Filomat 34:10 (2020), 3175–3190
https://doi.org/10.2298/FIL2010175I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Chains of Three-Dimensional Evolution Algebras: a Description

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Abstract. In this paper we describe locally all the chains of three-dimensional evolution algebras (3-dimensional CEAs). These are families of evolution algebras with the property that their structure matrices with respect to a certain natural basis satisfy the Chapman-Kolmogorov equation. We do it by describing all 3-dimensional CEAs whose structure matrices have a fixed rank equal to 3, 2 and 1, respectively. We show that arbitrary CEAs are locally CEAs of fixed rank. Since every evolution algebra can be regarded as a weighted digraph, this allows us to understand and visualize time-dependent weighted digraphs with 3 nodes.

1. Introduction

Evolution algebras are a particular type of genetic algebras introduced a few years ago to enlighten the non-Mendelian genetic. Their foundations were developed in [17], a pioneering monograph where many strong connections of evolution algebras with other mathematical fields (such as graph theory, stochastic processes, group theory, dynamic systems, mathematical physics, among others) are established. Algebraically, evolution algebras are usually non-associative algebras (they are not even power-associative), and dynamically they represent discrete dynamical systems. In spite of the strong connections of these algebras with many branches of mathematics and other sciences, they are very easy to define, as we show below.

Along this paper, by an algebra we understand a linear space E over a field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) provided with a bilinear map $E \times E \to E$, named the product of E. All the algebras that we will consider in this work are finite-dimensional.

An **evolution algebra** is an algebra E provided with a basis $B = \{e_1, e_2, \ldots, e_n\}$ such that $e_ie_j = 0$, if $i \neq j$. Such a basis is said to be a **natural basis.** If $e_ie_i = \sum_k a_{ki}e_k$, then the **structure matrix** of E relative to B is defined as the matrix $M_B(A)$ whose $i-$th column is given by the coefficients of e_ie_i respect to B. Therefore,

$$M_B(A) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

2010 *Mathematics Subject Classification*. 13J30, 13M05;

Keywords. evolution algebra, Chapman-Kolmogorov equation, rank, chain.
Received: 07 March 2020; Revised: 24 September 2020; Accepted: 24 November 2020
Communicated by Dijana Mosić

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Research supported by the Fund to Support innovative Development and Innovative ideas of Ministry of Innovative Development of the Republic of Uzbekistan and by the Spanish Project MTM2016-76327-C3-2-P (AEIFEDER, UE), and by Grupo de Investigación FQ199 of the Junta de Andalucía (Spain).

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Note that, whenever the matrix $M_B(A)$ is stochastic, this defines a Markov process. On the other hand, fixed a natural basis B, the dynamic nature of the evolution algebra E is given by the evolution operator $L_B : E \to E$ which is the unique linear operator on E such that $L_B(e_i) = e_i^2$. Therefore, the structure matrix $M_B(A)$ is precisely the matrix associated to L_B.

A main feature of an evolution algebra E is the following one: if we regard the elements of the natural basis B as the nodes of a graph, and the coefficient a_{ki} (where $e_ie_i = \sum_k a_{ki}e_k$) as a measure of the interaction of the node e_i over the node e_k then we obtain that the evolution algebra E is canonically associated to a weighted digraph $G_B(A)$ whose set of nodes is $\{e_1, e_2, \ldots, e_n\}$ and whose adjacency matrix is $M_B(A)^t$. (In graph theory the adjacency matrix of a weighted graph is usually obtained by writing in the i-th row the weight of the respective arcs starting in the node e_i. Because of this the adjacency matrix is the transpose the structure matrix).

Recall that a complex weighted digraph (or a complex network) with n nodes (where $n \in \mathbb{N}$) is defined as $G = (V, E, A)$ where $V = \{v_1, \ldots, v_n\}$ is the set of nodes, $E \subseteq V \times V$ is the edge set and $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$ is the adjacency matrix, under the convention that if $(v_i, v_j) \notin E$ then $a_{ij} = 0$. Therefore, by identifying B with V, it is clear that every complex networks has canonically associated an evolution algebra with natural basis B, and vice versa.

