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On the L-Grundy Domination Number of a Graph

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Abstract. In this paper, we continue the study of the L-Grundy domination number of a graph introduced and first studied in [Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77]. A vertex in a graph dominates itself and all vertices adjacent to it, while a vertex totally dominates another vertex if they are adjacent. A sequence of distinct vertices in a graph G is called an L-sequence if every vertex v in the sequence is such that v dominates at least one vertex that is not totally dominated by any vertex that precedes v in the sequence. The maximum length of such a sequence is called the L-Grundy domination number, $\gamma_{\rm gr}^L(G)$, of G. We show that the L-Grundy domination number of every forest G on n vertices equals n, and we provide a linear-time algorithm to find an L-sequence of length n in G. We prove that the decision problem to determine if the L-Grundy domination number of a split graph G is at least k for a given integer k is NP-complete. We establish a lower bound on $\gamma_{\rm gr}^L(G)$ when G is a regular graph, and investigate graphs G on n vertices for which $\gamma_{\rm gr}^L(G) = n$.

1. Introduction

In the last decade, four Grundy domination invariants were introduced [2, 6, 7], which arise from two standard domination invariants, the domination number and the total domination number. The idea behind these invariants is to determine the worst case that can appear during online creation of (total) dominating sets. When building a dominating set, we may form a sequence D by adding vertices one by one, and require that when a vertex x is added to D it dominates at least one vertex that was not dominated before x was added. The length of a longest such sequence in a graph G is the *Grundy domination number*, $\gamma_{\rm gr}(G)$. The Grundy domination number of a graph was introduced in [6] and has subsequently attracted much attention [2, 3, 5, 9]. In [6] it was shown that the decision version of the problem is NP-complete, even when restricted to chordal graphs, and the Grundy domination number was established for some well-known classes of graphs. The concept was further studied in [5], where exact formulas for Grundy domination numbers of Sierpiński graphs were proven, and a linear algorithm for determining these numbers in arbitrary interval graphs was presented. The Grundy domination number was also studied in Kneser graphs [9] and graph products [3, 15].

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Total domination, introduced for the first time in [10], is an intensively studied area of graph theory; see the monograph surveying the topic [13]. A vertex in a graph totally dominates another vertex if they are adjacent. Let D be a sequence of vertices such that every vertex in the sequence totally dominates at least one vertex that was not totally dominated by preceding vertices in the sequence. Such sequence is an *open neighborhood sequence*, and the maximum length of an open neighborhood sequence in a graph G is the *Grundy total domination number*, $\gamma_{\rm gr}^t(G)$. The Grundy total domination number of a graph G was introduced in [7], where it was also proved that the decision version of the Grundy total domination number is NP-complete, even when restricted to bipartite graphs. (One of the motivations for studying Grundy total domination is the total domination game introduced in [12].) In addition, several lower and upper bounds for the Grundy total domination number were presented and the concept was also studied in regular graphs. Grundy total domination number was investigated in some well-known graph classes, such as trees [8], Kneser graphs [9] and graph products [4]. For split graphs it was shown that the problem is NP-complete [8]. In [11] the graphs with equal total and Grundy total domination numbers were studied.

A connection of Grundy domination number to the zero forcing number of a graph was presented in [2], and led to the introduction of additonal two Grundy domination invariants, namely the Z-Grundy domination and L-Grundy domination number of a graph. The Z-Grundy domination number is dual to the concept of the zero forcing number of a graph, as introduced in [1]. The latter presents a useful lower bound for the minimum rank of a graph. Recently, all four types of Grundy domination invariants were studied by Lin [14], and for each of them a minimum rank type parameter was considered, which is an upper bound for the Grundy type parameter.

Let $S = (v_1, \ldots, v_k)$ be a sequence of vertices of a graph G. The corresponding set of vertices from the sequence S will be denoted by $\widehat{S}$. Let $S' = (u_1, \ldots, u_m)$ be another sequence of vertices of G, with $\widehat{S} \cap \widehat{S'} = \emptyset$. The *concatenation* of S and S' is defined as the sequence $S \oplus S' = (v_1, \ldots, v_k, u_1, \ldots, u_m)$.

Two vertices v and w are *neighbors* in G if they are adjacent; that is, if $vw \in E(G)$. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N_G(v)$, whereas the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. If the graph G is clear from the context, we omit the subscript G and write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively.

Given a graph G and a sequence $S = (v_1, \ldots, v_k)$ of distinct vertices from G, for every $i \in [k] \setminus \{1\}$ we define the set $\Phi_S(v_i)$ by

$$\Phi_S(v_i) := N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j).$$

The sequence S is called an *L-sequence* if $\Phi_S(v_i) \neq \emptyset$ for every $i \in [k] \setminus \{1\}$. We say that the vertex v_i *footprints* the vertices from the set $\Phi_S(v_i)$, and that v_i is the *footprinter* of every vertex $u \in \Phi_S(v_i)$. That is, v_i footprints a vertex u if either $u = v_i$ or v_i totally dominates u, and in both cases the vertex u is not totally dominated by any of the vertices that precede v_i in the sequence. Thus we note that it is possible that some vertex v_i in the L-sequence S footprints only itself and at the same time it does not totally dominate any vertex not already totally dominated by vertices that precede v_i in the sequence. We also note that if a vertex footprints itself, then it can be footprinted later by some other vertex.

