Filomat 34:10 (2020), 3191–3204
https://doi.org/10.2298/FIL2010191A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Operator Matrices and Their Weyl Type Theorems

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Abstract. We denote the collection of the 2×2 operator matrices with $(1,2)$-entries having closed range by $\mathcal{S}$. In this paper, we study the relations between the operator matrices in the class $\mathcal{S}$ and their component operators in terms of the Drazin spectrum and left Drazin spectrum, respectively. As some application of them, we investigate how the generalized Weyl's theorem and the generalized a-Weyl's theorem hold for operator matrices in $\mathcal{S}$, respectively. In addition, we provide a simple example about an operator matrix in $\mathcal{S}$ satisfying such Weyl type theorems.

1. Introduction

If $\mathcal{H}$ is a complex Hilbert space and we decompose $\mathcal{H}$ as a direct sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, each bounded linear operator T can be expressed as the operator matrix form

$$T = \begin{pmatrix} A & C \\ Z & B \end{pmatrix}$$

with respect to the space of decomposition, where A, B, C, Z are operators from $\mathcal{H}_i$ into $\mathcal{H}_j$ for $i, j = 1, 2$. We shall write $N(T)$ and $R(T)$ for the null space and the range of a bounded linear operator T on $\mathcal{H}$, respectively. Our goal is to find various connections between T and its components. However, it is not easy to find the relations between them without any conditions. So we begin with the following notation.

Notation 1.1. *Throughout this paper, we denote the collection $\mathcal{S}$ as follows:*

$$\mathcal{S} = \left\{ \begin{pmatrix} A & C \\ Z & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K} \mid R(C) \textit{ is closed} \right\}. \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 47A10, 47A53, 58B15
Keywords. 2×2 operator matrices; Browder essential approximate point spectrum; generalized Weyl's theorem; generalized a-Weyl's theorem; generalized a-Browder's theorem
Received: 11 October 2019; Accepted:11 February 2020
Communicated by Dragan S. Djordjević
Corresponding author: Ji Eun Lee
The first author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Science and ICT(NRF-2017R1C1B1006538). The second author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT)(2019R1F1A1058633). The third author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT)(2019R1A2C1002653).
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The class $\mathcal{S}$ is unexpectedly large. For example, if C is a semi-Fredholm operator or semi-regular, i.e., $N(C) \subset \cap_{n\in\mathbb{N}} R(C^n)$ and $R(C)$ is closed, then the operator matrices $\begin{pmatrix} A & C \\ Z & B \end{pmatrix}$ are in the class $\mathcal{S}$. For another example, if for given $x \in \mathcal{H}$ there exists $k > 0$ and a $y \in \mathcal{H}$ such that (i) $Cx = Cy$ and (ii) $\|y\| \le k\|Cx\|$, then $R(C)$ is closed. Hence the operator matrices $\begin{pmatrix} A & C \\ Z & B \end{pmatrix}$ are in the class $\mathcal{S}$.

Lemma 1.2. *([5]) If $M = \begin{pmatrix} A & C \\ Z & B \end{pmatrix} \in \mathcal{S}$, then M has the following matrix representation;*

$$M = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & 0 & C_1 \\ Z & B_1 & B_2 \end{pmatrix} \tag{2}$$

which maps from $\mathcal{H} \oplus N(C) \oplus N(C)^\perp$ to $R(C)^\perp \oplus R(C) \oplus \mathcal{K}$ where $C_1 = C|_{N(C)^\perp}$, $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$, $A_2 = P_{R(C)} A|_{\mathcal{H}}$, B_1 denotes a mapping B from $N(C)$ into $\mathcal{K}$, B_2 denotes a mapping B from $N(C)^\perp$ into $\mathcal{K}$, $P_{R(C)^\perp}$ denotes the projection of $\mathcal{H}$ onto $R(C)^\perp$, and $P_{R(C)}$ denotes the projection of $\mathcal{H}$ onto $R(C)$.

Weyl's theorem for upper triangular operator matrices has been studied by many authors (see [5]-[7], [13], [16], [18]-[20]). This paper is organized as follows. In Section 3, we study the relations between the operator matrices in the class $\mathcal{S}$ and their component operators regarding the left Drazin spectrum. In Section 4, we also explore how the generalized Weyl's theorem and the generalized a-Weyl's theorem hold for operator matrices in $\mathcal{S}$, respectively. As some applications of them, we give a simple example of operator matrices in $\mathcal{S}$ which satisfy Weyl type theorems.

2. Preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, and $\sigma_s(T)$ denote the spectrum, the point spectrum, the approximate point spectrum, and the surjective spectrum of T, respectively. For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

We now simply review several notions of various spectra, which are used in this paper. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* (resp., *lower semi-Fredholm*) if it has closed range and finite dimensional null space (resp., its range has finite co-dimension). If $T \in \mathcal{L}(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm* and *index of a semi-Fredholm operator* T is defined by $i(T) := \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. For $T \in \mathcal{L}(\mathcal{H})$ and a nonnegative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ where $T_0 = T$. If for some integer n the range $R(T^n)$ is closed and T_n is upper (resp., lower) semi-Fredholm, then T is called *upper* (resp., *lower*) *semi-B-Fredholm*. Moreover, if T_n is Fredholm, then T is called *B-Fredholm*. An operator T is called *semi-B-Fredholm* if it is upper or lower semi-B-Fredholm. Let $T \in \mathcal{L}(\mathcal{H})$ and let

$$\Delta(T) := \{n \in \mathbb{N} : m \in \mathbb{N} \text{ and } m \ge n \Rightarrow (R(T^n) \cap N(T)) \subseteq (R(T^m) \cap N(T))\}.$$

Then the *degree of stable iteration* $\operatorname{dis}(T)$ of T is defined as $\operatorname{dis}(T) := \inf \Delta(T)$. Let T be semi-B-Fredholm and let d be the degree of stable iteration of T. It follows from [10, Proposition 2.1] that T_m is semi-Fredholm and $i(T_m) = i(T_d)$ for each $m \ge d$. This enables us to define the *index of semi-B-Fredholm* T as the index of semi-Fredholm T_d. Let $BF(\mathcal{H})$ be the class of all B-Fredholm operators. In [8], he studied this class of operators and he proved [8, Theorem 2.7] that an operator $T \in \mathcal{L}(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$ where T_1 is Fredholm and T_2 is nilpotent. It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let $\mathcal{A}$ be a unital algebra. We say that an element $x \in \mathcal{A}$ is *Drazin invertible of degree k* if there exists an element $a \in \mathcal{A}$ such that $x^k ax = x^k$, $axa = a$, and $xa = ax$. Let $a \in \mathcal{A}$. Then the *Drazin spectrum* is defined by

$$\sigma_D(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}.$$

It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that (see [17, Corollary 2.2]) $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is nilpotent. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index 0. We review some spectra as follows;

(1) the *semi-B-Fredholm spectrum* $\sigma_{SBF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-}B\text{-Fredholm}\}$,

(2) the *B-Fredholm spectrum* $\sigma_{BF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Fredholm}\}$,

(3) the *B-Weyl spectrum* $\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\}$.

