A Closure Operator for the Digital Plane

Josef Šlapal

IT4Innovations Centre of Excellence, Brno University of Technology, Božetěchova 2, 612 66 Brno, Czech Republic

Abstract. We introduce and study a closure operator on the digital plane $\mathbb{Z}^2$. The closure operator is shown to provide connectedness that allows for a digital analogue of the Jordan curve theorem. This enables using the closure operator for structuring the digital plane in order to study and process digital images. An advantage of the closure operator over the Khalimsky topology on $\mathbb{Z}^2$ is demonstrated, too.

1. Introduction

Since the digital images are digital approximations of the real ones, to be able to study them, we need to equip the digital plane $\mathbb{Z}^2$ with a structure that would behave analogously to the Euclidean topology on the real plane. In particular, such a structure is required to satisfy a digital analogue of the Jordan curve theorem (recall that the Jordan curve theorem states that every simple closed curve in the Euclidean plane separates this plane into precisely two connected components). In digital images, the digital simple closed curves satisfying a digital analogue of the Jordan curve theorem (the so-called Jordan curves) represent borders of imaged objects. The classical, graph-theoretic approach to the problem of providing the digital plane with a convenient structure is based on using the well-known binary relations of 4-adjacency and 8-adjacency on $\mathbb{Z}^2$ - cf. [7–9, 12]. Since neither of the two binary relations itself allows for an analogue of the Jordan curve theorem, a combination of them has to be used - one for Jordan curves and the other for their complements. To eliminate this disadvantage, a topological approach to the problem was proposed in [5] that is based on employing the so-called Khalimsky topology as the basic structure on the digital plane for the study of digital images. The topological approach was then developed by many authors - see, e.g., [2, 6, 11].

In [10], a combination of the classical and topological approaches was used to obtain connectedness structures on $\mathbb{Z}^2$, namely closure operators induced by sets of walks in a graph with the vertex set $\mathbb{Z}^2$. In the present note, we use solely the topological approach. We directly define a closure operator on $\mathbb{Z}^2$ and show that it allows for an analogue of the Jordan curve theorem. This enables the operator to be used for studying and processing digital images. We also show that the closure operator defined provides a richer variety of Jordan curves than the Khalimsky topology.

2. Preliminaries

By a *closure operator c* on a set X, we mean a map $c \colon \exp X \to \exp X$ (where $\exp X$ denotes the power set of X) which is

(i) grounded (i.e., $c\emptyset = \emptyset$),

(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq cA$), and

(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow cA \subseteq cB$).

The pair (X, c) is then called a *closure space* and every subset $A \subseteq X$ with $cA = A$ is said to be *closed*. A subset $A \subseteq X$ is said to be *open* if its complement $X - A$ is closed. Closure spaces were studied by E. Čech in [1] (who called them topological spaces).

A closure operator c on X that is

(iv) additive (i.e., $c(A \cup B) = cA \cup cB$ whenever $A, B \subseteq X$) and

(v) idempotent (i.e., $ccA = cA$ whenever $A \subseteq X$)

is called a *Kuratowski closure operator* or a *topology* and the pair (X, c) is called a *topological space*.

Given a cardinal $n > 1$, a closure operator c on a set X and the closure space (X, c) are called an S_n-*closure operator* and an S_n-*closure space* (briefly, an S_n-*space*), respectively, if the following condition is satisfied:

$A \subseteq X \Rightarrow cA = \bigcup \{cB; B \subseteq A, \operatorname{card} B < n\}$.

S_2-topologies (S_2-topological spaces) are called *Alexandroff topologies* (*Alexandroff spaces*). Of course, any S_2-closure operator is additive and any S_n-closure operator is an S_m-closure operator whenever $n < m$. Evidently, if $n \leq \aleph_0$, then any additive S_n-closure operator is an S_2-closure operator. We will use the fact that every S_n-closure operator c on a set X is given by determining cA for all non-empty subsets $A \subseteq X$ with $\operatorname{card} A < n$.

For closure spaces, we use some concepts that are natural extensions of certain basic concepts known for topological spaces - see, e.g., [3]. In particular, a closure space (Y, d) is said to be a *subspace* of a closure space (X, c) if $dA = cA \cap Y$ for every subset $A \subseteq X$. We then say that Y is a subspace of (X, c). Further, a closure space (X, c) is *connected* if $\emptyset$ and X are the only subsets of X that are both closed and open. Given a closure space (X, c), a subset $A \subseteq X$ is said to be *connected* in (X, c) if A is a connected subspace of (X, c). And a *component* of a closure space (X, c) is a maximal (with respect to set inclusion) connected subset of the space. We will employ the obvious fact that the union of a sequence of connected subsets of a closure space is connected if every pair of consecutive terms of the sequence has a nonempty intersection.