The *L-Grundy domination number*, $\gamma_{\rm gr}^L(G)$, of the graph G is the length of a longest L-sequence. Given an L-sequence S, the corresponding set $\widehat{S}$ of vertices will be called an *L-set* (the requirement that all vertices in S are distinct prevents the creation of an infinite sequence by repetition of one and the same vertex). An L-sequence S of length $\gamma_{\rm gr}^L(G)$ in G is called a *$\gamma_{\rm gr}^L$-sequence* of G. If S is a $\gamma_{\rm gr}^L$-sequence of a graph G, then $\widehat{S}$ is a dominating set of G and if G has no isolated vertices then $\widehat{S}$ is also a total dominating set of G. In particular,

$$\gamma_{\rm gr}^L(G) \geq \gamma(G) \quad \text{and} \quad \gamma_{\rm gr}^L(G) \geq \gamma_t(G),$$

where $\gamma(G)$ and $\gamma_t(G)$ denote the domination number and total domination number of G, respectively.

In the next section we establish the necessary notation and graph theory terminology. In Section 3, we prove a lower bound for the L-Grundy domination number in regular graphs, in a similar way as it was done for Grundy total domination in [7]. In Section 4 we consider upper bounds for the L-Grundy domination number in arbitrary graphs. It is clear that if G is a graph, then the bound $\gamma_{\rm gr}^L(G) \le n(G)$ is sharp, but the question remains if it can be improved by involving other parameters. We pose the following conjecture, where $n(G)$ and $\delta(G)$ denote the order and minimum degree, respectively, of the graph G.

Conjecture 1.1. *If G is a graph, then $\gamma_{\rm gr}^L(G) \le n(G) - \delta(G) + 1$ always holds.*

In Section 5, we prove that somewhat surprisingly for every forest T, we have $\gamma_{\rm gr}^L(T) = n(T)$. Further, we prove that every graph G with $\gamma_{\rm gr}^L(G) = n(G)$ satisfies $\delta(G) = 1$, implying that Conjecture 1.1 is true for graphs with $\gamma_{\rm gr}^L(G) = n(G)$. It was shown in [2] that the decision version of L-Grundy domination number is NP-complete even when restricted to bipartite graphs. In Section 6, we prove that this holds also when restricted to split graphs.

2. Basic Concepts and Notation

This section is devoted to notational and preliminary issues. For notation and terminology, we will typically follow [13]. Specifically, let G be a (a directed or undirected) graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G will be denoted by $n(G)$ and $m(G)$, respectively. A *loop* is an edge in G that joins a vertex to itself. A *multiple edge* in G presents two or more edges that are incident to the same two vertices of G; in particular, a *double edge* means two edges that join the same pair of vertices. Unless explicitly stated otherwise, graphs in this article will be without loops and multiple edges.

The *open neighborhood* of a set $S \subseteq V(G)$ is the set of all neighbors of vertices in S, denoted $N_G(S)$, whereas the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. If A is a set of vertices in G, then the set of neighbors of a vertex v that belong to the set A in G is denoted by $N_A(v; G)$; that is, $N_A(v; G) = N_G(v) \cap A$. For a set $S \subseteq V(G)$, we let $N_A(S; G) = \cup_{v \in S} N_A(v; G)$. If the graph G is clear from the context, we omit the label G and write $N(S)$, $N[S]$, $N_A(v)$ and $N_A(S)$ rather than $N_G(S)$, $N_G[S]$, $N_A(v; G)$ and $N_A(S; G)$, respectively.

The *degree* of a vertex v in G is denoted by $d_G(v) = |N_G(v)|$. A *k-regular graph* is a graph in which every vertex has degree k. A graph G is *regular* if it is k-regular for some integer $k \ge 0$. The maximum degree among all the vertices of G is denoted by $\Delta(G)$. If the graph G is clear from the context, we omit the subscript G and write $d(v)$ rather than $d_G(v)$.

A *directed graph* (or *digraph*) $D = (V(D), A(D))$ is a graph where the edge set $A(D)$ is a set of ordered pairs of vertices, called *arcs* or *directed edges*. If (x, y) is an arc of D, then we say that the arc (x, y) is *from x to y* in D, and that x is *adjacent to y*, and y is *adjacent from x*. The *indegree* of a vertex v is the number of arcs incoming to a vertex; that is, the number of vertices adjacent to v. The *outdegree* of v is the number of arcs outgoing from a vertex; that is, the number of vertices adjacent from v. A *directed cycle* C is a connected digraph where all vertices have indegree and outdegree equal to 1.

A *forest* is a graph with no cycles. A *tree* is a connected forest. An *independent set* in a graph is a set of pairwise non-adjacent vertices. The *independence number* of a graph G, denoted by $\alpha(G)$, is the cardinality of the largest independent set in G. An independent set of size $\alpha(G)$ will be called an α-set of G. A *complete graph* is a graph whose vertices are pairwise adjacent. The complete graph with n vertices is denoted by K_n. A set $C \subseteq V(G)$ is a *clique* in a graph G if its induced subgraph is complete. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets. A *complete bipartite graph* is a bipartite graph, such that every pair of vertices in the two independent sets are adjacent. If the two maximal independent sets are of size n and m (where $n, m \in \mathbb{N}$), then the complete bipartite graph is denoted by $K_{n,m}$.

3. Lower Bounds in Regular Graphs

The Grundy total domination number of regular graphs was studied in [7], where the following lower bound for regular graph was proven.

Theorem 3.1. ([7]) *For $k \ge 3$, if G is a connected, k-regular graph different from $K_{k,k}$, then $\gamma_{\rm gr}^t(G) \ge \frac{n(G)}{k-1}$ with strict inequality if $k \ge 5$.*

In [11] it was proved that every Grundy total dominating sequence of a bipartite graph has the same number of vertices in each partite set of the graph.

Theorem 3.2. ([11]) *If G is a bipartite graph with bipartition $A \cup B$, then the Grundy total domination number of G is even and for any Grundy total dominating sequence $S = (v_1, \ldots, v_{2k})$ we have $|A \cap \hat{S}| = |B \cap \hat{S}| = k$.*

Since $\gamma_{\rm gr}^t(G) \le \gamma_{\rm gr}^L(G)$, a lower bound for the Grundy total domination number is also a lower bound for the L-Grundy domination number. For a regular graph, the following result improves this lower bound on the L-Grundy domination number, which follows from Theorem 3.1.

