Filomat 34:10 (2020), 3217–3227
https://doi.org/10.2298/FIL2010217G

Published by Faculty of Sciences and Mathematics, University of Niεs, Serbia
Availableat :http : //www.pmf.ni.ac.rs/filomat

Enveloping Actions and Duality Theorems for Partial Twisted Smash Products

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Abstract. In this paper, we first generalize the theorem about the existence of an enveloping action to a partial twisted smash product. Then we construct a Morita context between the partial twisted smash product and the twisted smash product related to the enveloping action. Finally, we present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products.

1. Introduction

Partial group actions were considered first by Exel in the context of operator algebras and they turned out to be a powerful tool in the study of C^*-algebras generated by partial isometries on a Hilbert space in [11]. A treatment from a purely algebraic point of view was given recently in [8], [9], [10]. In particular, the algebraic study of partial actions and partial representations was initiated in [9] and [10], motivating investigations in diverse directions. Now, the results are formulated in a purely algebraic way independent of the C^*-algebraic techniques which originated them.

The concepts of partial actions and partial coactions of Hopf algebras on algebras were introduced by Caenepeel and Janssen in [6]. In which they put the Galois theory for partial group actions on rings into a broader context, namely, the partial entwining structures. In particular, partial actions of a group G determine partial actions of the group algebra kG in a natural way. Further developments in the theory of partial Hopf actions were done by Lomp in [14].

Alves and Batista extended several results from the theory of partial group actions to the Hopf algebra setting, they constructed a Morita context relating the fixed point subalgebra for partial actions of finite dimensional Hopf algebras, and constructed the partial smash product in [1]. Later, they constructed a Morita context between the partial smash product and the smash product related to the enveloping action, defined partial representations of Hopf algebras and showed some results relating partial actions and partial representations in [2].

2010 *Mathematics Subject Classification*. 16T05; 16T15
Keywords. Enveloping action, partial twisted smash product, Morita context, duality theorem.
Received: 17 October 2019; Accepted: 10 August 2020
Communicated by Dragana Cvetković-Ilić
Research supported by the NSF of China (No. 11761017, 11901245) and the Guizhou Provincial Science and Technology Foundation (No. [2020]1Y005), the Youth Project for Natural Science Foundation of Guizhou provincial department of education (No. KY[2018]155) and the Innovative exploration and academic seedling project of Guizhou University of Finance and Economics (No. [2017]5736-023).
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Furthermore, they proved a dual version of the globalization theorem: every partial coaction of a Hopf algebra admits an enveloping coaction. They explored some consequences of globalization theorems in order to present versions of the duality theorems of Cohen-Montgomery and Blattner-Montgomery for partial Hopf actions in [3]. Recently, they introduced partial representations of Hopf algebras and gave the paradigmatic examples of them, namely, the partial representation defined from a partial action and the partial representation related to the partial smash product in [5]. Alves, Batista, Dokuchaev and Paques introduced the notion of a twisted partial Hopf action as a unified approach for twisted partial group actions, partial Hopf actions and twisted actions of Hopf algebras, they established the conditions on partial cocycles in order to construct partial crossed products, and explored the relations between partial crossed products with partial cleft extensions of algebras in [4]. Chen and Wang introduced the twisted partial coactions of Hopf algebras and studied their properties in [7]. Recently, the first author introduced the notion of partial representation of partial twisted smash products and explored its relationship with partial actions of Hopf algebras in [13].

Therefore, one is prompted to ask several questions:

• How can we give an enveloping action for such a partial twisted smash product ?

• How can we construct a Morita context between the partial twisted smash product $\underline{A \circledast H}$ and the twisted smash product $B \circledast H$?

• How can we explore some consequences of globalization theorems in order to present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products?

The aim of this paper is to answer these questions.

In Section 3, we prove the existence of an enveloping action for such a partial twisted smash product. In Section 4, we construct a Morita context between the partial twisted smash product $\underline{A \circledast H}$ and the twisted smash product $B \circledast H$, where H is a Hopf algebra which acts partially on the unital algebra A, B is an enveloping action for partial actions. This result can also be found in [2] for the context of partial group actions. In Section 5, we explore some consequences of globalization theorems in order to present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products.

2. Preliminaries

Throughout the paper, let k be a fixed field and all algebraic systems are supposed to be over k. Let M be a vector space over k and let id_M the usual identity map. For the comultiplication Δ in a coalgebra C with a counit ε_C, we use the Sweedler-Heyneman's notation (see Sweedler [15]): $\Delta(c) = c_{(1)} \otimes c_{(2)}$, for any $c \in C$.

We first recall some basic results and propositions that we will need later from [1],[2] and [13].

2.1. Left module algebra Let H be a Hopf algerba and B an algebra. B is said to be a left H-module algebra if there exists a k-linear map $\triangleright = \{\triangleright : H \otimes B \to B\}$ satisfying the following conditions:

$$h \triangleright (ab) = (h_{(1)}) \triangleright a)(h_{(2)} \triangleright b),$$
$$1_H \triangleright a = a,$$
$$h \triangleright (g \triangleright a) = hg \triangleright a,$$

for all $h, g \in H$ and $a, b \in B$.