Now we define the next sets;

$$SBF_+(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : T \text{ is upper semi-}B\text{-Fredholm}\},$$

$$SBF_-(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : T \text{ is lower semi-}B\text{-Fredholm}\},$$

$$SBF_+^-(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : T \in SBF_+(\mathcal{H}) \text{ and } i(T) \leq 0\},$$

$$SBF_-^+(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : T \in SBF_-(\mathcal{H}) \text{ and } i(T) \geq 0\},$$

$$LD(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\},$$

$$RD(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : q(T) < \infty \text{ and } R(T^{q(T)}) \text{ is closed}\}.$$

By definitions, we recall the *upper semi-B-essential approximate point spectrum* $\sigma_{SBF_+^-}(T)$, the *lower semi-B-essential approximate point spectrum* $\sigma_{SBF_-^+}(T)$, the *left Drazin spectrum* $\sigma_{LD}(T)$, and the *right Drazin spectrum* $\sigma_{RD}(T)$ given by

$$\begin{aligned}
\sigma_{SBF_+^-}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{H})\},\\
\sigma_{SBF_-^+}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_-^+(\mathcal{H})\},\\
\sigma_{LD}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathcal{H})\},\\
\sigma_{RD}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \notin RD(\mathcal{H})\}.
\end{aligned}$$

It is well known that

$$\begin{gathered}
\sigma_{SBF_+^-}(T) \cup \sigma_{SBF_-^+}(T) = \sigma_{BW}(T),\\
\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{acc} \sigma_a(T) \subseteq \sigma_D(T),\\
\sigma_{SBF_-^+}(T) \subseteq \sigma_{RD}(T) = \sigma_{SBF_-^+}(T) \cup \operatorname{acc} \sigma_s(T) \subseteq \sigma_D(T).
\end{gathered}$$

The notation $p_0(T)$ (resp., $p_0^a(T)$) denotes the set of all poles (resp., left poles) of T, while $\pi_0(T)$ (resp., $\pi_0^a(T)$) is the set of all eigenvalues of T which is an isolated point in $\sigma(T)$ (resp., $\sigma_a(T)$). We say that *generalized Browder's theorem* for T if $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$, *generalized a-Browder's theorem* for T if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = p_0^a(T)$, *generalized Weyl's theorem* for T if $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, and *generalized a-Weyl's theorem* for T if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi_0^a(T)$. It is well known that

$$\begin{array}{ccc}
\text{generalized } a\text{-Weyl's theorem} & \Longrightarrow & \text{generalized Weyl's theorem}\\
\Downarrow & & \Downarrow\\
\text{generalized } a\text{-Browder's theorem} & \Longrightarrow & \text{generalized Browder's theorem.}
\end{array}$$

3. The filling in holes of the spectra for operator matrices

Throughout this section, whenever $M = \begin{pmatrix} A & C \\ Z & B \end{pmatrix}$ *is in the class* $\mathcal{S}$ *in Notation 1.1, we denote M by the matrix representation as* (2) *for every* $Z \in \mathcal{L}(\mathcal{H}, \mathcal{K})$.

Let $M = \begin{pmatrix} A & C \\ Z & B \end{pmatrix}$ be an operator matrix in the class $\mathcal{S}$. Since $R(C)$ is closed, $C_1 = C|_{N(C)^\perp} : N(C)^\perp \to R(C)$ is invertible. For a given complex number λ, using the representation of Lemma 1.2, we write $M - \lambda$ as follows;

$$\begin{aligned} M - \lambda &= \begin{pmatrix} A_1 - \lambda & 0 & 0 \\ A_2 - \lambda & 0 & C_1 \\ Z & B_1 - \lambda & B_2 - \lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (B_2 - \lambda)C_1^{-1} \end{pmatrix} \begin{pmatrix} B_1 - \lambda & \triangle_\lambda & 0 \\ 0 & A_1 - \lambda & 0 \\ 0 & 0 & C_1 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2 - \lambda) & 0 & I \end{pmatrix} \end{aligned} \tag{3}$$

where $A_1 - \lambda = P_{R(C)^\perp}(A - \lambda)|_{\mathcal{H}}$, $A_2 - \lambda = P_{R(C)}(A - \lambda)|_{\mathcal{H}}$, $B_1 - \lambda = (B - \lambda)|_{N(C)}$, $B_2 - \lambda = (B - \lambda)|_{N(C)^\perp}$ and $\triangle_\lambda = Z - (B_2 - \lambda)C_1^{-1}(A_2 - \lambda)$ (see [5, Page 714] for more details). Note that

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & B_2C_1^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}A_2 & 0 & I \end{pmatrix} \text{ are invertible.} \tag{4}$$

From now on, we use the above matrix representation (3) for $M - \lambda$. We start with the filling in holes problem for the Drazin spectrum of the operator matrices in the class $\mathcal{S}$.

Lemma 3.1. *Let* $S, T, A,$ *and* $B \in \mathcal{L}(\mathcal{H})$. *Then the following statements hold.*

(i) If ST is Drazin invertible and T is invertible, then S is also Drazin invertible.

(ii) If A and B are invertible, and $T = ASB$, *then T is Drazin invertible if and only if S is Drazin invertible.*

Proof. (i) Since $U = ST$ is Drazin invertible, 0 is a pole of the resolvent operator U^{-1} of order p. Moreover, $R(U^p)$ is closed and

$$\mathcal{H} = R(U^p) \oplus N(U^p). \tag{5}$$

On a direct sum of $\mathcal{H}$ we can write U by $U = U_1 \oplus U_2$, where U_1 is invertible on $R(U^p)$ and U_2, the restriction of U to $N(U^p)$, is nilpotent of order p. Suppose that T is invertible. From (5), we have

$$T^{-1}(\mathcal{H}) = \mathcal{H} = R(U^p) \oplus N(U^p).$$

So $S(\mathcal{H}) = UT^{-1}(\mathcal{H}) = (U_1 \oplus U_2)(R(U^p) \oplus N(U^p))$ where U_1 is invertible on $R(U^p)$ and U_2 is nilpotent of order p. Therefore S is Drazin invertible.

(ii) Suppose that A and B are invertible, and $T = ASB$. If T is Drazin invertible, it follows from (i) that AS is Drazin invertible. By [21, Theorem 2.3], SA is also Drazin invertible. Since A is invertible, S is Drazin invertible again from (i). The converse implication is satisfied by the same way. □

Remark 3.2. In general, we observe that even though ST is Drazin invertible and S is invertible, T may not be Drazin invertible. For example, let U be the unilateral shift operator on $l^2(\mathbb{N})$. Since the spectrum $\sigma(U)$ of U is the closed unit disc, both U and U^* are not Drazin invertible. If S and T have the following operator matrix forms;

$$S = I \oplus \begin{pmatrix} I & U \\ 0 & I \end{pmatrix} \text{ and } T = I \oplus \begin{pmatrix} -U & 0 \\ I & 0 \end{pmatrix},$$

then $ST = I \oplus \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ is Drazin invertible and S is invertible. However, $S^{-1}(ST) = T$ is not Drazin invertible.

