,s)$, let $z^*=\frac{1}{\bar{z}}$ and

$$F(z)=f(z^*),\qquad z\in\mathbb{D}_e.$$

Hence, $F\in C^1(\mathbb{D}_e)$ and satisfies (a) and (b). Let $a\in\mathbb{D}$. Using the fact that $|\bar{\partial}F(z)|=|f'(z^*)||z^*|^2$, making change of variables $z=w^*$, and combining with (1) ⇔ (2), we deduce that

$$\begin{aligned}&\int_{\mathbb{D}_e}|\bar{\partial}F(z)|^p(|z|^2-1)^{p-2}(|\varphi(z)|^2-1)^s dA(z)\\=&\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{p-2}(|\varphi(w)^*|^2-1)^s dA(w)\\=&\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{p-2}(\frac{1}{|\varphi(w)|^2}-1)^s dA(w)<\infty.\end{aligned}$$

(3) ⇒ (1). Let $z\in\mathbb{D}$ and $R>1$. Using Cauchy-Green formula we obtain

$$f(z)=\frac{1}{2\pi i}\int_{|w|=R}\frac{g(w)}{w-z}d\zeta-\frac{1}{\pi}\int_{1<|w|<R}\frac{\bar{\partial}g(w)}{w-z}dA(w).$$

Notice the fact that

$$\int_{|w|=R}\frac{g(w)}{(w-z)^2}dw\to 0,\ \ as\ \ R\to\infty.$$

We deduce

$$f'(z)=-\frac{1}{\pi}\int_{\mathbb{D}_e}\frac{\bar{\partial}g(w)}{(w-z)^2}dA(w).$$

Let G be defined by

$$G(z)=\begin{cases}\bar{\partial}g(z), & z\in\mathbb{D}_e,\\ 0, & z\in\mathbb{D}.\end{cases}$$

Let T denote the Calderón-Zygmund operator defined by

$$Tg(z)=p.v.\int_C\frac{g(w)}{(w-z)^2}dA(w).$$

It is not hard to see that

$$f'(z)=-\frac{1}{\pi}(TG)(z),\ \ z\in\mathbb{D}.$$

Hence, using the boundedness of Calderón-Zygmund operators (see Remark 1) and Lemma 1, we deduce that

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \left| \frac{1}{|\varphi_a(z)|^2} - 1 \right|^s dA(z) \\
=&\frac{1}{\pi} \int_{\mathbb{D}} |(TG)(z)|^p (1-|z|^2)^{p-2} \left| \frac{1}{|\varphi_a(z)|^2} - 1 \right|^s dA(z) \\
\lesssim& \int_{\mathbb{C}} |(TG)(z)|^p \left|1-|z|^2\right|^{p-2} \left| \frac{1}{|\varphi_a(z)|^2} - 1 \right|^s dA(z) \\
\lesssim& \int_{\mathbb{C}} |G(z)|^p \left|1-|z|^2\right|^{p-2} \left| \frac{1}{|\varphi_a(z)|^2} - 1 \right|^s dA(z) \\
\lesssim& \int_{\mathbb{D}_e} |\bar{\partial} g(z)|^p \left(|z|^2-1\right)^{p-2} \left(|\varphi_a(z)|^2 - 1\right)^s dA(z) < \infty.
\end{aligned}$$

The proof is completed. □

Remark 2. Such function F is said to be a pseudonalytical extension of f, clearly it is not uniquelly determined by f.

Remark 3. T is also known as Ahlfors - Beoruling operator, which appears in discussions related to different topics in complex analysis, like Beltrami equation.

Given a function $v \in L^\infty(\partial\mathbb{D})$, the associated Toeplitz operator T_v is defined by

$$(T_v f)(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{v(\xi) f(\xi)}{\xi - z} d\xi, \qquad f \in H^1, z \in \mathbb{D}.$$

Recall that a subspace X of H^1 is said to have the K-property if $T_{\overline{\psi}}(X) \subset X$ for any $\psi \in H^\infty$.

Corollary 1. Let $p > 1$, $0 < s < 1$ and $p + s > 2$. The $F(p, p-2, s)$ has the K-property.