2.2. Partial left module algebra Let H be a Hopf algerba and A an algebra. A is said to be a partial left H-module algebra if there exists a k-linear map $\rightharpoonup = \{\rightharpoonup : H \otimes A \to A\}$ satisfying the following conditions:

$$h \rightharpoonup (ab) = (h_{(1)}) \rightharpoonup a)(h_{(2)} \rightharpoonup b),$$
$$1_H \rightharpoonup a = a,$$
$$h \rightharpoonup (g \rightharpoonup a) = (h_{(1)} \rightharpoonup 1_A)(h_{(2)}g \rightharpoonup a),$$

for all $h, g \in H$ and $a, b \in A$.

2.3. Right module algebra Let H be a Hopf algebra and B an algebra, B is said to be a right H-module algebra if there exists a k-linear map $\triangleleft = \{\triangleleft : B \otimes H \to B\}$ satisfying the following conditions:

$$(ab) \triangleleft h = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)}),$$
$$a \triangleleft 1_H = a,$$
$$(a \triangleleft g) \triangleleft h = a \triangleleft gh,$$

for all $h, g \in H$ and $a, b \in B$.

2.4. Partial right module algebra Let H be a Hopf algebra and A an algebra, A is said to be a partial right H-module algebra if there exists a k-linear map $\leftharpoonup = \{\leftharpoonup : A \otimes H \to A\}$ satisfying the following conditions:

$$(ab) \leftharpoonup h = (a \leftharpoonup h_{(1)})(b \leftharpoonup h_{(2)}),$$
$$a \leftharpoonup 1_H = a,$$
$$(a \leftharpoonup g) \leftharpoonup h = (1_A \leftharpoonup h_{(1)})(a \leftharpoonup gh_{(2)}),$$

for all $h, g \in H$ and $a, b \in A$.

2.5. H-bimodule algebra Let H be a Hopf algebra and B an algebra. B is called a H-bimodule algebra if the following conditions hold:

(i) B is not only a left H-module algebra with the left module action $\triangleright$, but also a right H-module algebra with the right module action $\triangleleft$.

(ii) These two module structure maps satisfy the compatibility condition, i.e., $(h \triangleright a) \triangleleft g = h \triangleright (a \triangleleft g)$ for all $a \in B$ and $h, g \in H$.

2.6. Partial H-bimodule algebra Let H be a Hopf algebra and A an algebra. A is called a partial H-bimodule algebra if the following conditions hold:

(i) A is not only a partial left H-module algebra with the partial left module action $\rightharpoonup$, but also a partial right H-module algebra with the partial right module action $\leftharpoonup$.

(ii) These two partial module structure maps satisfy the compatibility condition, i.e., $(h \rightharpoonup a) \leftharpoonup g = h \rightharpoonup (a \leftharpoonup g)$ for all $a \in A$ and $h, g \in H$.

3. Enveloping actions

In the context of partial actions of Hopf algebras, it is proved that a partial action of a Hopf algebra on a unital algebra A admits an enveloping action (B, θ) if and only if each of the ideals $\theta(A) \trianglerighteq B$ is a unital algebra in [2]. In this section, we mainly extend this famous result to partial twisted smash products.

Recall from [13] that let H be a Hopf algebra with an antipode S and A a partial H-bimodule algebra. We first propose a multiplication on the vector space $A \otimes H$:

$$(a \circledast h)(b \circledast g) = a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \circledast h_{(2)}g,$$

for all $a, c \in A$ and $g, h \in H$. It is obvious that the multiplication is associative. In order to make it to be an unital algebra, we project onto the

$$\underline{A \circledast H} = (A \otimes H)(1_A \otimes 1_H).$$

Then we can deduce directly the form and the properties of typical elements of this algebra

$$\underline{a \circledast h} = a(h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)})) \otimes h_{(2)},$$

and finally verify that the product among typical elements satisfy

$$(\underline{a \circledast h})(\underline{b \circledast g}) = \underline{a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \circledast h_{(2)}g}, \tag{3. 1}$$

for all $h, g \in H$ and $a, b \in A$. Then $\underline{A \circledast H}$ is an associative algebra with a multiplication given by Eq.(3.1) and with the unit $\underline{1_A \circledast 1_H}$, and call it by a partial twisted smash product, where 1_A is the unit of A.

Example 3.1. *As a k-algebra, the four dimensional Hopf algebra H_4 is generated by two symbols c and x which satisfy the relations $c^2 = 1$, $x^2 = 0$ and $xc + cx = 0$. The coalgebra structure on H_4 is determined by*

$$\Delta(c) = c \otimes c,\ \ \Delta(x) = x \otimes 1 + c \otimes x,\ \varepsilon(c) = 1,\ \varepsilon(x) = 0.$$

Then H_4 has the basis l (identity), c, x, cx. We now consider the dual H_4^* of H_4. We have $H_4 \cong H_4^*$ (as Hopf algebras) via

$$1 \mapsto 1^* + c^*,\ \ c \mapsto 1^* - c^*,\ \ x \mapsto x^* + (cx)^*,\ \ cx \mapsto x^* - (cx)^*.$$