If (X, c) is a closure space and $A \subseteq X$ a subset such that the subspace $X - A$ has exactly two components (say B and C), then A is said to *separate* (X, c) into precisely two components (namely, B and C).

We will work with some basic graph-theoretic concepts only - we refer to [4] for them. By a *graph* we understand an undirected simple graph without loops, i.e., a pair $G = (V, E)$ where V is set and $E \subseteq \{\{x, y\}; \ x, y \in V, \ x \neq y\}$. The elements of V are called *vertices* of G and those of E are called *edges* of G. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. Recall that a *walk* in G is a (finite) sequence $(x_0, x_1, ..., x_n)$, n a non-negative integer, of vertices of G such that, for every $i \in \{0, 1, ..., n-1\}$, x_i and x_{i+1} are adjacent. If, moreover, the vertices $x_0, x_1, ..., x_n$ are pair-wise different, then $(x_0, x_1, ..., x_n)$ is called a *path*. A sequence $(x_0, x_1, ..., x_n)$ of vertices of a graph G is called a *circle* in G if $n > 2$, $x_0 = x_n$, and $(x_0, x_1, ..., x_{n-1})$ is a path in G.

We will employ also simple directed graphs without loops that will be called *directed graphs* for short. Thus, a *directed graph* is a pair $G = (V, E)$ where, unlike the (undirected) graphs, $E \subseteq \{(x, y) \in V \times V; \ x \neq y\}$. If x and y are vertices of G, i.e., elements of V, we say that there is an edge from x to y if $(x, y) \in E$. If (V, E) is a directed graph, then its *symmetrization* is the (undirected) graph (V, E') where $E' = \{\{x, y\}; \ (x, y) \in E$ or $(y, x) \in E\}$.

Definition 2.1. Let c be a closure operator on $\mathbb{Z}^2$. A circle $(z_0, z_1, ..., z_n)$ in a graph G with the vertex set $\mathbb{Z}^2$ is said to be a *simple closed curve* in G with respect to c if it is a minimal (with respect to set inclusion) circle in G that is a connected subset of $(\mathbb{Z}^2, c)$, i.e., if, for every circle $(t_0, t_1, ..., t_m)$ in G with $\{t_0, t_1, ..., t_{m-1}\} \subseteq \{z_0, z_1, ..., z_{n-1}\}$, we have $\{t_0, t_1, ..., t_{m-1}\} = \{z_0, z_1, ..., z_{n-1}\}$ or $\{t_0, t_1, ..., t_{m-1}\}$ is not a connected subset of $(\mathbb{Z}^2, c)$. A simple closed curve in G with respect to c is called a *Jordan curve* (with respect to c) if it separates the space $(\mathbb{Z}^2, c)$ into exactly two components.
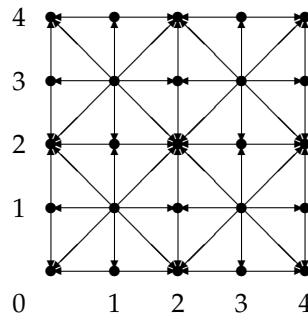
Figure 1: The Khalimsky topology on $\mathbb{Z}^2$.

Let $z = (x, y) \in \mathbb{Z}^2$ be a point. We put

$H(z) = \{(x + i, y); \ i \in \{-1, 0, 1\}\}$,
$V(z) = \{(x, y + i); \ i \in \{-1, 0, 1\}\}$,
$D(z) = \{(x + i, y + i); \ i \in \{-1, 0, 1\}\}$,
$D'(z) = \{(x + i, y - i); \ i \in \{-1, 0, 1\}\}$.

Next, we put

$A(z) = H(z) \cup V(z) \cup D(z) \cup D'(z))$.

In the literature, the points of $H(z) \cup V(z)$ and $A(z)$ different from z are said to be *4-adjacent* and *8-adjacent* to z, respectively. In this note, for all $z \in \mathbb{Z}^2$, each of the sets $H(z)$, $V(z)$, $D(z)$, $D'(z)$ will be called a *basic segment*. Note that basic segments may be considered to be digital (three-element) line segments (where $H(z)$ is oriented horizontally, $V(z)$ is oriented vertically, and $D(z)$ and $D'(z)$ are oriented diagonally in $\mathbb{Z}^2$).