Proof. The proof is similar to [9, Theorem 2]. For completeness, we give the proof. Suppose that $f \in F(p, p-2, s)$, $h \in H^\infty$. We need to show that

$$g_1 =: T_{\overline{h}} f \in F(p, p-2, s).$$

Since, by definition of Toeplitz operator, g_1 is the orthogonal projection of $f\overline{h}$ onto H^2, then, we have

$$f\overline{h} = g_1 + \overline{g_2},$$

where $g_2 \in H_0^2$. Therefore, we obtain that

$$g_1 = f\overline{h} - \overline{g_2} \qquad a.e. \text{ on } \partial\mathbb{D}.$$

From Theorem 1, we know that there is a function $F \in C^1(\mathbb{D}_e)$ satisfying (a), (b) and (c). We let

$$H(z) =: \overline{h(z^*)}, \quad G_2 =: \overline{g_2(z^*)}, \quad G_1(z) =: F(z)H(z) - G_2(z) \quad z \in \mathbb{D}_e.$$

Hence, using the fact that

$$F|_{\partial\mathbb{D}} = f, \quad H|_{\partial\mathbb{D}} = \overline{h}, \quad G_2|_{\partial\mathbb{D}} = \overline{g_2},$$

we get

$$G_1|_{\partial\mathbb{D}} = g_1.$$

Since H and G_2 are holomorphic in $\mathbb{D}_e$, we obtain

$$\overline{\partial}H = 0, \quad \overline{\partial}G_2 = 0.$$

Thus, we have $\overline{\partial}G_1 = H \cdot \overline{\partial}F$ on $\mathbb{D}_e$. Furthermore,

$$|\overline{\partial}G_1| \leq \|H\|_\infty |\overline{\partial}F|.$$

It is clear that G_1 is C^1-smooth in $\mathbb{D}_e$ and bounded at ∞. Using the fact of above and Theorem 1, we easy to get (a), (b) and (c) hold true with G_1 and g_1 in place of F and f. The proof is completed. □

4. Closure of $F(p, p-2, s)$ spaces in $\mathcal{A}_\omega$

Let us recall the following result.

Lemma 2. [4, Lemma 7] *Let ω be a regular majorant. Suppose that $f \in \mathcal{A}$. Then $f \in \Lambda_\omega$ if and only if there exists a bounded function $g \in C^1(\mathbb{D}_e)$ satisfying*

$$\lim_{r\to 1^+} g(re^{i\theta}) = f(e^{i\theta});$$

$$\sup_{z\in\mathbb{D}_e} \frac{(|z|^2-1)}{\omega(|z|^2-1)} |\overline{\partial}g(z)| < \infty.$$

Moreover,

$$\|f\|_{\Lambda_\omega} \approx \inf_g \sup_{z\in\mathbb{D}_e} \frac{(|z|^2-1)}{\omega(|z|^2-1)} |\overline{\partial}g(z)|.$$

Lemma 3. *Let ω be a regular majorant. Then*

$$\int_{\mathbb{D}_e} \frac{\omega(|w|^2-1)}{(|w|^2-1)|w-z|^2} dA(w) \lesssim \frac{\omega(1-|z|^2)}{(1-|z|^2)}, \quad z \in \mathbb{D}.$$

Proof. Making change of variable $w = \frac{1}{v}$, $v \in \mathbb{D}$, we have

$$\int_{\mathbb{D}_e} \frac{\omega(|w|^2-1)}{(|w|^2-1)|w-z|^2} dA(w) = \int_{\mathbb{D}} \frac{\omega(\frac{1-|v|^2}{|v|^2})}{\frac{1-|v|^2}{|v|^2}\frac{|1-vz|^2}{|v|^2}} \frac{1}{|v|^4} dA(v)$$
$$= \int_{\mathbb{D}} \frac{\omega(\frac{1-|v|^2}{|v|^2})}{(1-|v|^2)|1-vz|^2} dA(v)$$
$$\lesssim \int_0^1 \frac{\omega(\frac{1-r^2}{r^2})}{(1-r^2)(1-r^2|z|^2)} rdr.$$

Let $t = \frac{1-r^2}{r^2}$. Then $r^2 = \frac{1}{1+t}$ and $rdr = \frac{-dt}{2(t+1)^2}$. We obtain

$$\int_0^1 \frac{\omega(\frac{1-r^2}{r^2})}{(1-r^2)(1-r^2|z|^2)} rdr$$
$$\lesssim \int_0^\infty \frac{\omega(t)}{t[t+(1-|z|^2)]} dt$$
$$= \int_0^{1-|z|^2} \frac{\omega(t)}{t[t+(1-|z|^2)]} dt + \int_{1-|z|^2}^\infty \frac{\omega(t)}{t[t+(1-|z|^2)]} dt.$$

Note that

$$\int_0^\delta \frac{\omega(t)}{t}dt + \delta\int_\delta^\infty \frac{\omega(t)}{t^2}dt \le C(\omega)\cdot\omega(\delta).$$

We have

$$\int_0^\delta \frac{\omega(t)}{t}dt \lesssim \omega(\delta)$$

and

$$\delta\int_\delta^\infty \frac{\omega(t)}{t^2}dt \lesssim \omega(\delta).$$

Thus,

$$\int_0^{1-|z|^2} \frac{\omega(t)}{t[t+(1-|z|^2)]}dt \le \frac{1}{(1-|z|^2)}\int_0^{1-|z|^2}\frac{\omega(t)}{t}dt \lesssim \frac{\omega(1-|z|^2)}{(1-|z|^2)}$$

and

$$\int_{1-|z|^2}^\infty \frac{\omega(t)}{t[t+(1-|z|^2)]}dt \le \int_{1-|z|^2}^\infty \frac{\omega(t)}{2t^2}dt \lesssim \frac{\omega(1-|z|^2)}{(1-|z|^2)}.$$