Theorem 3.3. *For $k \ge 3$ if G is a k-regular connected bipartite graph different from $K_{k,k}$, then*

$$\gamma_{\rm gr}^L(G) \ge \frac{k\,n(G)}{2(k-1)}.$$

Proof. Let $n = n(G)$. We construct an L-sequence of length $\frac{nk}{2(k-1)}$ in the k-regular connected bipartite graph G of order n as follows. Let A and B be the partite sets of the bipartite graph G. Since G is k-regular, we note that $|A| = |B| = \frac{n}{2}$. First, we put in the sequence S all vertices from the set A in an arbitrary order. We note that each such vertex footprints itself and possibly some vertices in B. Secondly, we continue the sequence by adding to it vertices from B such that each added vertex from B footprints a vertex of A. Thus, the vertices in B that belong to the sequence S totally dominate the set A. Theorem 3.2 implies that the number of vertices from B in S is $\frac{1}{2}\gamma_{\rm gr}^t(G)$. By Theorem 3.1, we have $\gamma_{\rm gr}^t(G) \ge \frac{n}{k-1}$ for every k-regular bipartite graph G different from $K_{k,k}$, implying that

$$\gamma_{\rm gr}^L(G) \ge |A| + \frac{1}{2}\left(\frac{n}{k-1}\right) = \frac{n}{2} + \frac{1}{2}\left(\frac{n}{k-1}\right) = \frac{nk}{2(k-1)},$$

as desired. □

We remark that the bound in Theorem 3.3 is best possible in the case when $k = 3$, as may be seen by considering the graph $G = K_{4,4} - M$, where M is a perfect matching of $K_{4,4}$. In this case, G is a 3-regular graph of order $n = 8$ and $\gamma_{\rm gr}^L(G) = 6 = \frac{nk}{2(k-1)}$.

Remark 3.4. *For $k \ge 1$, the following holds.*
(a) *If $G = K_{k+1}$, then $\gamma_{\rm gr}^L(G) = 2$.*
(b) *If $G = K_{k,k}$, then $\gamma_{\rm gr}^L(G) = k + 1$.*

We establish next a lower bound on the L-Grundy domination number of a k-regular connected graph that is different from $K_{k,k}$ and K_{k+1}.

Theorem 3.5. *For $k \ge 3$, if G is a k-regular connected graph different from $K_{k,k}$ and K_{k+1}, then*

$$\gamma_{\rm gr}^L(G) \ge \frac{2(k-2)n(G) + (4-k)\alpha(G)}{(k-1)^2}.$$

Proof. Let G be a k-regular connected graph of order n different from $K_{k,k}$ and K_{k+1}. Let A be an α-set of G and $B = V(G) \setminus A$. Let $B_1 \cup B_2 \cup \cdots \cup B_k$ be a partition of B, such that each vertex of B_i has exactly i neighbors in A for $i \in [k]$. Let $n_i = |B_i|$ for $i \in [k]$, and let $E(A, B)$ be the set of edges from A to B. Since $|E(A, B)| = k\alpha(G) = n_1 + 2n_2 + \cdots + kn_k$ and $n_1 + n_2 + \cdots + n_k = n - \alpha(G)$ we get

$$n_1 = 2n - (2 + k)\alpha(G) + \sum_{i=3}^{k}(i-2)n_i. \qquad (1)$$

Let $A_1 = N(B_1) \cap A$. Since each vertex in B_1 has exactly one neighbor in A_1, we note that no two vertices in A_1 have a common neighbor in B_1. We state this formally as follows.

Observation 3.6. *If $a, a' \in A_1$, then the set of neighbors of a in B_1 is disjoint with the set of neighbors of a' in B_1.*

We proceed further with the following claim.

Claim 3.7. $|A_1| \geq \frac{n_1}{k-1}$.

Proof. Suppose that there exists $a \in A_1$ that has k neighbors $x_1, \dots, x_k$ in B_1. Since G does not contain a subgraph isomorphic to K_{k+1}, there exists $x_i, x_j \in \{x_1, \dots, x_k\}$ such that $x_i x_j \notin E(G)$, implying that $(A \setminus \{a\}) \cup \{x_i, x_j\}$ is an independent set larger than A, a contradiction. Hence every $a \in A_1$ has at most $k-1$ neighbors in B_1 and $N(a) \cap B_1$ is a clique. Therefore, $(k-1)|A_1| \geq |B_1| = n_1$ and, consequently, $|A_1| \geq \frac{n_1}{k-1}$. (□)

We construct an L-sequence S as follows. Initially, we let the sequence consist of all vertices from A in any order. Since when $a \in A$ is added to S it footprints at least itself, the resulting sequence S is an L-sequence of length $\alpha(G)$. Let $A_1 = \{a_1, \dots, a_{|A_1|}\}$. For each $a_i \in A_1$, let b_i be an arbitrary neighbor of a_i in B_1 and let $B' = \{b_i : i \in [|A_1|]\}$. From Observation 3.6 we note that B' contains $|A_1|$ pairwise different vertices, and so $\{a_1 b_1, \dots, a_{|A_1|} b_{|A_1|}\}$ is a matching in G. We now add to S all vertices from B' in any order. Since when $b_i \in B'$ is added to S it footprints the vertex $a_i \in A_1$, the sequence S is still an L-sequence. The length of the resulting sequence S is $\alpha(G) + |A_1|$.