For any $z = (x, y) \in \mathbb{Z}^2$, we put

$$v\{z\} = \begin{cases} A(z) & \text{if } x, y \text{ are odd,} \\ H(z) & \text{if } x \text{ is odd and } y \text{ is even,} \\ V(z) & \text{if } x \text{ is even and } y \text{ is odd,} \\ \{z\} & \text{otherwise.} \end{cases}$$

Evidently, $z \in v\{z\}$ and the conjunction of $u \in v\{t\}$ and $z \in v\{u\}$ implies $z \in v\{t\}$ whenever $z, u, t \in \mathbb{Z}^2$. Because of this fact, it is easy to see that, putting $v\emptyset = \emptyset$ and $vB = \bigcup_{z \in B} v\{z\}$ whenever $\emptyset \neq B \subseteq \mathbb{Z}^2$, we get an Alexandroff topology v on $\mathbb{Z}^2$. This topology is simply the *Khalimsky topology* discussed [5], i.e., the topology on $\mathbb{Z}^2$ obtained as the topological product of two copies of the topology on $\mathbb{Z}$ generated by the subbase (of open sets) $\{\{2k - 1, 2k, 2k + 1\}; \ k \in \mathbb{Z}\}$. The Alexandroff topological space $(\mathbb{Z}^2, v)$ is called the *Khalimsky plane*. The Khalimsky topology v is demonstrated in Figure 1. For any point $z \in \mathbb{Z}^2$, a point $t \in \mathbb{Z}^2, t \neq z$, belongs to $v\{z\}$ if and only if there is an edge from z to t in the directed graph a section of which is displayed in Figure 1. The directed graph shows that the Khalimsky plane is connected.

We denote by K the *connectedness graph* of the Khalimsky topology, i.e., the graph with the vertex set $\mathbb{Z}^2$ such that, for all $z, t \in \mathbb{Z}^2$, z and t are adjacent if and only if they are different and $\{z, t\}$ is a connected subset of the Khalimsky plane. Of course, K is the symmetrization of the graph shown in Figure 1.

The following statement (Jordan curve theorem for the Khalimsky plane) immediately follows from [5]:

Theorem 2.2. *Every simple closed curve C in the graph K having at least four points is a Jordan curve with respect to the Khalimsky topology v and has the property that its union with any of the two components of the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, v)$ is connected.*

It is evident that a circle C in the graph K is a simple closed curve with respect to the Khalimsky topology if and only if, for every point $z \in C$, C has exactly two points adjacent to z. Hence, simple closed curves
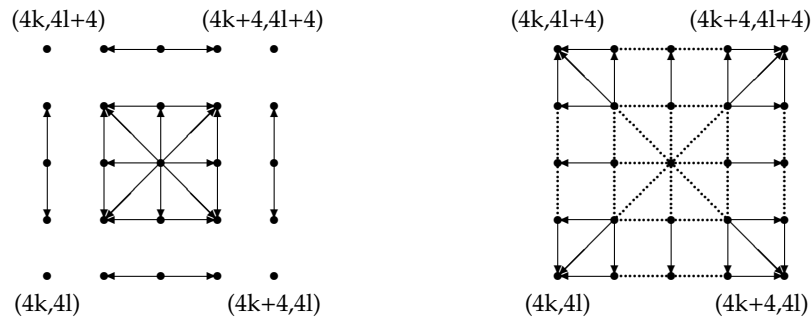
Figure 2: The closure operator w.

with respect to the Khalimsky topology may never turn at the acute angle $\frac{\pi}{4}$ (in fact, they may turn only at the points whose coordinates have the same parity). The Jordan curves with respect to the Khalimsky topology v determined in Theorem 2.2 will be briefly called Jordan curves in the Khalimsky plane $(\mathbb{Z}^2, v)$. It is a disadvantage of the Khalimsky topology that Jordan curves in the Khalimsky plane may not turn at the acute angle $\frac{\pi}{4}$. Therefore, we will introduce a closure operator on $\mathbb{Z}^2$ which does not have this disadvantage, i.e., which allows the Jordan curves in a graph with the vertex set $\mathbb{Z}^2$ to turn, at some points, at the acute angle $\frac{\pi}{4}$.

3. An S_3-Closure Operator on $\mathbb{Z}^2$

For every point $z \in \mathbb{Z}^2$, we put

$$w\{z\} = \begin{cases} H(z) \text{ if } z = (4k+2, y) \text{ where } k \in \mathbb{Z} \text{ and } y \neq 4l+2 \text{ for every } l \in \mathbb{Z}, \\ V(z) \text{ if } z = (x, 4l+2) \text{ where } l \in \mathbb{Z} \text{ and } x \neq 4k+2 \text{ for every } k \in \mathbb{Z}, \\ A(z) \text{ if } z = (4k+2, 4l+2), \ k, l \in \mathbb{Z}, \\ \{z\} \text{ otherwise} \end{cases}$$