Lemma 3.3. *[6, Lemma 2.2] Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be given. Then the following implication holds for every $C \in B(\mathcal{K}, \mathcal{H})$.*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \text{ is right Drazin invertible } \Longrightarrow B \text{ is right Drazin invertible.}$$

Lemma 3.4. *[12, Theorem 2.3] Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be given operators such that the ranges $R(A)$ and $R(B)$ are closed. Then, the range $R(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix})$ is closed for every $C \in B(\mathcal{K}, \mathcal{H})$ if and only if at least one of $\dim N(A^*)$ and $\dim N(B)$ is finite.*

Lemma 3.5. *Let $M \in \mathcal{S}$. Then, with the same notation as in (3), the following statements hold.*
(i) *If M is Drazin invertible, then B_1 is left Drazin invertible and A_1 is right Drazin invertible.*
(ii) *If $\dim N(M) < \infty$ and both A_1 and B_1 are left Drazin invertible, then so is M.*

Proof. If $M \in \mathcal{S}$, then M can be written as

$$M = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & B_2C_1^{-1} \end{pmatrix} \begin{pmatrix} B_1 & \triangle_0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}A_2 & 0 & I \end{pmatrix},$$

where $A_1 = P_{R(C)^\perp}A|_{\mathcal{H}}$, $A_2 = P_{R(C)}A|_{\mathcal{H}}$, $B_1 = B|_{N(C)}$, $B_2 = B|_{N(C)^\perp}$ and $\triangle_0 = Z - B_2C_1^{-1}A_2$.

(i) Suppose that M is Drazin invertible. By Lemma 3.1, $\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}$ is Drazin invertible. Since $\sigma_{RD}(T) \subset \sigma_D(T)$ for an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$, it follows that $\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}$ is right Drazin invertible. By Lemma 3.3, we know that A_1 is right Drazin invertible. Also, since $\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}$ is left Drazin invertible, it follows from [2, Theorem 2.1] that $\begin{pmatrix} B_1^* & 0 \\ \triangle_0^* & A_1^* \end{pmatrix}$ is right Drazin invertible. But, there exists a unitary operator $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} B_1^* & 0 \\ \triangle_0^* & A_1^* \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A_1^* & \triangle_0^* \\ 0 & B_1^* \end{pmatrix}.$$

Hence $\begin{pmatrix} B_1^* & 0 \\ \triangle_0^* & A_1^* \end{pmatrix}$ and $\begin{pmatrix} A_1^* & \triangle_0^* \\ 0 & B_1^* \end{pmatrix}$ are similar. Again by Lemma 3.3, B_1^* is right Drazin invertible. Therefore B_1 is left Drazin invertible.

(ii) Suppose that both A_1 and B_1 are left Drazin invertible. Then $p(A_1) < \infty$, $p(B_1) < \infty$, and $R(T^{p(A_1)+1})$ and $R(T^{p(B_1)+1})$ are closed. Let $k := p(B_1) < \infty$ and $l := p(A_1) < \infty$. Then we can choose $n := \max\{k, l\}$. By [11, Lemma 2.2], we know that $p(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}) \leq 2n < \infty$. In fact, it is known that for each positive integer k,

$$\dim N(M^k) = \dim N\left(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}^k\right).$$

However, $\dim N(M) < \infty$, thus M has finite ascent. So it suffices to show that the range of the next operator matrix is closed;

$$\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}^{2n+1} = \begin{pmatrix} {B_1}^{2n+1} & {B_1}^{2n}\triangle_0 + \cdots + \triangle_0 {A_1}^{2n} \\ 0 & {A_1}^{2n+1} \end{pmatrix}.$$

On the other hand, since $R({A_1}^{l+1})$ and $R({B_1}^{k+1})$ are closed, it follows from [2, Lemma 1.1] that both $R({A_1}^{2n+1})$ and $R({B_1}^{2n+1})$ are closed for $n := \max\{k, l\}$. Since A_1 has finite ascent, it is well known that $\dim N(A_1) < \infty$, so this implies from [2, Remark 2.3] that $\dim N({A_1}^{2n+1}) < \infty$. Hence it follows from Lemma 3.4 that $R\Big[\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}^{2n+1}\Big]$ is closed. Therefore $R(M^{2n+1})$ is closed, so this means that M is left Drazin invertible. □

In [22], Zhang et al. discussed that $\sigma_D(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}) \cup \mathcal{Q} = \sigma_D(A) \cup \sigma_D(B)$, where $\mathcal{Q}$ is the union of certain holes in $\sigma_D(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix})$ which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$. So it is naturally to ask what is exactly the set $\mathcal{Q}$ for 2×2 operator matrices M in the class $\mathcal{S}$. From this argument, we proved the following theorem.

Theorem 3.6. *For $M \in \mathcal{S}$, then the following property holds:*

$$\sigma_D(A_1) \cup \sigma_D(B_1) = \sigma_D(M) \cup \mathcal{Q}$$

where $\sigma_D(\cdot)$ denotes the Drazin spectrum of $\cdot$ and $\mathcal{Q}$ is the union of certain of the holes in $\sigma_D(M)$ which happen to be subsets of $\sigma_D(A_1) \cap \sigma_D(B_1)$.

Proof. We first show that for $M \in \mathcal{S}$,

$$[\sigma_D(B_1) \cup \sigma_D(A_1)] \setminus [\sigma_D(B_1) \cap \sigma_D(A_1)] \subset \sigma_D(M) \subset \sigma_D(B_1) \cup \sigma_D(A_1). \tag{6}$$

Indeed, let $\lambda \notin \sigma_D(B_1) \cup \sigma_D(A_1)$. Then both $B_1 - \lambda$ and $A_1 - \lambda$ are Draizin invertible, so they have finite ascent and descent. It follows from [14, Lemma 2.5] that $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ has also finite ascent and descent, so this is Drazin invertible. From Lemma 3.1, M is also Drazin invertible. To show the first inclusion, we let $\lambda \in [\sigma_D(B_1) \cup \sigma_D(A_1)] \setminus \sigma_D(M)$. Then $M - \lambda$ is Drazin invertible. Again from Lemma 3.1 and (3), we get that $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda & 0 \\ 0 & A_1 - \lambda & 0 \\ 0 & 0 & C_1 \end{pmatrix}$ is Drazin invertible. Since C_1 is invertible, it follows that $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is Drazin invertible. Note that

$$\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A_1 - \lambda \end{pmatrix} \begin{pmatrix} I & \triangle_\lambda \\ 0 & I \end{pmatrix} \begin{pmatrix} B_1 - \lambda & 0 \\ 0 & I \end{pmatrix}. \tag{7}$$

If $A_1 - \lambda$ is Drazin invertible, then it follows from (7) that $B_1 - \lambda$ is Drazin invertible. Similarly, if $B_1 - \lambda$ is Drazin invertible, then so is $A_1 - \lambda$. This means that $\lambda \in \sigma_D(B_1) \cap \sigma_D(A_1)$. Thus (6) can be proved.