Claim 3.8. *There exists a sequence $S' = (s_1, \dots, s_\ell)$ of vertices in $B \setminus B_1$, such that $N_{A \setminus A_1}(\widehat{S'}) = A \setminus A_1$ and each $s_i \in S'$ footprints at least one and at most $k-1$ vertices in $A \setminus A_1$ for $i \in [\ell]$; that is, each $s_i \in S'$ is adjacent to at least one and at most $k-1$ vertices from $A \setminus (A_1 \cup (N_{A \setminus A_1}(\{s_1, \dots, s_{i-1}\}))$.*

Proof. For the purpose of contradiction, suppose to the contrary that we already chose a subset $X \subseteq \widehat{S'}$ of vertices in $B \setminus B_1$ that satisfies the statement of the claim, but in $B_X = B \setminus (B_1 \cup X)$ every vertex has either no neighbor in $A_X = A \setminus N_A(B_1 \cup X)$ (which is not yet empty set) or has all its k neighbors in A_X. Let B'_X be the set of vertices from B_X with no neighbor in A_X and let B''_X be the set of vertices from B_X with all k neighbors in A_X. We note that $B_X = B'_X \cup B''_X$. In this case, the vertices in the set $A_X \cup B''_X$ induce a bipartite k-regular subgraph of G, which implies that G is disconnected, a contradiction. (□)

By Claim 3.8, there exists a sequence $S' = (s_1, \dots, s_\ell)$ of vertices in $B \setminus B_1$, such that $N_{A \setminus A_1}(\widehat{S'}) = A \setminus A_1$ and each $s_i \in S'$ footprints at least one and at most $k-1$ vertices in $A \setminus A_1$ for $i \in [\ell]$. We now consider the expanded sequence $S = S \oplus S'$. The resulting sequence S is an L-sequence. Thus, letting $\alpha = \alpha(G)$ we have

$$
\begin{aligned}
\gamma_{\rm gr}^{L}(G) &\geq |A| + |B'| + \ell \\
&= |A| + |A_1| + \ell \\
&\overset{(Claim\ 3.8)}{\geq} |A| + |A_1| + \tfrac{|A \setminus A_1|}{k-1} \\
&= \left(\tfrac{k}{k-1}\right)|A| + \left(\tfrac{k-2}{k-1}\right)|A_1|. \\
&\overset{(Claim\ 3.7)}{\geq} \left(\tfrac{k}{k-1}\right)\alpha + \left(\tfrac{k-2}{k-1}\right)\tfrac{n_1}{k-1} \\
&\overset{(Equation\ 1)}{=} \left(\tfrac{k}{k-1}\right)\alpha + \left(\tfrac{k-2}{(k-1)^2}\right)\left(2n - (2+k)\alpha + \sum_{i=3}^{k}(i-2)n_i\right) \\
&= \frac{1}{(k-1)^2}\left(2(k-2)n + (4-k)\alpha + (k-2)\sum_{i=3}^{k}(i-2)n_i\right).
\end{aligned}
$$

Since $n_i \geq 0$ for $i \in [k] \setminus \{1, 2\}$, this implies that

$$
\gamma_{\rm gr}^{L}(G) \geq \frac{1}{(k-1)^2}\left(2(k-2)n + (4-k)\alpha\right),
$$

which is the desired lower bound on the L-Grundy domination number. □

If G is a k-regular graph on n vertices, then by double counting the edges joining a maximum independent set A in G and its complement $V(G) \setminus A$, we have $\alpha(G) = |A| \le \frac{1}{2}n$. Further if G is a bipartite graph, then both partite sets of G have cardinality $\frac{1}{2}n$, implying that $\alpha(G) = \frac{1}{2}n$. Thus in the proof of Theorem 3.5, we can choose in this case the α-set A to be either of the two partite sets of G, implying that $n_1 = n_2 = \cdots = n_{k-1} = 0$ and $n_k = n - \alpha(G) = \frac{1}{2}n$. Hence, as an immediate consequence of the proof of Theorem 3.5 we have

$$\gamma_{\rm gr}^L(G) \ge \frac{1}{(k-1)^2}\Big(2(k-2)n + (4-k)\alpha + (k-2)^2 n_k\Big) = \left(\frac{k}{2(k-1)}\right)n.$$

Theorem 3.3 is therefore a corollary of Theorem 3.5. In the special case of Theorem 3.5 when $k = 3$, we have the following lower bound on the L-Grundy domination number of a cubic graph.

Corollary 3.9. *If G is a cubic graph different from $K_{3,3}$ and K_4, then*

$$\gamma_{\rm gr}^L(G) \ge \frac{1}{2}n(G) + \frac{1}{4}\alpha(G).$$

4. General Upper Bound

From the definition of the Grundy domination number, Grundy total domination number and L-Grundy domination number it follows that L-Grundy domination number of a graph G is bounded below by $\gamma_{\rm gr}^t(G)$ and $\gamma_{\rm gr}(G) + 1$, as first observed in [2]. The Grundy domination number and Grundy total domination number have trivial upper bounds in terms of the minimum degree of a graph G, namely $\gamma_{\rm gr}(G) \le n(G) - \delta(G)$ and $\gamma_{\rm gr}^t(G) \le n(G) - \delta(G) + 1$. In the case of the L-Grundy domination number, it is not necessary true that $n(G) - \delta(G)$ is an upper bound. Although we strongly suspect that $\gamma_{\rm gr}^L(G) \le n(G) - \delta(G) + 1$, we have yet to prove that this upper bound always holds since there exist vertices that are footprinted twice, first by itself and then by one of its neighbors. However, we were able to establish the following result.

Theorem 4.1. *If G is a graph with $\gamma_{\rm gr}^L(G) = n(G)$, then $\delta(G) \le 1$.*

Proof. For the purpose of contradiction, let G be a graph of order n satisfying $\gamma_{\rm gr}^L(G) = n$ and suppose, to the contrary, that $\delta(G) \ge 2$. In particular, G has no isolated vertex. Let S be a $\gamma_{\rm gr}^L$-sequence of a graph $G = (V, E)$ and for $x \in S$ let $i(x)$ denotes the position of x in S. Since $\gamma_{\rm gr}^L(G) = n$, we note that $|S| = n$.