Here $\{1^*, c^*, x^*, (cx)^*\}$ denote the dual basis of $\{1, c, x, cx\}$. Let $T = 1^* - c^*$, $P = x^* + (cx)^*$, $TP = x^* - (cx)^*$. We get another basis $\{1, T, P, TP\}$ of H_4^*. Recall from [6] if A is the subalgebra $k[x]$ of H_4, it is shown that A is a right partial H_4-comodule algebra with the coaction

$$\rho(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes c + 1 \otimes cx),\ \rho^r(x) = \frac{1}{2}(x \otimes 1 + x \otimes c + x \otimes cx).$$

By similar way we can show that A is a left partial H_4-comodule algebra with the coaction

$$\rho(1) = \frac{1}{2}(1 \otimes 1 + c \otimes 1 + cx \otimes 1),\ \ \rho^l(x) = \frac{1}{2}(1 \otimes x + c \otimes x + cx \otimes x).$$

It is clear that A is a partial H_4-bicomodule algebra. Then A is a partial H_4^*-bimodule algebra via

$$f \rightharpoonup a = \sum < f, a_{[1]} > a_{[0]},\ a \leftharpoonup g =< g, a_{[-1]} > a_{[0]},\ a \in A, f, g \in H^*.$$

Therefore we can obtain the partial twisted smash product $k[x] \circledast H_4^*$, we only consider the elements P, T of H_4^* as follows, then

$$\begin{aligned}(x \otimes T)(x \otimes P) &= x(T_{(1)} \rightharpoonup x \leftharpoonup S^*(T_{(3)})) \otimes T_{(2)}P \\ &= \sum x(T \rightharpoonup x \leftharpoonup S^*(T)) \otimes TP \\ &= \sum x < T, \frac{1}{2}(1 + c + cx) > x < T, \frac{1}{2}(1 + c + cx) > \otimes TP = 0.\end{aligned}$$

Definition 3.2. *Let H be a Hopf algebra and A, B be two partial H-bimodule algebras. A morphism of algebras $\theta : A \to B$ is said to be a morphism of partial H-bimodule algebras if $\theta(h \rightharpoonup a \leftharpoonup k) = h \rightharpoonup \theta(a) \leftharpoonup k$ for all $h, k \in H$ and $a \in A$. If, in addition, θ is an isomorphism, the partial actions are called equivalent.*

Motivated by the ideas of [2], we have the following lemmas of partial H-bimodule algebras.

Lemma 3.3. *Let H be a Hopf algebra, B a H-bimodule algebra and A an ideal of B with unity 1_A. Then H acts partially on A by $h \rightharpoonup a = 1_A(h \triangleright a), a \leftharpoonup h = (a \triangleleft h)1_A$, for all $a \in A, b \in B$ and $h \in H$.*

Lemma 3.4. *Let H be a Hopf algebra and A an algebra. Then $(A, \rightharpoonup, \leftharpoonup)$ is a partial H-bimodule algebra.*

Recall from [2] that if B is an H-module algebra and A is a right ideal of B with unity 1_A, the induced partial action on A is called admissible if $B = H \triangleright A$.

Definition 3.5. *Let H be a Hopf algebra, B an H-bimodule algebra and A an ideal of B with unit 1_A. The induced partial actions on A is called admissible if $B = H \triangleright A \triangleleft H$.*

Definition 3.6. *Let A be a partial H-bimodule algebra. An enveloping action for A is a pair (B, θ), where*
(a) B is an H-bimodule algebra;
(b) The map $\theta : A \to B$ is a monomorphism of algebras;
(c) The sub-algebra $\theta(A)$ is an ideal in B;
(d) The partial action on A is equivalent to the induced partial action on $\theta(A)$;
(e) The induced partial action on $\theta(A)$ is admissible.

As we know, it is straightforward that $\mathrm{Hom}(H\otimes H, A)$ has an structure of H-bimodule given by

$$(h \triangleright f)(k\otimes k') = f(kh\otimes k')$$

and

$$(h \triangleleft f)(k\otimes k') = f(k\otimes hk').$$

Lemma 3.7. *Let $\varphi : A \to Hom(H\otimes H, A)$ be the map given by $\varphi(a)(h\otimes k) = h \rightharpoonup a \leftharpoonup k$, then we have*
(i) φ is a linear injective map and an algebra morphism;
(ii) $(\varphi(1_A) \triangleleft h') * (h \triangleright \varphi(a))) = \varphi((1_A \leftharpoonup h')(h \rightharpoonup a))$ for any $h, h' \in H$ and $a \in A$;
(iii) $(\varphi(b) \triangleleft h') * (h \triangleright \varphi(a))) = \varphi((b \leftharpoonup h')(h \rightharpoonup a))$ for any $h, h' \in H$ and $a, b \in A$.

Proof. It is easy to see that φ is linear, because the partial action is bilinear. Since $\varphi(a)(1_H\otimes 1_H) = a$, it follows that it is also injective. For any $a, b \in A$ and $h, k \in H$, we have

$$\begin{aligned} &\varphi(ab)(h\otimes k) = h \rightharpoonup (ab) \leftharpoonup k) \\ = \;& [h_{(1)} \rightharpoonup a \leftharpoonup k_1][h_{(2)} \rightharpoonup b \leftharpoonup k_2] \\ = \;& \varphi(a)(h_{(1)}\otimes k_{(1)})\varphi(b)(h_{(2)}\otimes k_{(2)}) = \varphi(a) * \varphi(b)(h\otimes k). \end{aligned}$$

Therefore φ is multiplicative.