Next, we claim that for $M \in \mathcal{S}$, we have

$$\eta(\sigma_D(M)) = \eta(\sigma_D(B_1) \cup \sigma_D(A_1)), \tag{8}$$

where $\eta(K)$ denotes the polynomially convex hull of the compact set $K \subset \mathbb{C}$. Indeed, if $M - \lambda$ is Drazin invertible, then $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is Drazin invertible. By Lemma 3.5(i), we get that

$$\sigma_{LD}(B_1) \cup \sigma_{RD}(A_1) \subset \sigma_D(M).$$

Since $int(\sigma_D(M)) \subset int(\sigma_D(A_1) \cup \sigma_D(B_1))$ by (6), where the interior of a set S is denoted by $int(S)$, it follows from the previous fact and punctured neighborhood theorem ([22]) that

$$\begin{aligned} \partial(\sigma_D(B_1) \cup \sigma_D(A_1)) &\subset \partial(\sigma_D(B_1)) \cup \partial(\sigma_D(A_1)) \\ &\subset \sigma_{LD}(B_1) \cup \sigma_{RD}(A_1) \subset \sigma_D(M). \end{aligned}$$

Therefore it follows from (6) that (8) can be proved, so that the passage from $\sigma_D(B_1) \cup \sigma_D(A_1)$ to $\sigma_D(M)$ is the filling in certain of the holes in $\sigma_D(B_1) \cap \sigma_D(A_1)$. Hence this completes the proof of this theorem. □

We recall that $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. Then we get the next corollary.

Corollary 3.7. *Let $M \in \mathcal{S}$. Assume that one of the following statements holds.*
(i) *$\sigma_D(B_1) \cap \sigma_D(A_1)$ has no interior points.*
(ii) *$\sigma_i(M) = \sigma_i(B_1) \cup \sigma_i(A_1)$ for $\sigma_i \in \{\sigma, \sigma_b, \sigma_w\}$.*
(iii) *B_1^* or A_1 has the single valued extension property.*
Then we have

$$\sigma_D(M) = \sigma_D(B_1) \cup \sigma_D(A_1). \tag{9}$$

Proof. If (i) holds, the proof is clear from Theorem 3.6.

Suppose that (ii) holds. It suffices to show that this inclusion $\sigma_D(B_1) \cup \sigma_D(A_1) \subset \sigma_D(M)$ are satisfied since $\sigma_D(M) \subset \sigma_D(B_1) \cup \sigma_D(A_1)$ by Theorem 3.6. Without loss of generality, it is enough to show that if $0 \notin \sigma_D(M)$, then $0 \notin \sigma_D(B_1) \cup \sigma_D(A_1)$. If $0 \notin \sigma_D(M)$, then M is Drazin invertible and $0 \in iso\sigma(M)$. Since $0 \in iso\sigma(M)$, then there exists $\epsilon > 0$ such that for $0 < |\lambda| < \epsilon$, $M - \lambda$ is invertible. If $\sigma(M) = \sigma(B_1) \cup \sigma(A_1)$ holds, then both $B_1 - \lambda$ and $A_1 - \lambda$ are invertibel for $0 < |\lambda| < \epsilon$, so this implies that B_1 and A_1 are Drazin invertible. We now suppose that $\sigma_b(M) = \sigma_b(B_1) \cup \sigma_b(A_1)$ holds. Then from [7], $B_1 - \lambda$ is left invertible and $A_1 - \lambda$ is right invertible for any $0 < |\lambda| < \epsilon$. Thus both the ascent of $B_1 - \lambda$ and the descent of $A_1 - \lambda$ are zero. Since $M - \lambda$ is invertible for $0 < |\lambda| < \epsilon$, it follows that $0 \notin \sigma_b(M) = \sigma_b(B_1) \cup \sigma_b(A_1)$, so $B_1 - \lambda$ and $A_1 - \lambda$ are Browder. Hence they are invertible for $0 < |\lambda| < \epsilon$. This means that $0 \notin \sigma_D(B_1) \cup \sigma_D(A_1)$. So it remains to prove that $\sigma_w(M) = \sigma_w(B_1) \cup \sigma_w(A_1)$ holds. The proof follows from [1, Theorem 3.4] by the similar method.

Finally, assume (iii) holds. Without loss of generality, if $0 \notin \sigma_D(M)$, then there exists $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, $0 < |\lambda| < \epsilon$, $M - \lambda$ is invertible. Thus $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is also invertible. So, $B_1 - \lambda$ is left invertible and $A_1 - \lambda$ is right invertible. Since B^* or A_1 has the single valued extension property, both $A_1 - \lambda$ and $B_1 - \lambda$ are invertible for $0 < |\lambda| < \epsilon$. This means that B_1 and A_1 are Drazin invertible. Hence $\sigma_D(A_1) \cup \sigma_D(B_1) \subset \sigma_D(M)$. □

Let $\rho_D(T) = \mathbb{C} \setminus \sigma_D(T)$ be the *Drazin resolvent set* of $T \in \mathcal{L}(\mathcal{H})$. Now we apply the main result in [22, Theorem 3.1] to full matrix version $M \in \mathcal{S}$.

Corollary 3.8. *Suppose that $M \in \mathcal{S}$. Then the following relation holds;*

$$\bigcap_{Z \in \mathcal{L}(\mathcal{H},\mathcal{K})} \sigma_D(M) \subseteq \left(\bigcap_{Z \in \mathcal{L}(\mathcal{H},\mathcal{K})} \sigma(M) \right) \setminus [\rho_D(B_1) \cap \rho_D(A_1)].$$

Moreover, if one of the following conditions holds;
(1) $\sigma(B_1) \cap \sigma(A_1) = \emptyset$; (2) $int(\sigma_p(A_1)) = \emptyset$;
(3) $int(\sigma_p(B_1^*)) = \emptyset$; (4) $\sigma_s(A_1) = \sigma(A_1)$; (5) $\sigma_a(B_1) = \sigma(B_1)$,
then we have

$$\bigcap_{Z \in \mathcal{L}(\mathcal{H},\mathcal{K})} \sigma_D(M) = \left(\bigcap_{Z \in \mathcal{L}(\mathcal{H},\mathcal{K})} \sigma(M) \right) \setminus [\rho_D(B_1) \cap \rho_D(A_1)]$$

Proof. The proof follows from [22, Theorem 3.1]. □

Motivated by Theorem 3.6, we have a similar development for the left Drazin spectrum. So we first recall the following lemma. Here, we say that $\eta(K)$ denotes the *polynomially convex hull* of the compact set $K \subset \mathbb{C}$.

Lemma 3.9. *([6]) Let $T \in \mathcal{L}(\mathcal{H})$. Then*

$$\eta(\sigma_\tau(T)) = \eta(\sigma_D(T))$$

holds for $\sigma_\tau \in \{\sigma_{SBF}, \sigma_{BW}\sigma_{LD}, \sigma_{RD}\}$.

Lemma 3.10. *If $M \in \mathcal{S}$, then the following properties hold;*

$$\eta(\sigma_\tau(M)) = \eta(\sigma_\tau(B_1) \cup \sigma_\tau(A_1)), \tag{10}$$

where $\sigma_\tau \in \{\sigma_{SBF}, \sigma_{BW}, \sigma_{LD}, \sigma_{RD}\}$.