We construct next a directed graph D as follows. Let $V(D) = V$, where $xy \in A(D)$ if and only if x footprints y with respect to the sequence S in G. Since $|S| = n$, we note that each vertex of D has outdegree at least 1 (noting that loops are also possible, which contribute 1 to the outdegree of each vertex). Once the sequence S is constructed, all vertices are footprinted, implying that each vertex of D has indegree at least 1. Since the graph G has no isolated vertex, we note that if a vertex footprints itself, then it will also be subsequently footprinted by one of its neighbors. Let G' be the (undirected) graph obtained from D by removing the orientation of each arc in D to form an edge. Next, we list some properties of directed graph D and the graph G'.

Claim 4.2. *If we delete from G' all loops and multiple edges, then the resulting graph is a forest.*

Proof. Let F be the graph obtained from G' by deleting all loops and multiple edges. For the purpose of contradiction, suppose to the contrary that F has a cycle C. Let C' be a subgraph of the directed graph D obtained from C by restoring the orientation of the edges $xy \in E(C)$. Since each vertex is footprinted by exactly one vertex different from itself, the oriented cycle C' is a directed cycle in D (in which every vertex has indegree 1 and outdegree 1). Let $V(C') = \{x_1, x_2, \ldots, x_\ell\}$ where $\ell \ge 3$ and (x_i, x_{i+1}) is an arc from x_i to x_{i+1} for $i \in [k]$ and where addition is taken modulo k. Renaming vertices of C' if necessary, we may assume that x_1 is the vertex with the smallest index in S; that is, among all vertices in $V(C')$, the vertex x_1 appears first in S. According to our orientation of the directed cycle C', the vertex x_i is footprinted by the vertex x_{i-1} for $i \in [\ell] \setminus \{1\}$. In particular, the vertex x_ℓ is footprinted by the vertex $x_{\ell-1}$. However since x_1 is a neighbor of

x_ℓ and the vertex x_1 is in S before $x_{\ell-1}$, the vertex x_1 footprints x_ℓ, implying that x_ℓ is footprinted by at least two vertices different from itself, a contradiction. (□)

It follows from Claim 4.2 that the directed graph D has a forest structure (with addition of loops and multiple edges). Therefore we will call D a *pseudo forest*. We note that D has no vertices with outdegree 0 and all vertices of D have indegree 1 or 2. Further we note that if we delete all loops in D, then all vertices of the resulting directed graph have indegree 1.

We call a vertex x in D that has outdegree 1 and such that $(x, x) \in A(D)$ a *leaf of type 1* in D, and we call a vertex x that has outdegree 1 and such that $(x, y) \in A(D)$ where $x \neq y$ and $(y, x) \in A(D)$ a *leaf of type 2* in D. We call a leaf of type 1 in D or a leaf of type 2 in D a *leaf* of D.

Recall that a *component* of a graph is a maximal connected subgraph of the graph. In particular, we note that a component of a graph is by definition connected. A component T of a pseudo forest D is called a *p*seudo tree. Let T be a pseudo tree in D, and let x be a vertex of T with the smallest index in S. We call the vertex x the *root* of the component T. Let T' be the (undirected) multigraph obtained from T by removing the orientation of each arc in T and removing any resulting loops.

Claim 4.3. *The following holds for the pseudo tree T with the root x:*

(a) $(x, x) \in A(D)$,

(b) *the multigraph T' has exactly one multiple edge, namely a double edge incident with the vertex x.*

Proof. Since G has no isolated vertex and each vertex that footprints itself is also footprinted by one of its neighbors, we note that $n(T) \geq 2$. Since T is a component of D, we note that the vertex x footprints itself and all its neighbors $N = \{x_1, \ldots, x_k\}$ in G that are not adjacent to any vertex that precedes x in S. Hence, $(x, x) \in A(T)$, which proves Part (a).

Further, $(x, x_i) \in A(T)$ for all $i \in [k]$. Since $\gamma_{\rm gr}^L(G) = n$, the vertex from N with the smallest index in S footprints x. Renaming indices if necessary, we may assume that x_1 is the vertex in N with the smallest index in S that footprints x, and so $(x_1, x) \in A(T)$ but no other vertex in N footprints x. Thus, T' has at least one multiple edge, namely the edge xx_1 (corresponding to the arcs (x, x_1) and (x_1, x) in T).

Suppose now that there is a multiple edge yz in T' different from the edge xx_1. As observed earlier, no other vertex in $N \setminus \{x_1\}$ footprints x, implying that $x \neq y$ and $x \neq z$. Since yz is a multiple edge in T', we note that $(y, z) \in A(T)$ and $(z, y) \in A(T)$; that is, y footprints z and z footprints y. Renaming y and z if necessary, we may assume that the distance from x to y in T' is less than or equal to the distance from x to z in T'. Let $P\colon x, y_1, \ldots, y_\ell = y$ be a shortest path from x to y in T'. In the component T, we note that $(x, y_1) \in A(T)$ and $(y_i, y_{i+1}) \in A(T)$ for $i \in [\ell - 1]$. This implies that x footprints y_1, and if $\ell \geq 2$, then the vertex y_i footprints y_{i+1} for all $i \in [\ell - 1]$. Since $d_{T'}(x, y) \leq d_{T'}(x, z)$, we note that $z \notin V(P)$. This implies that the vertex y is footprinted by at least two vertices different from itself, namely the vertices $y_{\ell-1}$ and z, a contradiction. Hence, the multiple edge xx_1 is the only multiple edge in T'. This proves Part (b). (□)

Claim 4.4. *If u and v are vertices in D with $(u, u) \in A(D)$ and $(v, v) \in A(D)$, then u and v are not adjacent in G.*

Proof. Suppose, to the contrary, that $uv \in E(G)$. Renaming vertices if necessary, we may assume that $i(u) < i(v)$. Hence when v is added to S, the vertex v is already footprinted, implying that $(v, v) \notin A(D)$, a contradiction. (□)

As a consequence of Claim 4.4, we have the following results.