The study of complex networks is a very active area of scientific research focused on real-world networks such as computer networks, technological networks, brain networks and social networks. Nowadays, it is clear that many real-life problems can be better modeled by using time-dependent weighted digraphs. These are networks whose edges and associated weights change over the time. Nevertheless a consistent and vast theoretical framework for time varying weighted digraphs is still under development.

In time-dependent digraphs, it is usual to consider that the edges are activated in a sequence of time intervals. This means that the weight of an edge (e_i, e_j) can be represented by intervals of time $[s, t]$ with $0 < s < t$. This is the philosophy of the chains of evolution algebras (evolution algebras and weighted digraphs are canonically related) introduced in [1].

For $0 \leq a < b \leq \infty$, and $n \in \mathbb{N}$, we define an n-dimensional **chain of evolution algebras** (**CEA** for short) on $[a, b]$ as

$$\mathcal{E}_{[a,b]} := \{E^{[s,t]} : a < s < t < b\},$$

where every $E^{[s,t]}$ is an n-dimensional evolution algebra provided with a structure constant matrix $M^{[s,t]}$ (respect to a prefixed natural basis B) satisfying that

$$M^{[s,w]} = M^{[s,t]}M^{[t,w]} \quad \text{(Chapman-Kolmogorov equation)}, \tag{1}$$

for any $a < s < t < w < b$. In this case we say that

$$\mathcal{M}_{[a,b]} = \{M^{[s,t]} : a < s < t < b\}$$

is the **Chapman-Kolmogorov chain** (**C-K chain** for short) associated to $\mathcal{E}_{[a,b]}$.

We point out that if $\mathcal{E}_{[a,b]}$ is an n-dimensional CEA on a bounded interval $[a, b]$ and if $\mathcal{E}_{[b,c]}$ is an n-dimensional CEA on an interval $[b, c]$ then, by providing $E^{[s,b]}$ and $E^{[b,t]}$ for $a < s < b < t < c$, it turns out that $\mathcal{E}_{[a,b]}$ and $\mathcal{E}_{[b,c]}$ determine uniquely an n-dimensional CEA on $[a, c]$ as follows:

$$\mathcal{E}_{[a,c]} := \{E^{[s,t]} : a < s < t < c\}, \tag{2}$$

where, if $a < s < t < b$ (respectively if $b < s < t < c$) then, $E^{[s,t]}$ is given by the corresponding evolution algebra in $\mathcal{E}_{[a,b]}$, (respectively in $\mathcal{E}_{[b,c]}$), meanwhile if $a < s < b < t < c$ then, $E^{[s,t]}$ is defined as the evolution algebra whose structure matrix (respect to B) is $M^{[s,t]} := M^{[s,b]}M^{[b,t]}$.

To understand the meaning of the Chapman-Kolmogorov equation from the dynamic nature of a CEA, $\mathcal{E}_{[a,b]}$, note that (1) means that $L_B^{[s,t]} = L_B^{[s,\tau]}L_B^{[\tau,t]}$, for every $0 < s < \tau < t$, where $L_B^{[s,t]}$ is the evolution operator of the evolution algebra $E^{[s,t]}$.

From the point of view of the weighted digraphs associated to the evolution algebras $E^{[s,t]}$ of the CEA $\mathcal{E}_{[a,b]}$, the Chapman-Kolmogorov equation means that the adjacency matrix of the graph associated to $E^{[s,t]}$ can be obtained from the product of the adjacency matrices of the weighted digraphs associated to $E^{[\tau,t]}$ and $E^{[s,\tau]}$ (as the adjacency matrix is the transpose of the structure matrix). Consequently, the weighted digraph associated to the time interval $[s,t]$ can be obtained from the corresponding graphs associated to the time intervals $[s,\tau]$ and $[\tau,t]$, for every $s<\tau<t$.