Proof. Let $\sigma_\tau \in \{\sigma_{SBF}, \sigma_{BW}, \sigma_{LD}, \sigma_{RD}\}$. By Lemma 3.9, we obtain that $\eta(\sigma_\tau(B_1) \cup \sigma_\tau(A_1)) = \eta(\sigma_D(B_1) \cup \sigma_D(A_1))$ for $B_1 \in \mathcal{L}(N(C))$ and $A_1 \in \mathcal{L}(\mathcal{H})$. Thus by [22, Theorem 2.9], we have

$$\begin{aligned} \eta(\sigma_\tau(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix})) &= \eta(\sigma_D(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix})) \\ &= \eta(\sigma_D(B_1) \cup \sigma_D(A_1)) = \eta(\sigma_\tau(B_1) \cup \sigma_\tau(A_1)). \end{aligned}$$

Therefore we have the desired result. □

Theorem 3.11. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Then the next equality holds;*

$$\sigma_{LD}(B_1) \cup \sigma_{LD}(A_1) = \sigma_{LD}(M) \cup \mathcal{W},$$

where $\mathcal{W}$ is the union of the certain of the holes in $\sigma_{LD}(M)$ which happen to be subsets of $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$.

Proof. We first show that

$$\sigma_{LD}(B_1) \subseteq \sigma_{LD}(M) \subseteq \sigma_{LD}(B_1) \cup \sigma_{LD}(A_1). \tag{11}$$

Since the second inclusion in (11) holds from Lemma 3.5(ii), we only need to show the first inclusion in (11). Suppose that $M - \lambda$ is left Drazin invertible. Since $\dim N(M - \lambda) < \infty$ for every $\lambda \in \mathbb{C}$, it follows from (3) and (4) that $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is left Drazin invertible. By [2, Theorem 2.1] and Lemma 3.5(i), $B_1 - \lambda$ is left Drazin invertible. Consequently, (11) is proved and this implies that

$$(\sigma_{LD}(B_1) \cup \sigma_{LD}(A_1)) \setminus \sigma_{LD}(M) \subseteq \sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1). \tag{12}$$

Therefore, from Lemma 3.5(ii) the passage from $\sigma_{LD}(B_1) \cup \sigma_{LD}(A_1)$ to $\sigma_{LD}\left(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}\right)$ is the filling in certain of the holes in $\sigma_{LD}\left(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}\right)$. Moreover, the certain of the holes in $\sigma_{LD}\left(\begin{pmatrix} B_1 & \triangle_0 \\ 0 & A_1 \end{pmatrix}\right)$ should occur in $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ from (12). This proof is complete. □

In the following example, we observe an operator matrix M satisfying the assumptions in Theorem 3.11.

Example 3.12. Let U be the unilateral shift on $l^2(\mathbb{N})$. We denote an operator matrix M on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ as follows;

$$M := \begin{pmatrix} UU^* & -I \\ I & I \end{pmatrix}.$$

Since the identity operator has closed range, M belongs to $\mathcal{S}$. If $x \oplus y \in N(M - \lambda)$ where $x = (x_n)$ and $y = (y_n)$ for $n = 1, 2, 3, \cdots$, then it follows from a simple calculation that

$$\begin{cases} \lambda x_1 + y_1 = 0 \\ x_1 + (1-\lambda) y_1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1-\lambda) x_i - y_i = 0 \\ x_i + (1-\lambda) y_i = 0 \quad \text{for} \quad i = 2, 3, \cdots, \end{cases}$$

so that $x \oplus y = 0 \oplus 0$ for every $\lambda \in \mathbb{C}$. This means that every λ is a complex value of finite multiplicity for M.

Remark 3.13. Suppose that $M \in \mathcal{S}$. Then we have the similar result as Theorem 3.11 in terms of the right Drazin spectrum. This means that the following equality is satisfied.

$$\sigma_{RD}(B_1) \cup \sigma_{RD}(A_1) = \sigma_{RD}(M) \cup \mathcal{W},$$

where $\mathcal{W}$ is the union of the certain of the holes in $\sigma_{RD}(M)$ which happen to be subsets of $\sigma_{RD}(B_1) \setminus \sigma_{RD}(A_1)$.

Corollary 3.14. *Let $M \in \mathcal{S}$. Then the following statements hold.*

(i) If every λ is a complex value of finite multiplicity for M and $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points, then

$$\sigma_{LD}(M) = \sigma_{LD}(A_1) \cup \sigma_{LD}(B_1).$$

(ii) If $\sigma_{RD}(B_1) \setminus \sigma_{RD}(A_1)$ has no interior points, then

$$\sigma_{RD}(M) = \sigma_{RD}(A_1) \cup \sigma_{RD}(B_1).$$

In [3, Theorem 2.5], Aiena et al. proved that every left Drazin invertible operator $T \in \mathcal{L}(\mathcal{H})$ is equivalent to an upper semi-B-Fredholm operator having the single valued extension property at 0. So the next result comes from this argument and Corollary 3.14.

Theorem 3.15. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Suppose that $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points and both A_1 and B_1 have the single valued extension property. Then*

$$\sigma_{SBF_+^-}(M) = \sigma_{SBF_+^-}(A_1) \cup \sigma_{SBF_+^-}(B_1).$$

Proof. Let $\lambda \notin \sigma_{SBF_+^-}(M)$. Then $M - \lambda$ is upper semi-B-Fredholm and $i(M - \lambda) \leq 0$. Since M has the single valued extension property by [1], we have that M is left Drazin invertible. So it follows from Corollary 3.14 that $A_1 - \lambda$ and $B_1 - \lambda$ are also left Drazin invertible, and hence this means that they are upper semi-B-Fredholm and their indices are not positive, respectively. Consequently, we get the next inclusion,

$$\sigma_{SBF_+^-}(B_1) \cup \sigma_{SBF_+^-}(A_1) \subseteq \sigma_{SBF_+^-}(M). \tag{13}$$

To show the opposite inclusion of (13), let $\lambda \notin \sigma_{SBF_+^-}(B_1) \cup \sigma_{SBF_+^-}(A_1)$. Then $B_1 - \lambda$ and $A_1 - \lambda$ are left Drazin invertible from [3, Theorem 2.5]. Since $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points, $M - \lambda$ is left Drazin invertible. Therefore $\lambda \notin \sigma_{SBF_+^-}(M)$ and this implies that $\sigma_{SBF_+^-}(B_1) \cup \sigma_{SBF_+^-}(A_1) \supseteq \sigma_{SBF_+^-}(M)$. Hence the proof is completed. □

Like the case of the upper semi-B-essential approximate point spectrum, we also observe the similar results for the lower semi-B-essential approximate point spectrum and the B-weyl spectrum.

Remark 3.16. Let $M \in \mathcal{S}$. If $\sigma_{RD}(B_1) \setminus \sigma_{RD}(A_1)$ has no interior points, and both A_1^* and B_1^* have the single valued extension property, then we have the next equality.

$$\sigma_{SBF_-^+}(M) = \sigma_{SBF_-^+}(A_1) \cup \sigma_{SBF_-^+}(B_1).$$

As a consequence of Theorem 3.15, we get the following corollary.