Claim 4.5. *The following holds.*

(a) *If u and v are leaves of type* 1 *in D, then u and v are not adjacent in G.*

(b) *If y is a leaf of type* 1 *in D and x a root of a component T of D, then $xy \notin E(G)$.*

(c) *If T_1 and T_2 are components of D with roots x_1 and x_2, respectively, then $x_1x_2 \notin E(G)$.*

Claim 4.6. *If u is a leaf of type* 2 *in D which is adjacent to a vertex v in D, then $i(v) < i(u)$ and v is a root of the component in D that contains u.*

Proof. Let u be a leaf of type 2 in D. Thus, u has outdegree 1 and there exists exactly one vertex $v \in V(D)$ with $(u, v) \in A(D)$ and $(v, u) \in A(D)$. It follows from Claim 4.3 that v is a root of a component of D that contains the vertex u. By Claim 4.3(a), the vertex v footprints itself, implying that v is in S before u. (□)

Claim 4.7. *Let T_1 and T_2 be two components of order 2 in D with roots $x_1 \in V(T_1)$, $x_2 \in V(T_2)$ and leaves $y_1 \in V(T_1)$, $y_2 \in V(T_2)$. If $i(y_1) < i(y_2)$, then $x_2y_1 \notin E(G)$.*

Proof. Let x_1 be the root and y_1 the leaf of T_1, and let x_2 be the root and y_2 the leaf of T_2. Thus, x_1 footprints y_1, and x_2 footprints y_2. We note that y_i is a leaf of T_i of type 2 for $i \in [2]$ and hence y_1 footprints x_1 and y_2 footprints x_2. Suppose that $x_2y_1 \in E(G)$. Since $i(y_1) < i(y_2)$, x_2 is footprinted by y_1, a contradiction. (□)

Claim 4.8. *Let $P\colon xx_1x_2 \dots x_\ell$ be a directed path in T where $\ell \geq 2$, and so $(x, x_1) \in A(D)$ and $(x_i, x_{i+1}) \in A(D)$ for $i \in [\ell - 1]$. If x_ℓ is a leaf of type 1, then*

$$\max\{i(x_1), i(x_2), \dots, i(x_\ell)\} = i(x_{\ell-1}).$$

Proof. Since $P\colon xx_1x_2 \dots x_\ell$ is a directed path in T, we note that the vertex x footprints the vertex x_1, and the vertex x_i footprints the vertex x_{i+1} for $i \in [\ell - 1]$. Since x_ℓ is a leaf of type 1 in D, the vertex x_ℓ footprints only itself when it is added to S. This implies that $i(x_\ell) < i(x_{\ell-1})$. However, x_ℓ does not footprint $x_{\ell-1}$ since x_ℓ footprints only itself. Thus, $x_{\ell-1}$ is already footprinted when x_ℓ is added to S. Since $x_{\ell-2}$ footprints the vertex $x_{\ell-1}$, this implies that $i(x_{\ell-2}) < i(x_\ell)$. Since $x_{\ell-4}$ footprints the vertex $x_{\ell-3}$ (and $x_{\ell-3}$ is not footprinted by the vertex $x_{\ell-2}$), we have that $i(x_{\ell-4}) < i(x_{\ell-2})$. Continuing in this way, we deduce that the vertex $x_{\ell-2j}$ where $j \in \{0, 1, \dots, \lfloor \frac{\ell-1}{2} \rfloor\}$ is in S before $x_{\ell-1}$.

Since $x_{\ell-1}$ footprints x_ℓ and does not footprint $x_{\ell-2}$, as $x_{\ell-2}$ is footprinted by $x_{\ell-3}$, we note that $i(x_{\ell-3}) < i(x_{\ell-1})$. Since $x_{\ell-5}$ footprints the vertex $x_{\ell-4}$ (and $x_{\ell-4}$ is not footprinted by the vertex $x_{\ell-3}$), we have that $i(x_{\ell-5}) < i(x_{\ell-3})$. Analogously as before we deduce that the vertex $x_{\ell-(2j+1)}$ where $j \in \{1, \dots, \lfloor \frac{\ell-2}{2} \rfloor\}$ is in S before $x_{\ell-(2j-1)}$ (as $x_{\ell-2j}$ is footprinted by $x_{\ell-(2j+1)}$ and not by $x_{\ell-(2j-1)}$) and hence before $x_{\ell-1}$. Thus, every vertex in $V(P) \setminus \{x_{\ell-1}\}$ is in S before $x_{\ell-1}$. (□)

Let L be the set of vertices in D that are adjacent to at least one leaf in D. Among all vertices in L, let y have the largest index in S and let y_ℓ be a leaf of D that is adjacent to y, i.e., y footprints y_ℓ. Since $\delta(G) \geq 2$, there exists a vertex $z \in V(G)$ different from y that is adjacent to y_ℓ. The leaf y_ℓ can be of two types, which we deal with separately.

Case 1. The leaf y_ℓ is of type 1.
By Claim 4.5, the vertex z is not a leaf of type 1 in D and is not the root of a component of D.

Suppose that z is not a leaf of type 2 in D. Let T_z be a component of D that contains z and let $z, z_2, \dots, z_k$ be a directed path from z to a leaf z_k of type 1 in T_z (note that this path does not contain the root of T_z). Since y, and not z, is a footprinter of y_ℓ, we note that $i(y) < i(z)$. By Claim 4.8, we have that $i(z) < i(z_{k-1})$, implying that $i(y) < i(z_{k-1})$, contradicting our choice of the vertex y. Hence, z is a leaf of type 2 in D.