and, for every two-element subset $\{z, t\} \subseteq \mathbb{Z}^2$, we put $w\{z, t\} = w\{z\} \cup w\{t\} \cup \langle z, t \rangle$, where

$$\langle z, t \rangle = \begin{cases} H(z) \text{ if } z = (4k+2+i, l) \text{ and } t = (4k+2, l), \ k, l \in \mathbb{Z}, \ i \in \{-1, 1\}, \\ V(z) \text{ if } z = (k, 4l+2+i) \text{ and } t = (k, 4l+2), \ k, l \in \mathbb{Z}, \ i \in \{-1, 1\}, \\ D(z) \text{ if } z = (4k+2+i, 4l+2+i) \text{ and } t = (4k+2, 4l+2), \ k, l \in \mathbb{Z}, \ i \in \{-1, 1\}, \\ D'(z) \text{ if } z = (4k+2+i, 4l+2-i) \text{ and } t = (4k+2, 4l+2), \ k, l \in \mathbb{Z}, \ i \in \{-1, 1\}, \\ \{z, t\} \text{ otherwise.} \end{cases}$$

Evidently, we have $B \subseteq wB$ and $B \subseteq D \Rightarrow wB \subseteq wD$ whenever $B, D \subseteq \mathbb{Z}^2$ are subsets with $\text{card} B, \text{card} D < 3$. Because of this fact, it is easy to see that, putting $v\emptyset = \emptyset$ and $vB = \bigcup\{wD; \ D \subseteq B, \ \text{card} D < 3\}$ whenever $\emptyset \neq B \subseteq \mathbb{Z}^2$, we get an S_3-closure operator w on $\mathbb{Z}^2$.

The closure operator w is demonstrated in Figure 2. For any point $z \in \mathbb{Z}^2$, a point $u \in \mathbb{Z}^2$, $u \neq z$, belongs to $w\{z\}$ if and only if there is an edge from z to u in the directed graph demonstrated in the left part of Figure 2. If $\{z, t\} \subseteq \mathbb{Z}^2$ is a two-element subset, then a point $u \in \mathbb{Z}^2$ with $u \notin w\{z\} \cup w\{t\}$ belongs to $w\{z, t\}$ if and only if, in the directed graph demonstrated in the right part of Figure 2, z and t are the end points of a dotted line segment containing no other point of $\mathbb{Z}^2$ (the dotted line segments are not edges of the graph) and there is an edge from z or t to u such that the points z, t, u lie on a line (so that the set $\{z, t, u\}$ is a basic segment with $t \in w\{z\}$ or $z \in w\{t\}$ - cf. the directed graph in the left part of the figure).

A sequence S of pair-wise different points of $\mathbb{Z}^2$ is called a *w-connected element* if

(1) $S = (z_0, z_1)$ where $z_1 \in w\{z_0\}$ or $z_0 \in w\{z_1\}$ or

(2) $S = (z_0, z_1, z_2)$ where

     (a) $z_1 \in w\{z_0\}$ and $z_2 \in w\{z_0, z_1\}$ or

     (b) $z_1 \in w\{z_2\}$ and $z_0 \in w\{z_1, z_2\}$.

It is evident that every w-connected element is a connected subset of $(\mathbb{Z}^2, w)$.

Definition 3.1. A sequence $(z_0, z_1, ..., z_n)$ of points of $\mathbb{Z}^2$ is called a w-*walk* if there is an increasing sequence $(i_0, i_1, ..., i_m)$ of non-negative integers with $i_0 = 0$ and $i_m = n$ such that $(z_i| \ i_j \leq i \leq i_{j+1})$ is a w-connected element for every $j \in \{0, 1, ..., m-1\}$.

Note that every one-term sequence (with the term being a point of $\mathbb{Z}^2$) is a w-walk.

Example 3.2. The sequence $(z_0, z_1, ..., z_6) = ((1, 1), (2, 1), (2, 2), (3, 3), (4, 4), (3, 4), (2, 4))$ is a w-walk because the sequence $(i_0, i_1, i_2, i_3, i_4) = (0, 1, 2, 4, 6)$ has the property that $(z_i| \ i_j \leq i \leq i_{j+1})$ is a w-connected element for every $j \in \{0, 1, 2, 3\}$ - see Figure 2.