Corollary 3.17. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Suppose that $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ and $\sigma_{RD}(B_1) \setminus \sigma_{RD}(A_1)$ have no interior points and both A_1 and B_1 (or A_1^* and B_1^*) have the single valued extension property. Then*

$$\sigma_{BW}(M) = \sigma_{BW}(B_1) \cup \sigma_{BW}(A_1).$$

4. Weyl type theorems for operator matrices

In this section, we explore how generalized Weyl's theorem and generalized a-Weyl's theorem for $M \in \mathcal{S}$ hold. We characterize the operator matrices $M \in \mathcal{S}$ satisfying generalized Browder's theorem and generalized a-Browder's theorem, respectively, by means of localized single valued extension property under the condition which $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points. We start with the following lemma.

Lemma 4.1. *Let A_1, B_1 and M be the same notations as in* (3)*. Suppose A_1 and B_1 have the single valued extension property. Then M has also the single valued extension property.*

Proof. Let D be an open set in $\mathbb{C}$ and $f = f_1 \oplus f_2 \oplus f_3 : D \to \mathcal{H} \oplus N(C) \oplus N(C)^{\perp}$ be an analytic function such that

$$(M-\lambda)\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ f_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

on D. Since $\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (B_2-\lambda)C_1^{-1} \end{pmatrix}$ is invertible, it follows from (3) that

$$\begin{pmatrix} B_1-\lambda & \triangle_\lambda & 0 \\ 0 & A_1-\lambda & 0 \\ 0 & 0 & C_1 \end{pmatrix}\begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \\ g_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \\ g_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix}\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ f_3(\lambda) \end{pmatrix}$. Therefore, we get that

$$\begin{cases} (B_1-\lambda)g_1(\lambda) + \triangle_\lambda g_2(\lambda) = 0 \\ (A_1-\lambda)g_2(\lambda) = 0 \\ C_1 g_3(\lambda) = 0 \end{cases}$$

on D. Since C_1 is invertible, $g_3(\lambda) = 0$. Moreover, since A_1 and B_1 have the single valued extension property, it follows that $g_2(\lambda) = 0$ and $g_1(\lambda) = 0$. Therefore

$$0 = \begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \\ g_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix}\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ f_3(\lambda) \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix}$ is invertible, it follows that

$$\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ f_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

on D. Hence M has the single valued extension property. □

Now, we examine necessary and sufficiant conditions for which operator matrices in the class $\mathcal{S}$ satisfy generalized a-Browder's (resp., Browder's) theorem.

Theorem 4.2. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Suppose that $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points.*

(i) Generalized Browder's theorem holds for M if and only if either A_1 and B_1 or A_1^* and B_1^* have the single valued extension property at every $\lambda \notin \sigma_{BW}(M)$.

(ii) Generalized a-Browder's theorem holds for M if and only if A_1 and B_1 have the single valued extension property at every $\lambda \notin \sigma_{SBF_+^-}(M)$.

Proof. (i) Suppose that A_1 and B_1 have the single valued extension property at $\lambda \notin \sigma_{BW}(M)$. We will show that $\sigma_D(M) = \sigma_{BW}(M)$. Since $\sigma_{BW}(M) \subseteq \sigma_D(M)$ holds, we will prove the opposite inclusion. Let $\lambda \notin \sigma_{BW}(M)$. Then A_1 and B_1 have the single valued extension property at λ and so M has the single valued extension property at λ by Lemma 4.1. Since $M - \lambda$ is B-Weyl, it follows from [1, Theorem 3.4] and (3) that $p(M - \lambda) = q(M - \lambda) < \infty$. Thus $M - \lambda$ is Drazin invertible, and hence $\sigma_D(M) \subseteq \sigma_{BW}(M)$. Therefore generalized Browder's theorem holds for M. Assume that A_1^* and B_1^* have the single valued extension property at $\lambda \notin \sigma_{BW}(M)$. If $\lambda \notin \sigma_{BW}(M)$, then M^* has the single valued extension property at λ. Since $M - \lambda$ is B-Weyl, it follows that $\lambda \notin \sigma_D(M)$. Thus $\sigma_D(M) = \sigma_{BW}(M)$.

Conversely, we assume that generalized Browder's theorem holds for M. Then $\sigma_D(M) = \sigma_{BW}(M)$. Let $\lambda \notin \sigma_{BW}(M)$. Then $M - \lambda$ is Drazin invertible. Since $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points, it follows from Corollary 3.14 that $B_1 - \lambda$ and $A_1 - \lambda$ are both left Drazin invertible. Since $A_1 - \lambda$ is right Drazin invertible by Lemma 3.5(i), it follows that $A_1 - \lambda$ is Drazin invertible. Thus $B_1 - \lambda$ is also Drazin invertible by [22, Corollary 2.8]. Therefore A_1 and B_1, as well as, A_1^* and B_1^* have the single valued extension property at λ.

(ii) Suppose that A_1 and B_1 have the single valued extension property at $\lambda \notin \sigma_{SBF_+^-}(M)$. We will show that $\sigma_{LD}(M) = \sigma_{SBF_+^-}(M)$. Since $\sigma_{SBF_+^-}(M) \subseteq \sigma_{LD}(M)$ holds, we only show the opposite inclusion. Let $\lambda \notin \sigma_{SBF_+^-}(M)$. Then A_1 and B_1 have the single valued extension property at λ and so M has the single valued extension property at λ by Lemma 4.1. Since $M - \lambda$ is upper semi-B-Fredholm, it follows from [2, Theorem 2.5] that $M - \lambda$ is left Drazin invertible. Thus $\sigma_{LD}(M) = \sigma_{SBF_+^-}(M)$.

Conversely, we suppose that generalized a-Browder's theorem holds for M. Then $\sigma_{LD}(M) = \sigma_{SBF_+^-}(M)$. Let $\lambda \notin \sigma_{SBF_+^-}(M)$. Then $M - \lambda$ is left Drazin invertible. Since $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points, it follows from Corollary 3.14 that $A_1 - \lambda$ and $B_1 - \lambda$ are both left Drazin invertible. Therefore A_1 and B_1 have the single valued extension property at λ. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, *paranormal* if $||Tx||^2 \leq ||T^2x||||x||$ for all $x \in \mathcal{H}$, respectively.

Corollary 4.3. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. If one of the following statements holds;*
(i) *A has finite spectrum and B is paranormal,*
(ii) *$A = I$ and B is paranormal,*
then M satisfies the generalized a-Browder's theorem.

Proof. (i) Suppose that A has finite spectrum and B is paranormal. Since $M \in \mathcal{S}$, it follows that M has the following matrix representation as in (2) ;

$$M = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & 0 & C_1 \\ Z & B_1 & B_2 \end{pmatrix}.$$

Then B_1 is also paranormal. In this case, A_1 and B_1 have the single valued extension property. Moreover, $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points. Hence, from Theorem 4.2, M satisfies the generalized a-Browder's theorem.