For the third claim, we have the following calculation:

$$\begin{aligned} &\varphi((b \leftharpoonup h')(h \rightharpoonup a))(k\otimes k') \\ = \;& k \rightharpoonup (b \leftharpoonup h')(h \rightharpoonup a) \leftharpoonup k' \\ = \;& [k_{(1)} \rightharpoonup b \leftharpoonup h'k'_{(1)}][1_A \rightharpoonup k'_{(2)}][k_{(2)} \rightharpoonup h \rightharpoonup a \leftharpoonup k'_{(3)}] \\ = \;& [k_{(1)} \rightharpoonup b \leftharpoonup h'k'_{(1)}][k_{(2)}h \rightharpoonup a \leftharpoonup k'_{(2)}] \\ = \;& \varphi(b)(k_1\otimes h'k'_{(1)})\varphi(a)(k_{(2)}h\otimes k'_{(2)}) \\ = \;& (\varphi(b) \triangleleft h')(k_1\otimes k'_{(1)})(h \triangleright \varphi(a))(k_{(2)}\otimes k'_{(2)}) \\ = \;& \varphi((b \leftharpoonup h')(h \rightharpoonup a))(k\otimes k'). \end{aligned}$$

Therefore, $(\varphi(b) \triangleleft h') * (h \triangleright \varphi(a))) = \varphi((b \leftharpoonup h')(h \rightharpoonup a))$. One may obtain the second item by setting $b = 1_A$. □

Proposition 3.8. *Let $\varphi : A \to Hom(H\otimes H, A)$ be the map defined in Lemma 3.7 and $B = H \triangleright \varphi(A) \triangleleft H$ the H-submodule of $Hom(H\otimes H, A)$. Then*
(i) B is an H-module subalgebra of $Hom(H\otimes H, A)$;
(ii) $\varphi(A)$ is an ideal in B with unity $\varphi(1_A)$.

Proof. The proof is similar to the proof of [2]. □

By Lemma 3.7 and Proposition 3.8, we obtain the main result of this section.

Theorem 3.9. *Let A be a partial H-bimodule algebra and $\varphi : A \to Hom(H\otimes H, A)$ the map given by $\varphi(a)(h\otimes k) = h \rightharpoonup a \leftharpoonup k$. Assume that $B = H \triangleright \varphi(A) \triangleleft H$, then (B, φ) is an enveloping action of A.*

We will call (B, φ) the standard enveloping action of A.

Proposition 3.10. *Let A be a partial H-bimodule algebra and $\varphi : A \to Hom(H, A)$ the map given by $\varphi(a)(h\otimes k) = h \rightharpoonup a \leftharpoonup k$. Assume that $B = (\varphi(A), \triangleright, \triangleleft)$, then $\varphi(A)$ is an ideal of B if and only if*

$$k \rightharpoonup (h \rightharpoonup a) \leftharpoonup k' = [k_{(1)}h \rightharpoonup a \leftharpoonup k'_{(1)}][k_{(2)} \rightharpoonup 1_A \leftharpoonup k'_{(2)})].$$

Proof. Suppose that $\varphi(A)$ is an ideal of B. We know that

$$\varphi(h \rightharpoonup a) = \varphi(1_A) * (h \triangleright \varphi(a)) = (h \triangleright \varphi(a)) * \varphi(1_A).$$

Then, these two functions coincide for all $k, k' \in H$,

$$\varphi(h \rightharpoonup a)(k \otimes k') = (h \triangleright \varphi(a)) * \varphi(1_A)(k \otimes k').$$

The left-hand side of the previous equality leads to

$$\varphi(h \rightharpoonup a)(k \otimes k') = k \rightharpoonup (h \rightharpoonup a) \leftharpoonup k'.$$

While the right-hand side means

$$\begin{aligned}(h \triangleright \varphi(a)) * \varphi(1_A)(k \otimes k') &= (h \triangleright \varphi(a))(k_{(1)} \otimes k'_{(1)})\varphi(1_A)(k_{(2)} \otimes k'_{(2)})\\ &= \varphi(a)(k_{(1)}h \otimes k'_{(1)})\varphi(1_A)(k_{(2)} \otimes k'_{(2)})\\ &= [k_{(1)}h \rightharpoonup a \leftharpoonup k'_{(1)}][k_{(2)} \rightharpoonup 1_A \leftharpoonup k'_{(2)})].\end{aligned}$$

Conversely, suppose that the equality

$$k \rightharpoonup (h \rightharpoonup a) \leftharpoonup k' = [k_{(1)}h \rightharpoonup a \leftharpoonup k'_{(1)}][k_{(2)} \rightharpoonup 1_A \leftharpoonup k'_{(2)})]$$

holds for all $a \in A$ and $h, k, k' \in H$. Then $\varphi(1_A)$ is a central idempotent in B. Therefore $\varphi(A) = \varphi(1_A)B\varphi(1_A)$ is an ideal in B. □