Lemma 3.3. *A subset $B \subseteq \mathbb{Z}^2$ is connected in $(\mathbb{Z}^2, w)$ if and only if every pair of points $z, t \in B$ can be joined by a w-walk contained in B (i.e., there is a w-walk $(z_0, z_1, ..., z_n)$ such that $z_0 = z, z_n = t$ and $z_i \in B$ for every $i \in \{0, 1, ..., n\}$).*

Proof. If $B = \emptyset$, then the statement is trivial. Let $B \neq \emptyset$. If any two vertices from B can be joined by a w-walk contained in B, then B is clearly connected in $(\mathbb{Z}^2, w)$. Conversely, let B be connected in $(\mathbb{Z}^2, w)$ and suppose that there are vertices $z, t \in B$ which can not be joined by a w-walk contained in B. Let D be the set of all vertices from B which can be joined with z by a w-walk contained in B. Let $u \in wD \cap B$ be a vertex and assume that $u \notin D$. Then there is a point $s \in D$ with $u \in w\{s\}$ or there is a subset $\{s, r\} \subseteq D$ with $r \in w\{s\}$ and $u \in w\{s, r\}$. Thus, z and s can be joined by a w-walk $(z = z_0, z_1, ..., z_n = s)$ contained in B and also s and u can be joined by a w-walk contained in B, namely the w-connected element (s, u) or (s, r, u). It follows that z and u can be joined by the w-walk $(z = z_0, z_1, ..., z_n = s, u)$ or $(z = z_0, z_1, ..., z_n = s, r, u)$ contained in B, which is a contradiction. Therefore, $u \in D$, i.e., $wD \cap B = D$. Consequently, D is closed in the subspace B of $(\mathbb{Z}^2, w)$. Further, let $u \in w(B-D) \cap B$ be a vertex and assume that $u \in D$. Then $u \notin B-D$, hence there is a point $s \in B-D$ with $u \in w\{s\}$ or there is a subset $\{s, r\} \subseteq B - D$ with $r \in w\{s\}$ and $u \in w\{s, r\}$. We get a w-connected element (s, u) or (s, r, u). Since z can be joined with u by a w-walk $(z = z_0, z_1, ..., z_n = u)$ contained in B (because we have assumed that $u \in D$) and u can be joined with s by a w-walk contained in B, namely the w-connected element (u, s) or (u, r, s), z and s can be joined by the w-walk $(z = z_0, z_1, ..., z_n = u, s)$ or $(z = z_0, z_1, ..., z_n = u, r, s)$ contained in B. This is a contradiction with $s \notin D$. Thus, $u \notin D$, i.e., $w(B - D) \cap B = B - D$. Consequently, $B - D$ is closed in the subspace B of $(\mathbb{Z}^2, w)$. Hence, B is the union of the nonempty disjoint sets D and $B - D$ closed in the subspace B of $(\mathbb{Z}^2, w)$. But this is a contradiction because B is connected in $(\mathbb{Z}^2, w)$. Therefore, any two points of B can be joined by a w-walk contained in B. $\square$

We denote by H the graph with the vertex set $\mathbb{Z}^2$ such that, for all $z, t \in \mathbb{Z}^2$, z and t are adjacent in H if and only if they are different and one of the following two conditions is satisfied:

(1) $z \in w\{t\}$ or $t \in w\{z\}$,

(2) there is a point $u \in \mathbb{Z}^2$, $z \neq u \neq t$, such that either $z \in w\{u\}$ and $t \in w\{u, z\}$ or $t \in w\{u\}$ and $z \in w\{u, t\}$.

A section of the graph H is demonstrated in Figure 3.

Proposition 3.4. *The closure space $(\mathbb{Z}^2, w)$ is connected.*

Proof. It may easily be seen that every pair of points $z, t \in \mathbb{Z}^2$ may be joined by a path $(z_0, z_1, ..., z_n)$ in H. It is evident that every edge of H is an edge of the symmetrization of one of the two directed graphs demonstrated in Figure 2 (the set of edges of H is the union of the sets of edges of the symmetrizations of the two directed graphs). Clearly, for every $i \in \{0, 1, ..., n-1\}$, one of the following two conditions is satisfied:
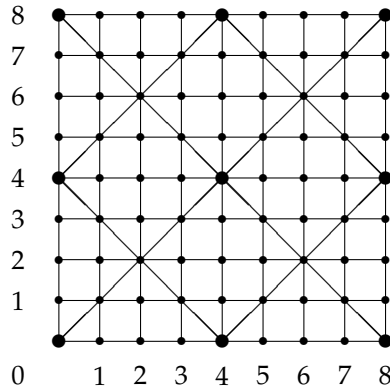
Figure 3: A section of the graph H.

  (i) (z_i, z_{i+1}) is a w-connected element,

 (ii) there is a w-connected element (t_0, t_1, t_2) such that $\{z_i, z_{i+1}\} = \{t_1, t_2\}$.