Since z is a leaf of type 2 in D, there exists a vertex $z' \in V(D)$ distinct from z such that $(z, z') \in A(D)$ and $(z', z) \in A(D)$, and the outdegree of z in D is 1. By Claim 4.6, the vertex z' is a root of the component T_z in D that contains z and $i(z') < i(z)$. Since y_ℓ is footprinted by y and not by z, we have $i(y) < i(z)$.

Suppose that $n(T_z) > 2$. In this case, we let $z, z', z_2, \dots, z_k$ be a directed path in T_z from z to a leaf z_k of type 1 in T_z. Since z' is footprinted by z (and not by z_2), the vertex z_2 is in S after z and thus after y. Thus Claim 4.8 implies that z_{k-1} is in S after z_2 and thus after y, a contradiction with the choice of y. Hence, $n(T_z) = 2$.

Recall that $i(y) < i(z)$. Since z' has only one neighbor in D, there exists a vertex $z_2 \in V(G)$ different from z that is adjacent to z' in G. Since z' is footprinted by z (and not z_2), the vertex z_2 is in S after z and thus after y. If the component T_2 of D that contains z_2 has order more than 2, then Claim 4.8 implies that there is a vertex from L which is in S after y, a contradiction. Hence, $n(T_2) = 2$.

By Claim 4.5, z_2 is a leaf of type 2 and the neighbor z_2' of z_2 in T_2 is a root of T_2. Since the degree of z_2' in G is at least 2, the vertex z_2' has a neighbor $z_3 \neq z_2$ in G, implying as before that z_3 is in S after z_2, noting that z_2' is footprinted by z_2. Claim 4.7 implies that this procedure will eventually produce a component T_i of order more than 2, i.e., $z_{i-1}, z_{i-1}' \in V(T_{i-1})$, $n(T_{i-1}) = 2$ and z_{i-1} is a leaf of type 2 in T_{i-1} and z_{i-1}' a root of T_{i-1}. Since the degree of z_{i-1}' in G is more than 1, there exists $z_i \in V(T_i)$ that is adjacent to z_{i-1}' in G. We therefore have $i(y) < i(z) < i(z_2) < \cdots < i(z_{i-1})$ and as z_{i-1}' is footprinted by z_{i-1} and not by z_i, this implies that $i(z_{i-1}) < i(z_i)$. Since T_i has order more than 2, Claim 4.8 guaranties the existence of a vertex from L that is in S after z_i and hence after y, a contradiction.

Case 2. The leaf y_ℓ is of type 2.
By Claim 4.3, the vertex y is the root of the component T_y that contains y_ℓ and $i(y) < i(y_\ell)$.

If $n(T_y) > 2$, then let z_2 be a neighbor of y in D different from y_ℓ and let $y, z_2, \ldots, z_k$ be a directed path from y to a leaf z_k of type 1 in T_y. By Claim 4.3, we note that the vertex y_ℓ is the only leaf of type 2 in T_y. Since y is footprinted by y_ℓ and not by z_2, we have $i(y_\ell) < i(z_2) \leq i(z_{k-1})$, where the last inequality follows from Claim 4.8. Hence, $i(y) < i(z_{k-1})$, contradicting the choice of the vertex y. Thus, $n(T_y) = 2$. Since y_ℓ is footprinted by y, we have $i(y) < i(z)$. If z is not a leaf of type 2 in a component of order 2, then we infer, using also Claim 4.8, that there exists a vertex from L that is in S after y, a contradiction. Therefore, z is a leaf of type 2 in T_1, with $n(T_1) = 2$. Hence the neighbor z' of z in T_1 has another neighbor in G. We continue in the same way as in Case 1 and obtain a contradiction. This completes the proof of Theorem 4.1. □

For every graph G we have $\gamma_{\rm gr}^L(G) \leq n(G)$. In particular, if $\delta(G) \leq 1$, then $\gamma_{\rm gr}^L(G) \leq n(G) - \delta(G) + 1$ holds. If $\delta(G) = 2$, then Theorem 4.1 implies that $\gamma_{\rm gr}^L(G) \leq n(G) - 1 = n(G) - \delta(G) + 1$. Thus, we have the following result.

Corollary 4.9. *If G is a graph with $\delta(G) \leq 2$, then $\gamma_{\rm gr}^L(G) \leq n(G) - \delta(G) + 1$.*

We believe that Corollary 4.9 holds for all graphs, and pose the following conjecture.

Conjecture 4.10. *If G is a graph, then $\gamma_{\rm gr}^L(G) \leq n(G) - \delta(G) + 1$.*

5. Forests

In this section we show that the L-Grundy domination number of any forest equals the number of vertices in the forest.

A *leaf* of a graph G is a vertex of degree 1 in G, while its only neighbor is a *support vertex*. We denote by $L(G)$ the set of leaves of G. For each vertex $u \in V(G)$, we let $L(u)$ be the set of leaf neighbors of u in G; that is, $L(u) = L(G) \cap N(u)$. We note that if $L(u) \neq \emptyset$, then u is a support vertex of G.

Theorem 5.1. *For any forest T,*

$$\gamma_{\rm gr}^L(T) = n(T).$$

Algorithm 1 returns a $\gamma_{\rm gr}^L$-sequence of an arbitrary forest T. The complexity of Algorithm 1 is $O(n(T))$.