In the connection between evolution algebras and Markov processes, the Chapman-Kolmogorov equation describes the fundamental relationship between the probability transitions (kernels). Indeed, a family of stochastic matrices satisfying the Chapman-Kolmogorov equation generates a Markov process (see e.g. [16]). However, there are many random processes which cannot be described by Markov processes of square stochastic matrices (see e.g. [3, 4, 7, 10]). To obtain a non-Markov process one can consider a solution of the Chapman-Kolmogorov equation which is not stochastic for some time, as it is done in [1, 12, 13] where a chain of evolution algebras (a CEA) is introduced and investigated. Later, this notion of CEA is generalized in [8] by means of a concept of flow of arbitrary finite-dimensional algebras (their matrices of structural constants are cubic matrices). In [2] Markov processes of cubic stochastic matrices (in a fixed sense), the so-called quadratic stochastic processes (QSPs), are studied.

In [14], chains generated by two-dimensional evolution algebras are described. In [1] several concrete examples of chains of evolution algebras are shown, analyzing their time-dynamics. In [12, Section 4] real CEAs of three-dimensional nilpotent evolution algebras are studied, and many examples of them are provided. Moreover, and application of CEAs to a population with possibility of twin birth can be found in [2].

The aim of this paper is describing all the 3-dimensional CEAs. Our approach shows the relevance of the **CEAs of rank r**, these are those whose associated C-K chain consists of matrices having a fixed rank r (equal to either $0,1,2$, or 3). This is to solve the Chapman-Kolmogorov equation (1) for the corresponding 3×3 structure matrices. In [12] this equation was solved when the given matrices $\{M^{[s,t]} : 0<s<t\}$ are upper-triangular.

In Sections 2, 3 and 4 of this paper, we describe completely all the CEAs having fixed rank equal to $3,2$ and 1 respectively (CEAs with zero rank are trivial). In Section 5 we show that these cases are certainly representative because an arbitrary CEA generated by 3-dimensional evolution algebras can be locally described by means of CEAs of fixed rank. More precisely, we will show that a bounded interval $[a,b]$ can be expressed as a union of intervals $[a,b]=\cup_{\lambda\in\Lambda}[a_\lambda,b_\lambda]$ satisfying the following conditions:

(i) these intervals are not overlapped in the meaning that $(a_\lambda,b_\lambda)\cap(a_\mu,b_\mu)=\emptyset$ if $\lambda,\mu\in\Lambda$ with $\lambda\neq\mu$,

(ii) for every $\lambda\in\Lambda$, the CEA $\mathcal{E}_{[a_\lambda,b_\lambda]} := \{E^{[s,t]} : a_\lambda<s<t<b_\lambda\}$ has a fixed rank r, equal to either $0,1,2$ or 3.

Moreover, beyond of giving this local description, we discuss when it is possible (or not) to provide a global description of an arbitrary 3-dimensional CEA in these terms, by justifying the reason for it.

Anyway, this approach will provide many examples of CEAs generated by 3-dimensional evolution algebras. Since an evolution algebra can be regarded as a complex network, this description might be helpful to understand time-dependent complex networks and to visualize many examples of them.

2. Determining the 3-dimensional CEAs of rank 3

Let $r\in\{1,2,3\}$. We say that the C-K chain $\mathcal{M}_{[a,b]}$ of a 3-dimensional CEA $\mathcal{E}_{[a,b]}$ has rank r if $rank\, M^{[s,t]}=r$, for every $M^{[s,t]}\in\mathcal{M}_{[a,b]}$ with $a<s<t<b$. In this case we say also that $\mathcal{E}_{[a,b]}$ is a CEA of rank r. In this section we will describe all the CEAs (equivalently all the C-K chains) of rank 3.

Examples of C-K chains $\mathcal{M}_{[a,b]}$ of 3×3 matrices of rank 3 are very easy to obtain, as we show next.

2.1. Prototype of C-K chain of rank 3

Choose a set of non-singular 3×3 matrices $\{M^{[a,t]} : a<t<b\}$,and put

$$M^{[u,v]} = M^{[a,u]^{-1}} M^{[a,v]},$$

for every $a < u < v < b$. Then

$$\mathcal{M}_{[a,b]} = \{M^{[u,v]} : a < u < v < b\}$$

defines a C-K chain of rank 3 and, hence, if $M^{[u,v]}$ is the structure matrix of the evolution algebra $E^{[u,v]}$ then $\mathcal{E}_{[a,b]} = \{E^{[u,v]} : a < u < v < b\}$ determines all the 3-dimensional CEAs $\mathcal{E}_{[a,b]}$ of rank 3.