Definition 3.11. *Let A be a partial H-bimodule algebra. An enveloping action (B, θ) of A is minimal if for every H-submodule M of B, $\theta(1_A)M\theta(1_A) = 0$ implies $M = 0$.*

Lemma 3.12. *Let $\varphi : A \to Hom(H \otimes H, A)$ be as above and consider the H-submodule $B = H \triangleright \varphi(A) \triangleleft H$. Then, (B, φ) is a minimal enveloping action of A.*

Proof. We only need to check that the minimality condition holds for cyclic submodules. Let $M = H \triangleright (\sum_{i=1}^{n} h_i \triangleright \varphi(a_i) \triangleleft k_i) \triangleleft H$, and suppose that $\theta(1_A)M\theta(1_A) = 0$. This means that for each $h, k \in H$,

$$\begin{aligned}0 &= \theta(1_A) * (\sum_{i=1}^{n} hh_i \triangleright \varphi(a_i) \triangleleft k_ik) * \theta(1_A)\\ &= \sum_{i=1}^{n} \varphi(hh_i \rightharpoonup a_i \leftharpoonup k_ik)\\ &= \varphi(\sum_{i=1}^{n}(hh_i \rightharpoonup a_i \leftharpoonup k_ik).\end{aligned}$$

Since φ is injective, then $\sum_{i=1}^{n}(hh_i \rightharpoonup a_i \leftharpoonup k_ik) = 0$. But

$$\sum_{i=1}^{n}(h_i \triangleright \varphi(a_i) \triangleleft k_i)(h \otimes k) = \sum_{i=1}^{n} \varphi(a_i))(hh_i \otimes k_ik) = \varphi(\sum_{i=1}^{n}(hh_i \rightharpoonup a_i \leftharpoonup k_ik) = 0,$$

for each $h, k \in H$. Hence we conclude that $\sum_{i=1}^{n} h_i \triangleright \varphi(a_i) \triangleleft k_i = 0$. □

By Definition 3.11 and Lemma 3.12, we have the following theorem.

Theorem 3.13. *Every partial H-bimodule algebra has a minimal enveloping action, and any two minimal enveloping actions of A are isomorphic as H-bimodule algebras.*

4. A Morita context

In this section, we will construct a Morita context between the partial twisted smash product $\underline{A \circledast H}$ and the twisted smash product $B \circledast H$, where B is an enveloping action for the partial twisted smash product.

Lemma 4.1. *Let A be a partial H-bimodule algebra and (B, θ) an enveloping action, then there is an algebra monomorphism from the partial twisted smash product $\underline{A \circledast H}$ into the twisted smash product $B \circledast H$.*

Proof. Define $\Phi : A \otimes H \to B \circledast H$ by $a \otimes h \mapsto \theta(a) \circledast h$ for $h, g \in H$ and $a, b \in A$. We first check that Φ is a morphism of algebras as follows:

$$\begin{aligned}
\Phi((a \otimes h)(b \otimes g)) &= \Phi(a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \otimes h_{(2)}g). \\
&= \theta(a(h_{(1)}) \rightharpoonup b \leftharpoonup S(h_{(3)})) \circledast h_{(2)}g \\
&= \theta(a)(h_{(1)} \triangleright \theta(b) \triangleleft S(h_{(3)})) \circledast h_{(2)}g \\
&= (\theta(a) \circledast h)(\theta(b) \circledast g) \\
&= \Phi(a \otimes h)\Phi(b \otimes g).
\end{aligned}$$

Next, we will verify that Φ is injective. For this purpose, take $x = \sum_{i=1}^{n} a_i \otimes h_i \in ker\Phi$ and choose $\{a_i\}_{i=1}^{n}$ to be linearly independent. Since θ is injective, we conclude that $\theta(a_i)$ are linearly independent. For each $f \in H^*$, $\sum_{i=1}^{n} \theta(a_i) f(h_i) = 0$, it follows that $f(h_i) = 0$, so $h_i = 0$. Therefore we have $x = 0$ and Φ is injective, as desired.

Since the partial twisted smash product $\underline{A \circledast H}$ is a subalgebra of $A \otimes H$, it is injectively mapped into $B \circledast H$ by Φ. A typical element of the image of the partial twisted smash product is

$$\begin{aligned}
\Phi((a \otimes h)(1_A \otimes 1_H)) &= \Phi(a \otimes h)\Phi(1_A \otimes 1_H) \\
&= (\theta(a) \circledast h)(\theta(1_A) \circledast 1_H) \\
&= \theta(a(h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}))) \circledast h_{(2)}g.
\end{aligned}$$

And this completes the proof. □

Take $M = \Phi(A \otimes H) = \{\sum_{i=1}^{n} \theta(a_i) \circledast h_i ; a_i \in A\}$ and take N as the subspace of $B \circledast H$ generated by the elements $(h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \otimes h_{(2)}$ with $h \in H$ and $a \in A$.