Proof. Let us start the proof by showing that the sequence S produced by Algorithm 1 is an L-sequence; i.e., every vertex in the sequence footprints some vertex in T. When $w \in V(T')$ such that $L(w) \neq \emptyset$ is chosen in some step of the algorithm (line 5 of the algorithm), then the vertex $u \in L(w)$ footprints itself and maybe also its unique neighbor w (for later purposes denoted also by w_i), and so the first part of the sequence S is an L-sequence. When T' contains just isolated vertices, each vertex of T' footprints itself. We note that all vertices that are added to S in the first and the second part (until line 13 of the algorithm) form an independent set. Hence, all these vertices are footprinted only by themselves and therefore, can be footprinted again by some other vertex. The last part is constructed from vertices w_i that are footprinted in the first part, only that they are listed in the reverse order. At the time vertex w_i appears in S, vertices in $L(w_i)$ (of a forest T' before w_i was removed), are footprinted only by themselves. Hence, w_i footprints vertices in $L(w_i)$. This yields that S is an L-sequence.

Clearly, the time complexity of the algorithm is $O(n(T))$ and all vertices from T are added to the sequence S. Hence, $|S| = n(T)$, $\gamma_{\rm gr}^L(T) = n(T)$ and S is a $\gamma_{\rm gr}^L$-sequence. □

Algorithm 1: A γ_{gr}^{L}-sequence of a forest.

Input: A forest T.
Output: A γ_{gr}^{L}-sequence S of T.

```
S = ();
T' = T;
i = 0;
while T' has non-isolated vertices do
    Choose w ∈ V(T'), such that L(w) ≠ ∅;
    S = S ⊕ (L(w));
    i = i + 1;
    w_i = w;
    T' = T' \ (L(w) ∪ {w})
while V(T') ≠ ∅ do
    Choose a vertex v ∈ V(T');
    S = S ⊕ (v);
    T' = T' \ {v}
while i > 0 do
    S = S ⊕ (w_i);
    i = i − 1;
```

6. Split graphs

A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. The decision version of L-Grundy domination is the following problem:

L-Grundy Domination Number Problem
Input: $G = (V, E)$, $k \in \mathbb{N}$.
Question: Is there a γ_{gr}^{L}-sequence of G of length at least k?

We prove that this problem remains NP-complete when restricted to split graphs.

In the proof, we will rely on the NP-completeness result [6, Corollary 8.5] concerning the Grundy Total Domination Number Problem. Given a graph G with no isolated vertices and a positive integer k the problem is to determine whether $\gamma_{\mathrm{gr}}^{t}(G) \geq k$. We also use the concept of the *t-footprinter* of a vertex, which is defined with respect to an open neighborhood sequence in G. If $S = (v_1, \ldots, v_k)$ is such a sequence, then v_i *t-footprints* the vertices in $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j)$, and so v_i is the *t-footprinter* of these vertices.

Theorem 6.1. L-Grundy Domination Number Problem *is NP-complete, even when restricted to split graphs.*

Proof. It is clear that the problem is in NP.

Given a graph $G = (V, E)$ with no isolated vertices, we construct the split graph $G' = (V_1 \cup V_2, E')$ as follows: $V_1 = \{v^1 : v \in V\}$ is an independent set, $V_2 = \{v^2 : v \in V\}$ induces a clique and $N_{G'}(v_i^1) = \{v^2 \in V_2 : v \in N_G(v_i)\}$.

We will prove that $\gamma_{\mathrm{gr}}^{L}(G') = n(G) + \gamma_{\mathrm{gr}}^{t}(G)$, which by the NP-completeness result on Grundy Total Domination Number Problem from [6] readily implies that the problem is NP-complete even when restricted to split graphs.

Let S^1 be any sequence of all vertices in V_1. If $(v_1, \ldots, v_\ell)$ is a γ_{gr}^{t}-sequence of G, then $S^1 \oplus (v_1^2, \ldots, v_\ell^2)$ is an L-sequence of G'. Hence $\gamma_{\mathrm{gr}}^{L}(G') \geq n(G) + \gamma_{\mathrm{gr}}^{t}(G)$.

Now, let $S = (w_1, \ldots, w_k)$ be a γ_{gr}^{L}-sequence of G'. Let $t = \min\{j : w_j \in V_2\}$ and $r = \min\{j : j > t, w_j \in V_2\}$. The vertex w_t L-footprints some vertex from V_1, while vertices from $(w_{r+1}, \ldots, w_k)$ L-footprint only vertices from V_1.

Now we distinguish two cases. Suppose the vertex w_r L-footprints some vertex in V_1. Hence, the subsequence of the vertices in S that are taken from $\widehat{S} \cap V_2$ forms an open neighborhood sequence in G' (all vertices t-footprint some vertex in V_1). Hence, the corresponding sequence forms an open neighborhood sequence in G and $|\widehat{S} \cap V_2| \leq \gamma_{\rm gr}^t(G)$. Thus, $\gamma_{\rm gr}^L(G') = |\widehat{S}| = |\widehat{S} \cap V_1| + |\widehat{S} \cap V_2| \leq n(G) + \gamma_{\rm gr}^t(G)$.

Suppose next that w_r does not L-footprint any vertex in V_1. In this case, the vertex w_r L-footprints only the vertex w_t. Hence, the subsequence of the vertices in S that are taken from $(\widehat{S} - w_r) \cap V_2$ form an open neighborhood sequence in G'. The corresponding sequence therefore forms an open neighborhood sequence in G and $|\widehat{S} \cap V_2| - 1 \leq \gamma_{\rm gr}^t(G)$. Let $u \in V_1 \cap N(w_t)$. We claim that $u \notin S$. Indeed, if u is in S before w_r, then w_r does not L-footprint w_t. If u is in S after w_r, then u does not L-footprint any vertex. This is a contradiction with S being a $\gamma_{\rm gr}^L$-sequence of G'. Hence, $|\widehat{S} \cap V_1| \leq |V_1| - 1 = n(G) - 1$. We again infer that $\gamma_{\rm gr}^L(G') = |\widehat{S}| = |\widehat{S} \cap V_1| + |\widehat{S} \cap V_2| \leq n(G) + \gamma_{\rm gr}^t(G)$. □

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