For every $i \in \{0, 1, ..., n-1\}$ satisfying condition (ii), in the sequence $(z_0, z_1, ..., z_n)$, we replace the subsequence (z_i, z_{i+1}) with the w-walk (t_1, t_0, t_1, t_2) if $(z_i, z_{i+1}) = (t_1, t_2)$ and with the w-walk (t_2, t_1, t_0, t_1) if $(z_i, z_{i+1}) = (t_2, t_1)$ (clearly, each of the two w-walks consists of two w-connected elements). Obviously, we obtain a w-walk joining z and t. Therefore, $(\mathbb{Z}^2, w)$ is connected by Lemma 3.3. □

Let $z = (x, y) \in \mathbb{Z}^2$ be a point such that $x = 4k + p$ and $y = 4l + q$ for some $k, l, p, q \in \mathbb{Z}$ with $p, q \in \{1, 3\}$. Then we define the *fundamental triangle T(z)* to be the subset of $\mathbb{Z}^2$ given as follows:

$$
T(z) = \begin{cases}
\{(r,s) \in \mathbb{Z}^2; 4k \le r \le 4k + 4, \ 4l \le s \le 4l + 4k + 4 - r\} & \text{if } x = 4k + 1, \ y = 4l + 1, \ k, l \in \mathbb{Z}, \\
\{(r,s) \in \mathbb{Z}^2; 4k \le r \le 4k + 4, \ 4l \le s \le 4l + r - 4k\} & \text{if } x = 4k + 3, \ y = 4l + 1, \ k, l \in \mathbb{Z}, \\
\{(r,s) \in \mathbb{Z}^2; 4k \le r \le 4l, \ 4l + 4k + 4 - r \le s \le 4l + 4\} & \text{if } x = 4k + 3, \ y = 4l + 3, \ k, l \in \mathbb{Z}, \\
\{(r,s) \in \mathbb{Z}^2; 4k \le r \le 4k + 4, \ 4l + r - 4k \le s \le 4l + 4\} & \text{if } x = 4k + 1, \ y = 4l + 3, \ k, l \in \mathbb{Z}.
\end{cases}
$$

Every fundamental triangle $T(z)$ consists of fifteen points and forms a digital right triangle obtained from a square with 5×5 points by dividing it by a diagonal. The (four types of) fundamental triangles $T(z)$ with the point z being marked by the bold dot are demonstrated (as subgraphs of the graph H) in the following figures:
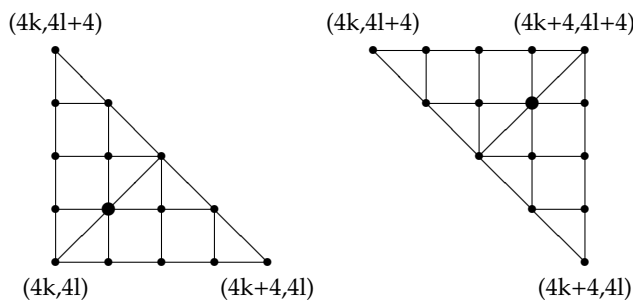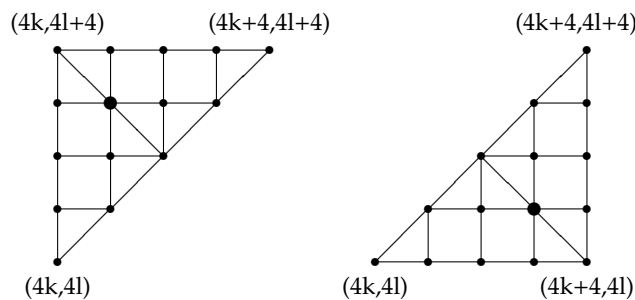
Note that each side of a fundamental triangle consists of five points and that two different fundamental triangles may have at most one side in common.

Remark 3.5. The following properties of fundamental triangles immediately follow from Lemma 3.3:

(1) Every fundamental triangle is connected (so that the union of two fundamental triangles having a common side is connected) in $(\mathbb{Z}^2, w)$.

(2) If we subtract from a fundamental triangle some of its sides, then the resulting set is still connected in $(\mathbb{Z}^2, w)$.

(3) If S_1, S_2 are fundamental triangles having a common side D, then the set $(S_1 \cup S_2) - M$ is connected in $(\mathbb{Z}^2, w)$ whenever M is the union of some sides of S_1 or S_2 different from D.

(4) Every connected subset of $\mathbb{Z}^2$ with at most two points is a subset of a fundamental triangle.