Proposition 4.2. *Let H be a Hopf algebra with an invertible antipode S and A a partial H-bimodule algebra. Suppose that $\theta(A)$ is an ideal of B, then M is a right $B \circledast H$ module and N is a left $B \circledast H$ module.*

Proof. In order to prove M is a right $B \circledast H$ module, let $\theta(a) \circledast h \in M$ and $b \circledast k \in B \circledast H$. Then

$$(\theta(a) \circledast h)(b \circledast k) = \theta(a)(h_{(1)}) \triangleright b \triangleleft S(h_{(3)})) \circledast h_{(2)}k.$$

Which lies in $\Phi(A \otimes H)$ because $\theta(A)$ is an ideal in B.

Now we show that N is a left $B \circledast H$ module. Let $(h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \circledast h_{(2)}$, where $h \in H$ is a generator of N, then we have

$$\begin{aligned}
& (b \circledast k)(h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \circledast h_{(2)}) \\
= \; & b(k_{(1)}h_{(1)}) \triangleright \theta(a) \triangleleft S(k_{(3)}h_{(3)})) \circledast k_{(2)}h_{(2)} \\
= \; & [(\varepsilon(k_{(1)}h_{(1)} \triangleright b)(k_{(2)}h_{(2)}) \triangleright \theta(a)] \triangleleft S(k_{(4)}h_{(4)})) \circledast k_{(3)}h_{(3)} \\
= \; & [((k_{(2)}h_{(2)}S(k_{(1)}h_{(1)}) \triangleright b(k_{(3)}h_{(3)}) \triangleright \theta(a)] \triangleleft S(k_{(5)}h_{(5)})) \circledast k_{(4)}h_{(4)} \\
= \; & [(k_{(2)}h_{(2)} \triangleright ((S(k_{(1)}h_{(1)}) \triangleright b)(k_{(2)}h_{(2)}) \triangleright \theta(a)] \triangleleft S(k_{(4)}h_{(4)})) \circledast k_{(3)}h_{(3)} \\
= \; & [(k_{(2)}h_{(2)}) \triangleright (S(k_{(1)}h_{(1)}) \triangleright b)\theta(a)] \triangleleft S(k_{(4)}h_{(4)})) \circledast k_{(3)}h_{(3)}.
\end{aligned}$$

Because $\theta(A)$ is an ideal of B, it follows that N is a left $B \circledast H$ module. □

By Proposition 4.2, we can define a left $\underline{A \circledast H}$ module structure on M and a right $\underline{A \circledast H}$ module structure on N induced by the monomorphism Φ as follows:

$$\begin{aligned} & (ah_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)}) \otimes h_{(2)}) \blacktriangleright (\theta(b)) \circledast k) \\ = \quad & (\theta(a)h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}) \circledast h_{(2)})(\theta(b)) \circledast k), \\ & ((k_{(1)}) \triangleright \theta(b) \triangleleft S(k_{(3)})) \circledast k_{(2)}) \blacktriangleleft (ah_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)}) \otimes h_{(2)}) \\ = \quad & ((k_{(1)}) \triangleright \theta(b) \triangleleft S(k_{(3)})) \circledast k_{(2)})(\theta(a)h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}) \circledast h_{(2)}). \end{aligned}$$

Proposition 4.3. *Under the same hypotheses of Proposition 4.2, M is indeed a left* $\underline{A \circledast H}$ *module with the map* $\blacktriangleright$ *and N is a right* $\underline{A \circledast H}$ *module with the map* $\blacktriangleleft$*.*

Proof. We first claim that $\underline{A \circledast H} \blacktriangleright M \subseteq M$. In fact,

$$\begin{aligned} & (ah_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)}) \otimes h_{(2)}) \blacktriangleright (\theta(b)) \circledast k) \\ = \quad & (\theta(a)(h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}) \circledast h_{(2)})(\theta(b)) \circledast k) \\ = \quad & \theta(a)(h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(5)}))(h_{(2)}) \rightharpoonup \theta(b) \leftharpoonup S(h_{(4)})) \circledast h_{(3)}k_{(2)} \\ = \quad & \theta(a)((h_{(1)}) \triangleright \theta(b) \triangleleft S(h_{(3)})) \circledast h_{(2)}k_{(2)}. \end{aligned}$$

Which lies inside M because $\theta(A)$ is an ideal of B.

Next, we verify that $N \blacktriangleleft \underline{A \circledast H} \subseteq N$. Use a similar method to N that is a left $B \circledast H$ module, and the equality holds because $\theta(1_A)$ is a central idempotent. □

The last ingredient for a Morita context is to define two bimodule morphisms. Now define

$$\sigma : N \otimes_{\underline{A \circledast H}} M \to B \circledast H \text{ and}$$

$$\tau : M \otimes_{B \circledast H} N \to \underline{A \circledast H} \cong \Phi(\underline{A \circledast H}).$$

As M, N and $\underline{A \circledast H}$ are viewed as subalgebras of $B \circledast H$, these two maps can be taken as the usual multiplication on $B \circledast H$. The associativity of the product assures us that these maps are bimodule morphisms and satisfy the associativity conditions. Therefore, we have the following results.