(ii) If $A = I$ and B is paranormal, then A_1 and B_1 are also paranormal. Moreover, in this case, $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points. In this case, since B_1 and A_1 have paranormal, they have the single valued extension property. Hence M satisfies the generalized a-Browder's theorem from Theorem 4.2. □

Example 4.4. Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Suppose that $\sigma(A) = \{0,1\}$ and B is a weighted shift defined by $Be_n = \beta_n e_{n+1}$ where $\beta_n = \frac{n+1}{n+2}$. Then B is clearly a hyponormal operator. Hence M satisfies generalized a-Browder's theorem from Corollary 4.3.

In [23, Theorem 3.1], Zguitti investigated how the generalized Weyl's theorem holds for upper triangular operator matrices. So we have naturally the next theorem for an operator matrix in the class $\mathcal{S}$.

Theorem 4.5. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Suppose that $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ have no interior points. Then the following statements hold.*
(i) *Suppose that A_1 and B_1 have the single valued extension property at every $\lambda \notin \sigma_{SBF_+^-}(M)$. If $B_1 \oplus A_1$ satisfies generalized a-Weyl's theorem, then so does M.*
(ii) *Assume that either A_1 and B_1 or A_1^* and B_1^* have the single valued extension property at $\lambda \notin \sigma_{BW}(M)$. If $B_1 \oplus A_1$ satisfies generalized Weyl's theorem, then so does M.*

Proof. (i) Let A_1 and B_1 have the single valued extension property at $\lambda \notin \sigma_{SBF_+^-}(M)$. Then generalized a-Browder's theorem holds for M by Theorem 4.2. This means that $\sigma_a(M) \setminus \sigma_{SBF_+^-}(M) \subseteq \pi_0^a(M)$. So we will show that $\pi_0^a(M) \subseteq \sigma_a(M) \setminus \sigma_{SBF_+^-}(M)$. Let $\lambda \in \pi_0^a(M)$. Then $\lambda \in \text{iso}\,\sigma_a(M)$ and $\alpha(M-\lambda) > 0$. We first prove that $\sigma_a(M) = \sigma_a(B_1 \oplus A_1)$. By [20, (12)], it is obvious that $\sigma_a(B_1) \subseteq \sigma_a(B_1 \oplus A_1) \subseteq \sigma_a(B_1) \cup \sigma_a(A_1)$. Thus we have

$$(\sigma_a(B_1) \cup \sigma_a(A_1)) \setminus \sigma_a(M) \subseteq \sigma_a(A_1) \setminus \sigma_a(B_1). \tag{14}$$

It follows from [20, Theorem 2] that the passage from $\sigma_a(M)$ to $\sigma_a(B_1) \cup \sigma_a(A_1)$ is the filling in certain of the holes in $\sigma_a(M)$. But (14) says that the filling in certain of the holes in $\sigma_a(M)$ should occur in $\sigma_a(A_1) \setminus \sigma_a(B_1)$. Since acc $\sigma_a(A_1) \subseteq \sigma_{LD}(A_1)$, if $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points then $\sigma_a(A_1) \setminus \sigma_a(B_1)$ also has no interior points. Thus $\sigma_a(M) = \sigma_a(B_1) \cup \sigma_a(A_1)$. It follows that $\sigma_a(M) = \sigma_a(B_1 \oplus A_1)$, and so $\lambda \in \text{iso}\,\sigma_a(B_1 \oplus A_1)$. Since $\alpha(M-\lambda) > 0$ and

$$N(B_1-\lambda) \oplus \{0\} \subseteq N(M-\lambda) \subseteq (B_1-\lambda)^{-1}(\triangle_0(N(A_1-\lambda))) \oplus N(A_1-\lambda),$$

it is obvious that $\alpha(B_1 \oplus A_1 - \lambda) > 0$, and hence $\lambda \in \pi_0^a(B_1 \oplus A_1)$. Since $B_1 \oplus A_1$ satisfies generalized a-Weyl's theorem, $\lambda \in \sigma_a(B_1 \oplus A_1) \setminus \sigma_{SBF_+^-}(B_1 \oplus A_1)$. Moreover, since the equality $\sigma_{LD}(B_1 \oplus A_1) = \sigma_{SBF_+^-}(B_1 \oplus A_1)$ holds, it follows that $B_1 \oplus A_1 - \lambda$ is left Drazin invertible. Since $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points, it follows from Corollary 3.14 that $\lambda \notin \sigma_{LD}(M)$. Thus $\lambda \in \sigma_a(M) \setminus \sigma_{SBF_+^-}(M)$, and hence $\pi_0^a(M) \subseteq \sigma_a(M) \setminus \sigma_{SBF_+^-}(M)$. Therefore generalized a-Weyl's theorem holds for M.

(ii) Suppose that either A_1 and B_1 have the single valued extension property at $\lambda \notin \sigma_{BW}(M)$. Then generalized Browder's theorem holds for M by Theorem 4.2 and so $\sigma(M) \setminus \sigma_{BW}(M) \subseteq \pi_0(M)$. So we will prove that $\pi_0(M) \subseteq \sigma(M) \setminus \sigma_{BW}(M)$. Let $\lambda \in \pi_0(M)$. Since $\lambda \in \text{iso}\,\sigma(M)$, there exists $\epsilon > 0$ such that $M-\mu$ is invertible for $0 < |\lambda-\mu| < \epsilon$. It follows from [19, Theorem 2] that $B_1-\mu$ is left invertible and $A_1-\mu$ is right invertible for $0 < |\lambda-\mu| < \epsilon$. But $\sigma_{LD}(B) \setminus \sigma_{LD}(A)$ has no interior points, hence $B-\mu$ is invertible for $0 < |\lambda-\mu| < \epsilon$. This implies by [19, Corollary 4] that $A-\mu$ is also invertible for $0 < |\lambda-\mu| < \epsilon$. Thus $\lambda \in \text{iso}\,\sigma(B_1 \oplus A_1)$. Since $\alpha(M-\lambda) > 0$, it is obvious that $\alpha(B_1 \oplus A_1 - \lambda) > 0$. Thus $\lambda \in \pi_0(B_1 \oplus A_1)$. Since $B_1 \oplus A_1$ satisfies generalized Weyl's theorem, we have that $\lambda \in \sigma(B_1 \oplus A_1) \setminus \sigma_{BW}(B_1 \oplus A_1)$. But $\sigma_D(B_1 \oplus A_1) = \sigma_{BW}(B_1 \oplus A_1)$, hence $B_1 \oplus A_1$ has the single valued extension property at λ. By Lemma 3.5(ii), $B_1-\lambda$ is left Drazin invertible, and so B_1 has the single valued extension property at λ. It follows that $A_1-\lambda$ is also the single valued extension property at λ. On the other hand, $A_1-\lambda$ is right Drazin invertible by Lemma 3.5(i). Thus $A_1-\lambda$ is Drazin invertible. Similarly, if A_1^* and B_1^* have the single valued extension property at $\lambda \notin \sigma_{BW}(M)$, then $B_1^* \oplus A_1^*$ has the single valued extension property at λ. Again by Lemma 3.5(i), $B_1-\lambda$ is Drazin invertible. Hence by [22, Theorem 2.9], $M-\lambda$ is Drazin invertible. Therefore $\lambda \in \sigma(M) \setminus \sigma_{BW}(M)$, and so $\pi_0(M) \subseteq \sigma(M) \setminus \sigma_{BW}(M)$. Consequently, generalized Weyl's theorem holds for M. □

As an application of Theorem 4.5, we get the following corollary.