Lemma 3.6. *For every circle C in the graph H that turns only at some of the points $(4k, 4l)$, $k, l \in \mathbb{Z}$, there are sequences $\mathcal{S}_F, \mathcal{S}_I$ of fundamental triangles, $\mathcal{S}_F$ finite and $\mathcal{S}_I$ infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the following two conditions are satisfied:*

(a) *Each term of $\mathcal{S}$, excluding the first one, has a common side with at least one of its predecessors.*

(b) *C is the union of those sides of fundamental triangles in $\mathcal{S}$ that are not shared by two different fundamental triangles from $\mathcal{S}$.*

Proof. Put $C_1 = C$ and let S_1^1 be an arbitrary fundamental triangle with $S_1^1 \cap C_1 \neq \emptyset$. For every $k \in \mathbb{Z}$, $1 \leq k$, if $S_1^1, S_2^1, ..., S_k^1$ are defined, let S_{k+1}^1 be a fundamental triangle with the following properties: $S_{k+1}^1 \cap C_1 \neq \emptyset$, S_{k+1}^1 has a side in common with S_k^1 which is not a subset of C_1 and $S_{k+1}^1 \neq S_i^1$ for all i, $1 \leq i \leq k$. Clearly, there will always be a (smallest) number $k \geq 1$ for which no such fundamental triangle S_{k+1}^1 exists. Denoting by k_1 this number, we have defined a sequence $(S_1^1, S_2^1, ..., S_{k_1}^1)$ of fundamental triangles. Let C_2 be the union of those sides of fundamental triangles in $(S_1^1, S_2^1, ..., S_{k_1}^1)$ that are disjoint from C_1 and are not shared by two different fundamental triangles in $(S_1^1, S_2^1, ..., S_{k_1}^1)$. If $C_2 \neq \emptyset$, we construct a sequence $(S_1^2, S_2^2, ..., S_{k_2}^2)$ of fundamental triangles in an analogous way to $(S_1^1, S_2^1, ..., S_{k_1}^1)$ by taking C_2 instead of C_1 (and obtaining k_2 analogously to k_1). Repeating this construction, we get sequences $(S_1^3, S_2^3, ..., S_{k_3}^3)$, $(S_1^4, S_2^4, ..., S_{k_4}^1)$, etc. We put $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, S_1^3, S_2^3, ..., S_{k_3}^3, ...)$ if $C_i \neq \emptyset$ for all $i \geq 1$ and $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, ..., S_1^l, S_2^l, ..., S_{k_l}^l)$ if $C_i \neq \emptyset$ for all i with $1 \leq i \leq l$ and $C_i = \emptyset$ for $i = l + 1$.

Further, let $S_1' = T(z)$ be a fundamental triangle such that $z \notin S$ whenever S is a term of $\mathcal{S}$. Having defined S_1', let $\mathcal{S}' = (S_1', S_2', ...)$ be a sequence of fundamental triangles defined analogously to $\mathcal{S}$ (by taking S_1' instead of S_1^1). Then one of the sequences $\mathcal{S}$, $\mathcal{S}'$ is finite and the other is infinite. Indeed, $\mathcal{S}$ is finite (infinite) if and only if its first term equals such a fundamental triangle $T(z)$ for which $z = (k, l) \in \mathbb{Z}^2$ has the property that the cardinality of the set $\{(x, l) \in \mathbb{Z}^2; x > k\} \cap C$ is odd (even). The same is true for $\mathcal{S}'$. If we put $\{\mathcal{S}_F, \mathcal{S}_I\} = \{\mathcal{S}, \mathcal{S}'\}$ where $\mathcal{S}_F$ is finite and $\mathcal{S}_I$ is infinite, then the conditions (a) and (b) are clearly satisfied. □
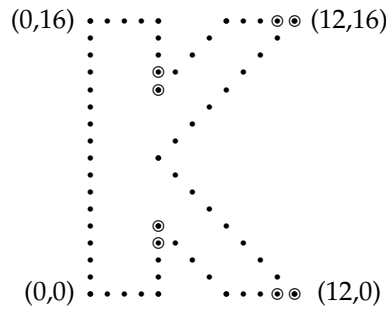
Now we are ready to prove the main result of this note:

Figure 4: A Jordan curve in $(\mathbb{Z}^2, w)$.

Theorem 3.7. *Every circle C in the graph H that turns only at some of the points $(4k, 4l)$, $k, l \in \mathbb{Z}$, is a Jordan curve with respect to the closure operator w and has the property that its union with any of the two components of the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, w)$ is connected.*

Proof. Clearly, every circle in the graph H that turns only at some of the points $(4k, 4l)$, $k, l \in \mathbb{Z}$, is a simple closed curve in $(\mathbb{Z}^2, w)$.