Proposition 4.4. *The partial twisted smash product* $\underline{A \circledast H}$ *is Morita equivalent to the twisted smash product* $B \circledast H$*.*

Theorem 4.5. *Let H be a Hopf algebra with invertible antipode, A a partial H-bimodule algebra,* (B, θ) *a unital enveloping action, and suppose that* $\theta(A)$ *is an ideal of B. Let M and N be the bimodules defined above. Then* $(\underline{A \circledast H}, B \circledast H, M, N, \sigma, \tau)$ *is a strict Morita context.*

Proof. By Proposition 4.4, we know $(\underline{A \circledast H}, B \circledast H, M, N, \sigma, \tau)$ is a Morita context. Next we need to show that σ and τ are surjective, or, equivalently, $MN = \Phi(\underline{A \circledast H})$ and $NM = B \circledast H$. Let us see first that $MN \subseteq \Phi(\underline{A \circledast H})$:

$$\begin{aligned} & (\theta(a) \circledast h)(\sum (k_{(1)} \triangleright \theta(b) \triangleleft S(k_{(3)})) \circledast k_{(2)}) \\ = \quad & \sum \theta(a)[h_{(1)} \triangleright (k_{(1)} \triangleright \theta(b) \triangleleft S(k_{(3)})) \triangleleft S(h_{(3)})] \circledast h_{(2)}k_{(2)} \\ = \quad & \sum \theta(a)[h_{(1)}k_{(1)} \triangleright \theta(b) \triangleleft S(h_{(3)}k_{(3)}))] \circledast h_{(2)}k_{(2)} \\ = \quad & \sum \theta(a)[h_{(1)}k_{(1)} \triangleright \theta(b) \triangleleft S(h_{(4)}k_{(4)}))][h_{(2)}k_{(2)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}k_{(3)}))] \circledast h_{(2)}k_{(2)} \\ = \quad & \sum \theta(a)[h_{(1)}k_{(1)} \triangleright \theta(b) \triangleleft S(h_{(2}k_{(2)}))][h_{(3)}k_{(3)} \triangleright \theta(1_A) \triangleleft S(h_{(4)}k_{(4)}))] \circledast h_{(2)}k_{(2)} \\ = \quad & \sum \theta(a(h_{(1)}k_{(1)} \triangleright b \triangleleft S(h_{(2}k_{(2)}))))[h_{(2)}k_{(2)} \triangleright \theta(1_A) \triangleleft S(h_{(4)}k_{(4)}))] \circledast h_{(3)}k_{(3)}, \end{aligned}$$

which is an element of $\Phi(\underline{A \circledast H})$.

Since

$$\sum \theta(a)(h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)})) \circledast h_{(2)} = (\theta(a) \circledast h)(\theta(1_A) \circledast 1_H),$$

and $\theta(1_A) \circledast 1_H \in N$, it follows that $MN = \Phi(\underline{A \circledast H})$.

In order to prove $NM = B \circledast H$, we only need to show that every element of the form $(h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(2)})) \circledast k$ is in NM, because this is a generating set for $B \circledast H$ as a vector space. In fact, we have

$$(h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(2)})) \circledast k = \sum ((h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \circledast h_{(2)})(\theta(1_A) \circledast S(h_{(4)})k).$$

This can be easily seen as follows:

$$\begin{aligned} & \sum ((h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \circledast h_{(2)})(\theta(1_A) \circledast S(h_{(4)})k) \\ = \ & (h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(5)}))(h_{(2)} \triangleright \theta(1_A) \triangleleft S(h_{(4)})) \circledast h_{(3)}S(h_{(6)})k \\ = \ & (h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \circledast h_{(2)}S(h_{(4)})k \\ = \ & (h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(4)})) \circledast h_{(2)}S(h_{(3)})k \\ = \ & (h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \circledast \varepsilon(h_{(2)})k \\ = \ & (h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(2)})) \circledast k. \end{aligned}$$

Therefore, $NM = B \circledast H$. □

5. Duality for partial twisted smash products

In this section, we explore some results of globalization theorems in order to present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products, generalizing the results of [14].

Let H be a Hopf algebra which is finitely generated and projective as k-module with dual basis $\{(b_i, p_i) \in H \otimes H^* | 1 \leq i \leq n\}$. Assume that H^* acts on H from the left by $f \rightarrow h = \sum h_{(1)} f(h_{(2)})$ and the right by $h \leftarrow f = \sum h_{(2)} f(h_{(1)})$, such that the smash product $H\#H^*$ can be considered as an algebra whose multiplication is given by

$$(h\#f)(k\#g) = \sum h(f_{(1)} \rightarrow k)\#f_{(2)} * g,$$

for any $h, k \in H$ and $f, g \in H^*$.

Lemma 5.1. *[15] Let H be a finite dimensional Hopf algebra. Then the linear maps*
(1) $\lambda : H\#H^* \rightarrow End(H)$, $\lambda(h\#f)(k) = h(f \rightarrow k)$,
(2) $\varphi : H^*\#H \rightarrow End(H)$, $\varphi(f\#h)(k) = (k \leftarrow f)h$,
are isomorphisms of algebras, where $h, k \in H$ *and* $f, g \in H^*$.