That is

$$\int_{\mathbb{D}_e} \frac{\omega(|w|^2-1)}{(|w|^2-1)|w-z|^2}dA(w) \lesssim \frac{\omega(1-|z|^2)}{(1-|z|^2)}.$$

The proof is completed. □

Theorem 2. *Let $p>1$, $0<s<1$, $p+s>2$ and ω be a regular majorant. If $f\in\mathcal{A}_\omega$, then the following statements are equivalent.*

(i) $f\in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega\cap F(p,p-2,s))$.

(ii) For any $\epsilon>0$,

$$\int_{\Omega_\epsilon(F)} \frac{\omega^p(|z|^2-1)}{(|z|^2-1)^2}(|\varphi_a(z)|^2-1)^s dA(z) < \infty,$$

where $\Omega_\epsilon(F)=\{z\in\mathbb{D}_e : \frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial F(z)|\ge\epsilon\}$ and F is pseudoanalytic extension of f.

Proof. (i) ⇒ (ii). Suppose that $f\in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega\cap F(p,p-2,s))\subseteq\mathcal{A}_\omega$. Then for any $\epsilon>0$, there exist a function $g\in\mathcal{A}_\omega\cap F(p,p-2,s)$, such that

$$\|f-g\|_{\Lambda_\omega}\le\frac{\epsilon}{2}.$$

From Lemma 2, there exist functions $F,G\in C^1(\mathbb{D}_e)$, such that

$$\frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial F-\bar\partial G| \lesssim \|f-g\|_{\Lambda_\omega}\le\frac{\epsilon}{2}.$$

Here F,G are its pseudoanalytic extension of f and g, respectively. Since

$$\frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial F| \lesssim \frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial F-\bar\partial G| + \frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial G|,$$

we have $\Omega_\epsilon(F)\subseteq\Omega_{\frac{\epsilon}{2}}(G)$. By Theorem 1, we can deduce that

$$\begin{aligned}
&\int_{\Omega_\epsilon(F)} \frac{\omega^p(|z|^2-1)}{(|z|^2-1)^2}(|\varphi_a(z)|^2-1)^s dA(z)\\
\le&\frac{2^p}{\epsilon^p}\int_{\Omega_{\frac{\epsilon}{2}}(G)} |\bar\partial G(z)|^p(|z|^2-1)^{p-2}(|\varphi_a(z)|^2-1)^s dA(z)\\
\le&\frac{2^p}{\epsilon^p}\int_{\mathbb{D}_e} |\bar\partial G(z)|^p(|z|^2-1)^{p-2}(|\varphi_a(z)|^2-1)^s dA(z)<\infty.
\end{aligned}$$

(ii) ⇒ (i). Let $f \in \mathcal{A}_\omega$. Using Cauchy-Green formula we obtain

$$f(z) = \frac{1}{2\pi i}\int_{|w|=R}\frac{F(w)}{w-z}d\zeta - \frac{1}{\pi}\int_{1<|w|<R}\frac{\overline{\partial}F(w)}{w-z}dA(w).$$

Noting the fact that $\int_{|w|=R}\frac{F(w)}{(w-z)^2}dw \to 0, \ \ as \ \ R\to\infty$, we obtain

$$f'(z) = -\frac{1}{\pi}\int_{\mathbb{D}_e}\frac{\overline{\partial}F(w)}{(w-z)^2}dA(w).$$

Let

$$f_1'(z) = -\frac{1}{\pi}\int_{\Omega_\epsilon(F)}\frac{\overline{\partial}F(w)}{(w-z)^2}dA(w)$$

and

$$f_2'(z) = -\frac{1}{\pi}\int_{\mathbb{D}_e\setminus\Omega_\epsilon(F)}\frac{\overline{\partial}F(w)}{(w-z)^2}dA(w).$$