Let C be a circle in the graph H that turns only at some of the points $(4k, 4l)$, $k, l \in \mathbb{Z}$. By Lemma 3.6, there are sequences $\mathcal{S}_F$ and $\mathcal{S}_I$ of fundamental triangles, $\mathcal{S}_F$ finite and $\mathcal{S}_I$ infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the conditions (a) and (b) are satisfied. Let S_F and S_I denote the union of all terms of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Then $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$. Let $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$ be the sequences obtained from $\mathcal{S}_F$ and $\mathcal{S}_I$ by subtracting C from each term of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Let S_F^* and S_I^* denote the union of all terms of $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$, respectively. Then S_F^* and S_I^* are connected by Remark 3.5(1)-(3) and it is clear that $S_F^* = S_F - C$ and $S_I^* = S_I - C$. So, S_F^* and S_I^* are the components of $\mathbb{Z}^2 - C$ by Remark 3.5(4) ($S_F - C$ is called the *inside* component and $S_I - C$ is called the *outside* component). Thus, C is a Jordan curve with respect to the closure operator w. The rest of the statement is evident. $\square$

Jordan curves in the graph H with respect to the closure operator w will be briefly called Jordan curves in the closure space $(\mathbb{Z}^2, w)$. The possible turning points of the Jordan curves in the closure space $(\mathbb{Z}^2, w)$ determined in Theorem 3.7 are the points represented by bold dots in Figure 3. Thus, the Jordan curves in $(\mathbb{Z}^2, w)$ have at least 12 points and they may turn, at some points, at the acute angle $\frac{\pi}{4}$.

Example 3.8. Consider the subset of $\mathbb{Z}^2$ shown in Figure 4, which represents the border of letter K. This set is a circle C in the graph H satisfying the condition of Theorem 3.7 and, therefore, it is a Jordan curve in $(\mathbb{Z}^2, w)$. But the set is not a Jordan curve in the Khalimsky plane because C turns at the acute angle $\frac{\pi}{4}$ at some (precisely four) of its points. In order that C be a Jordan curve in the Khalimsky plane, we have to delete the eight points denoted by the ringed dots. But this would cause a considerable deformation of C.

4. Conclusion

We have found a structure on the digital plane $\mathbb{Z}^2$, the closure operator w, which provides the plane with a connectedness allowing for a digital analogue of the Jordan curve theorem. This means that the closure operator w may be used as a background structure on the digital plane for the study and processing of digital pictures. An advantage of the Jordan curves in the closure space $(\mathbb{Z}^2, w)$ over the Jordan curves in the Khalimsky plane is that they may turn, at some points, at the acute angle $\frac{\pi}{4}$. Thus, the closure operator w provides a richer variety of Jordan curves than the Khalimsky topology. In then forthcoming research, we will focus on extending the closure operator w onto the digital space $\mathbb{Z}^3$ so that a digital analogue of the Jordan surface theorem (also known as 3D Jordan-Brouwer separation theorem) can be proved.

References

[1] E. Čech, Topological Spaces. In: Topological Papers of Eduard Čech. Academia, Prague, 1968, ch. 28, 436–472.

[2] U. Eckhardt, L.J. Latecki, Topologies for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$, Comput. Vision Image Understanding 90 (2003) 295–312.

[3] R. Engelking, General Topology, Państwowe Wydawnictwo Naukowe, Warszawa (1977).

[4] F. Harrary, Graph Theory, Addison-Wesley Publ. Comp., Reading, Massachussets, Menlo Park, California, London, Don Mills, Ontario (1969).

[5] E.D. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology Appl. 36 (1990) 1–17.

[6] C.O. Kiselman, Digital Jordan curve theorems, In: Borgefors, G., Nyström, I., Baja, G.S. (eds.), Discrete Geometry for Computer Imagery, Lect. Notes Comput. Sci., vol.1953, pp. 46–56. Springer, Heidelberg (2000).

[7] T.Y. Kong, W. Roscoe, A theory of binary digital pictures, Comput. Vision Graphics Image Process. 32 (1985) 221-243.

[8] T.Y. Kong, A. Rosenfeld, Digital topology: Introduction and survey, Comput. Vision Graphics Image Process 48 (1989) 357–393.

[9] A. Rosenfeld, Picture Languages, Academic Press, New York (1979).

[10] J. Šlapal, Closure operators on graphs for modeling connectedness in digital spaces, Filomat 32 (2018) 5011–5021.

[11] J. Šlapal, Jordan curve theorems with respect to certain pretopologies on $\mathbb{Z}^2$, In: Brlek, S., Reutenauer, Ch., Provencal, X. (eds.), Discrete Geometry for Computer Imagery. Lect. Notes Comput. Sci., vol. 5810, pp. 252–262. Springer, Heidelberg (2009).

[12] J. Šlapal, Graphs with a path partition for structuring digital spaces, Inform. Sci. 233 (2013) 305–312.