Corollary 4.6. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Then the following statements hold.*
(i) *If A and B are compact and isoloid and $B_1 \oplus A_1$ satisfies generalized Weyl's theorem, then M satisfies generalized a-Weyl's theorem.*
(ii) *If A is compact and isoloid and B is hyponormal, then M satisfies generalized Weyl's theorem.*

Proof. (i) Suppose that A and B are compact and isoloid. Then A_1, ${A_1}^*$, B_1, and ${B_1}^*$ have the single valued extension property from [1]. Moreover, $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points and $B_1 \oplus A_1$ satisfies generalized a-Weyl's theorem by [18]. Hence, from Theorem 4.5(i), M satisfies generalized a-Weyl's theorem.

(ii) Suppose that A is compact and isoloid and B is hyponormal. Then A_1 is decomposable and B_1 is also hyponormal. In this case, A_1 and B_1 have the single valued extension property. Moreover, $\sigma_{LD}(A_1) \setminus \sigma_{LD}(B_1)$ has no interior points and $B_1 \oplus A_1$ satisfies generalized Weyl's theorem by [18]. Hence, from Theorem 4.5(ii), M satisfies generalized Weyl's theorem. □

Finally, we investigate an equivalent condition so that the operator matrix $M \in \mathcal{S}$ satisfies generalized Weyl's theorem.

Theorem 4.7. *Let $M \in \mathcal{S}$ and let every λ be a complex value of finite multiplicity for M. Suppose A_1^* and B_1^* have the single valued extension property at $\lambda \notin \sigma_{BW}(M) \cup \sigma_{BW}(B_1 \oplus A_1)$. If A_1 and B_1 are isoloid, then the following statements are equivalent :*
(i) *Generalized Weyl's theorem holds for $B_1 \oplus A_1$.*
(ii) *Generalized Weyl's theorem holds for M.*

Proof. Suppose that A_1^* and B_1^* have the single valued extension property at $\lambda \notin \sigma_{BW}(M) \cup \sigma_{BW}(B_1 \oplus A_1)$. We will first show that $\sigma(M) = \sigma(B_1 \oplus A_1)$ and $\sigma_{BW}(M) = \sigma_{BW}(B_1 \oplus A_1)$. Let $\lambda \notin \sigma(B_1 \oplus A_1)$. Then $B_1 - \lambda$ is left invertible by [19, Theorem 2]. But B_1^* has the single valued extension property at λ, and hence $B_1 - \lambda$ is invertible by Remark 2.4. Thus $A_1 - \lambda$ is also invertible by [19, Corollary 4]. Consequently, $\lambda \notin \sigma(B_1 \oplus A_1)$, and so $\sigma(B_1 \oplus A_1) \subseteq \sigma(M)$. Since $\sigma(M) \subseteq \sigma(B_1 \oplus A_1)$ holds by [7, Lemma 3.1], we have $\sigma(M) = \sigma(B_1 \oplus A_1)$. If $\lambda \notin \sigma_{BW}(M)$, then M^* has the single valued extension property at λ, and hence $M - \lambda$ is Drazin invertible. By Lemma 3.5(i), $B_1 - \lambda$ is left Drazin invertible. Since B_1^* has the single valued extension property at λ, $B_1 - \lambda$ is Drazin invertible. It follows from [22, Corollary 2.8] that $A_1 - \lambda$ is also Drazin invertible. Hence $\lambda \notin \sigma_{BW}(B_1 \oplus A_1)$ and so $\sigma_{BW}(B_1 \oplus A_1) \subseteq \sigma_{BW}(M)$.

Conversely, suppose that $\lambda \notin \sigma_{BW}(B_1 \oplus A_1)$. Then A_1^* and B_1^* have the single valued extension property at λ, and so $B_1^* \oplus A_1^*$ has the single valued extension property at λ. Thus $\lambda \notin \sigma_D(B_1 \oplus A_1)$. It follows from Lemma 3.5(i) and [22, Theorem 2.9] that $M - \lambda$ is Drazin invertible. So $\lambda \notin \sigma_{BW}(M)$. Therefore $\sigma_{BW}(M) \subseteq \sigma_{BW}(B_1 \oplus A_1)$, and this implies that $\sigma_{BW}(M) = \sigma_{BW}(B_1 \oplus A_1)$.

Now, we shall prove that $\pi_0(M) = \pi_0(B_1 \oplus A_1)$. Let $\lambda \in \pi_0(M)$. Then $\lambda \in \text{iso}\, \sigma(M)$ and $\alpha(M - \lambda) > 0$. Since $\sigma(M) = \sigma(B_1 \oplus A_1)$, it follows that $\lambda \in \text{iso}\, \sigma(B_1 \oplus A_1)$. Moreover, since

$$N(B_1 - \lambda) \oplus \{0\} \subseteq N(M - \lambda) \subseteq (B_1 - \lambda)^{-1}(\triangle_0(N(A_1 - \lambda))) \oplus N(A_1 - \lambda),$$

it is clear that $\alpha(B_1 \oplus A_1 - \lambda) > 0$. Thus $\lambda \in \pi_0(B_1 \oplus A_1)$, and so $\pi_0(M) \subseteq \pi_0(B_1 \oplus A_1)$.

Conversely, let $\lambda \in \pi_0(B_1 \oplus A_1)$. Then $\lambda \in \text{iso}\, \sigma(B_1 \oplus A_1)$ and $\alpha(B_1 \oplus A_1 - \lambda) > 0$. Since A_1 and B_1 are isoloid, it follows that $\alpha(A_1 - \lambda) > 0$ and $\alpha(B_1 - \lambda) > 0$. Since $N(B_1 - \lambda) \oplus \{0\} \subseteq N(M - \lambda)$, $\alpha(M - \lambda) > 0$. But $\sigma(M) = \sigma(B_1 \oplus A_1)$, hence $\lambda \in \pi_0(M)$. Therefore $\pi_0(M) = \pi_0(B_1 \oplus A_1)$. Hence this completes the proof. □

Example 4.8. Let C be the bilateral shift given by $Ce_n = e_{n+1}$ on $L^2(\mu)$ with respect to $e_n(z) = z^n$ for $n \in \mathbb{Z}$. If $A = I$ and B is a multiplication operator on a Lebesgue space $L^2(\mu)$ where μ is a planar positive Borel measure with compact support. Then $\begin{pmatrix} A & C \\ Z & B \end{pmatrix} \in \mathcal{S}$. In this case, since A and B are normal, B_1 and A_1 are also normal and isoloid. Therefore, $B_1 \oplus A_1$ satisfies generalized Weyl's theorem. On the other hand, since B_1^* and A_1^* have the single-valued extension property, we conclude from Theorem 4.7 that $\begin{pmatrix} A & C \\ Z & B \end{pmatrix}$ satisfies generalized Weyl's theorem for every $Z \in \mathcal{L}(L^2(\mu), L^2(\mu))$.

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