Hence, $f'(z) = f_1'(z) + f_2'(z)$. By Lemma 3,

$$\begin{aligned}\frac{(1-|z|^2)}{\omega(1-|z|^2)}|f'(z)-f_1'(z)| =& \frac{(1-|z|^2)}{\omega(1-|z|^2)}|f_2'(z)|\\ \lesssim& \frac{(1-|z|^2)}{\omega(1-|z|^2)}\int_{\mathbb{D}_e\setminus\Omega_\epsilon(F)}\frac{|\overline{\partial}F(w)|}{|w-z|^2}dA(w)\\ =& \frac{(1-|z|^2)}{\omega(1-|z|^2)}\int_{\mathbb{D}_e\setminus\Omega_\epsilon(F)}\frac{\frac{(|w|^2-1)}{\omega(|w|^2-1)}|\overline{\partial}F(w)|}{\frac{(|w|^2-1)}{\omega(|w|^2-1)}|w-z|^2}dA(w)\\ \lesssim& \epsilon\frac{(1-|z|^2)}{\omega(1-|z|^2)}\int_{\mathbb{D}_e\setminus\Omega_\epsilon(F)}\frac{\omega(|w|^2-1)}{(|w|^2-1)|w-z|^2}dA(w)\\ \lesssim& \epsilon\frac{(1-|z|^2)}{\omega(1-|z|^2)}\int_{\mathbb{D}_e}\frac{\omega(|w|^2-1)}{(|w|^2-1)|w-z|^2}dA(w)\\ \lesssim& \epsilon\frac{(1-|z|^2)}{\omega(1-|z|^2)}\frac{\omega(1-|z|^2)}{(1-|z|^2)} = \epsilon,\end{aligned}$$

which implies that $f_1 \in \mathcal{A}_\omega$. Now, we are going to prove that $f_1 \in F(p, p-2, s)$.

Let

$$G(z) = \begin{cases}\overline{\partial}F(z), & z\in\Omega_\epsilon(F),\\ 0, & z\in\mathbb{C}\setminus\Omega_\epsilon(F),\end{cases}$$

and

$$Tg(z) = p.v.\int_C\frac{g(w)}{(w-z)^2}dA(w).$$

It is easy to see that $f_1'(z) = -\frac{1}{\pi}(TG)(z), \ \ z\in\mathbb{D}$. Hence, using the boundedness of the operator T and Lemma

1, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f_1'(z)|^p (1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
=&\frac{1}{\pi}\int_{\mathbb{D}} |(TG)(z)|^p (1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
\approx&\frac{1}{\pi}\int_{\mathbb{D}} |(TG)(z)|^p (1-|z|^2)^{p-2}\left(\frac{1}{|\varphi_a(z)|^2}-1\right)^s dA(z)\\
\lesssim&\int_{\mathbb{C}} |(TG)(z)|^p |1-|z|^2|^{p-2}\left|\frac{1}{|\varphi_a(z)|^2}-1\right|^s dA(z)\\
\lesssim&\int_{\mathbb{C}} |G(z)|^p |1-|z|^2|^{p-2}\left|\frac{1}{|\varphi_a(z)|^2}-1\right|^s dA(z)\\
\lesssim&\int_{\Omega_\epsilon(F)} \frac{\omega^p(|1-|z|^2|)}{|1-|z|^2|^2}\left|\frac{1}{|\varphi_a(z)|^2}-1\right|^s dA(z)\\
\lesssim&\int_{\Omega_\epsilon(F)} \frac{\omega^p(|z|^2-1)}{(|z|^2-1)^2}(|\varphi_a(z)|^2-1)^s dA(z)<\infty.
\end{aligned}$$

The proof is completed. □

Corollary 2. Let $0 < s < 1$, $p + s > 2$ and ω be a regular majorant. The $C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p-2, s))$ has the K-property.

Proof. Let $f \in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p-2, s))$, $\varphi \in H^\infty$ and g be the orthogonal projection of $f\overline{\varphi}$ onto H^2. Then $f\overline{\varphi} = g + \overline{j}$, where $j \in H^2_0$. By pseudoanalytic extension, similar to Corollary 1, there are functions G, F, Φ, J on $\mathbb{D}_e$ with

$$G = g,\ \ F = f,\ \ \Phi = \overline{\varphi},\ \ J = \overline{j},\ \ on\ \partial\mathbb{D},$$

such that $F\Phi = G + J$. Thus,

$$|\bar\partial G(z)| \le \|\varphi\|_\infty |\bar\partial F(z)|,\ \ z \in \mathbb{D}_e.$$

Combined with Theorem 2, we have

$$\begin{aligned}
&\int_{\{z\in\mathbb{D}_e:\frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial G(z)|\ge\epsilon\}} \frac{\omega^p(|z|^2-1)}{(|z|^2-1)^2}(|\varphi_a(z)|^2-1)^s dA(z)\\
\lesssim&\int_{\{z\in\mathbb{D}_e:\frac{(|z|^2-1)}{\omega(|z|^2-1)}|\bar\partial F(z)|\ge\frac{\epsilon}{\|\varphi\|_\infty}\}} \frac{\omega^p(|z|^2-1)}{(|z|^2-1)^2}(|\varphi_a(z)|^2-1)^s dA(z)<\infty.
\end{aligned}$$

That is $g \in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p-2, s))$. The proof is completed. □

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