The partial twisted smash product $\underline{A \circledast H}$ in Proposition 3.3 becomes naturally a right H-comodule algebra by

$$\rho = 1 \otimes \Delta : A \circledast H \otimes H \rightarrow A \otimes H \otimes H, \quad a \otimes h \mapsto a \otimes h_{(1)} \otimes h_{(2)}.$$

For $(a \otimes h)1_A \in \underline{A \circledast H}$, we have

$$\rho((a \otimes h)1_A) = a(h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(2)})) \otimes h_{(3)} \otimes h_{(4)},$$

which make $\underline{A \circledast H}$ into a right H-comodule algebra. Moreover, $\underline{A \circledast H}$ becomes a left H^* module algebra, where the action is defined by

$$f \cdot ((a \otimes h)1_A) = a(h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)}))\#(f \rightarrow h_{(2)}) = (a\#(f \rightarrow h))1_A.$$

for all $f \in H^*, h \in H, a \in A$.

Similar to [14], we can define a homomorphism $\phi : A \rightarrow A \otimes End(H)$ by

$$\phi(a) = \sum_{i=1}^{n} (b_{i(1)} \rightharpoonup a \leftharpoonup S(b_{i(2)})) \otimes \varphi(S^{-1}(p_i) \otimes 1_H).$$

Then ϕ is an algebra homomorphism.

Lemma 5.2. *Let $\psi : H\#H^* \to A \otimes End(H)$ be the map defined by $h\#f \mapsto 1 \otimes \lambda(h\#f)$ for all $h \in H$ and $f \in H^*$. Then we have*

$$\phi(1_A)\psi(h\#f)\phi(a) = \phi(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(3)}))\psi(h_{(2)}\#f).$$

Proof. For any $h \in H, f \in H^*$ and $a \in A$, we have

$$\begin{aligned}
& \phi(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(3)}))\psi(h_{(2)}\#f) \\
= & \sum_i p_i(h_{(1)})\phi(b_{i(1)} \rightharpoonup a \leftharpoonup S(b_{i(3)}))\psi(h_{(2)}\#f) \\
= & \sum_{i,j} b_{j(1)} \rightharpoonup (b_{i(1)} \rightharpoonup a \leftharpoonup S(b_{i(3)})) \leftharpoonup S(b_{j(3)}) \otimes \varphi(S^{-1}(p_j)\#1_H)\lambda(h \leftarrow p_i\#f) \\
= & \sum_{k,r}(b_{k(1)} \rightharpoonup 1_A \leftharpoonup S(b_{k(2)}))(b_{r(1)} \rightharpoonup a \leftharpoonup S(b_{r(2)})) \otimes \varphi(S^{-1}(p_k)\#1_H)\varphi(S^{-1}(p_{k(1)})\#1_H)\lambda(h \leftarrow p_{k(2)}\#f) \\
= & \phi(1_A)\sum_r(b_{r(1)} \rightharpoonup a \leftharpoonup S(b_{r(2)})) \otimes \varphi(S^{-1}(p_r)_{(2)}\#1_H)\lambda(h \leftarrow S(S^{-1}(p_r)_{(1)})\#f) \\
= & \phi(1_A)\sum_r(b_{r(1)} \rightharpoonup a \leftharpoonup S(b_{r(2)})) \otimes \lambda(h\#f)\varphi(S^{-1}(p_r)\#1_H) \\
= & \phi(1_A)\psi(h\#f)\phi(a).
\end{aligned}$$

The proof is completed. □

Theorem 5.3. *Let H be a finitely generated projective Hopf algebra and A a partial H-bimodule algebra. Then the map*

$$\Phi : A \otimes H\#H^* \to A \otimes End(H),\ a \otimes h\#f \mapsto \phi(a)\psi(h\#f)$$

is an algebra homomorphism. The image of the restriction to $\underline{A \circledast H}\#H^*$ lies inside $e(A \otimes End(H))e$, where e is the idempotent defined by

$$e = \sum_{i=1}^{n}(b_{i(1)} \rightharpoonup 1_A \leftharpoonup S(b_{i(2)})) \otimes \varphi(S^{-1}(p_i) \otimes 1_A).$$

Proof. For any $a, b \in A, h, k \in H$ and $f, g \in H^*$, we have

$$\begin{aligned}
\Phi(a \otimes h\#f)\Phi(b \otimes k\#g) &= \phi(a)\psi(h\#f)\phi(b)\psi(k\#g) \\
&= \phi(a)\phi(1_A)\psi(h\#f)\phi(b)\psi(k\#g) \\
&= \phi(a)\phi(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)}))\psi(h_{(2)}\#f)\psi(k\#g) \\
&= \phi(a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})))\psi(h_{(2)}(f_{(1)} \rightarrow k)\#f_{(2)} * g) \\
&= \Phi(\phi(a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)}))) \otimes h_{(2)}(f_{(1)} \rightarrow k)\#f_{(2)} * g) \\
&= \Phi((a \otimes h\#f)(b \otimes k\#g)).
\end{aligned}$$

Hence Φ is an algebra homomorphism. Since the image of the identity $1 = 1_A \circledast 1_H\#1_{H^*}$ of $\underline{A \circledast H}\#H^*$ under the map Φ is e, e is an idempotent. Moreover, for any $\gamma \in \underline{A \circledast H}\#H^*$, we have

$$\Phi(\gamma) = \Phi(1_A\gamma 1_A) \in e(A \otimes End(H))e,$$

as desired. And this completes the proof. □

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