2.2. Description of the C-K chains of rank 3

Let us determine all the C-K chains $\mathcal{M}_{[a,b]}$ associated to a 3-dimensional CEA $\mathcal{E}_{[a,b]}$ of rank 3.

Proposition 2.1. *Let $\mathcal{M}_{[a,b]}$ be a C-K chain of rank 3 associated to a 3-dimensional CEA $\mathcal{E}_{[a,b]}$. Then $\mathcal{M}_{[a,b]}$ is determined by the set of matrices*

$$\cup_{n\in\mathbb{N}}\{M^{[\alpha_n,t]} : a < t < b\},$$

where α_n is a strictly decreasing sequence in (a, b) such that $\alpha_n \to a$.

Proof. If $a < s < t < b$, let $n \in \mathbb{N}$ be such that $\alpha_n < s$. Since $M^{[\alpha_n,s]}M^{[s,t]} = M^{[\alpha_n,t]}$, we have that

$$M^{[s,t]} = M^{[\alpha_n,s]^{-1}}M^{[\alpha_n,t]},$$

as $M^{[\alpha_n,s]}$ is non-singular. Therefore all the matrices in $\mathcal{M}_{[a,b]}$ have been described. □

3. Determining the 3-dimensional CEAs of rank 2

In the next subsection we will provide a method to obtain 3-dimensional CEAs of rank 2 (and thus, C-K chains of 3×3 matrices of rank 2). Later, we will see that these examples are very relevant to describe all the C-K chains of rank 2.

3.1. Prototype of C-K chain of rank 2

Let α_n be a strictly decreasing sequence of real numbers such that $\alpha_n \to a$. Denote by $inv(\mathcal{M}_{2\times2}(\mathbb{K}))$ the set of all 2×2 invertible matrices with entries in $\mathbb{K}$. Consider a sequence of functions

$$\Psi_n : (\alpha_n, b) \to inv(\mathcal{M}_{2\times2}(\mathbb{K})) \quad \text{and} \quad \varphi_n : (\alpha_n, b) \to \mathcal{M}_{2\times1}(\mathbb{K}),$$

with the following property, for every $m, n \in \mathbb{N}$, and t, w such that $a < \alpha_m < \alpha_n < t < w < b$,

$$\begin{aligned} \Psi_n^{-1}(t)\Psi_n(w) &= \Psi_m^{-1}(t)\Psi_m(w), \\ \Psi_n^{-1}(t)\varphi_n(w) &= \Psi_m^{-1}(t)\varphi_m(w). \end{aligned} \tag{3}$$

(It is very easy to get examples of such applications. For instance, take arbitrary functions $\xi, \widetilde{\xi} : \mathbb{N} \to \mathbb{R}\backslash\{0\}$, and define $\Psi_{2n}(t) := \left(t^2 + 1\right)\xi(n)A$, $\varphi_{2n}(t) := t\xi(n)\mathbf{u}$, $\Psi_{2n+1}(t) := \left(t^2 + 1\right)\widetilde{\xi}(n)B$, and $\varphi_{2n+1}(t) := t\widetilde{\xi}(n)\mathbf{v}$ where $A, B \in inv(\mathcal{M}_{2\times2}(\mathbb{K}))$ and $\mathbf{u}, \mathbf{v} \in \mathcal{M}_{2\times1}(\mathbb{K})$ are such that $A^{-1}\mathbf{u} = B^{-1}\mathbf{v}$).

For $a < t < w < b$, define

$$\begin{aligned} P(t, w) &= \Psi_n^{-1}(t)\Psi_n(w) \\ u(t, w) &= \Psi_n^{-1}(t)\varphi_n(w), \end{aligned}$$

for some $n \in \mathbb{N}$ such that $\alpha_n < t$. Note that, by (3), we have that $P(t, w)$ and $u(t, w)$ do not depend on the particular n that we choose satisfying $\alpha_n < t$. Define

$$M^{[t,w]} = \begin{pmatrix} P(t, w)_{2\times2} & u(t, w)_{2\times1} \\ 0_{1\times2} & 0_{1\times1} \end{pmatrix}.$$

We claim that $\{M^{[t,w]} : a < t < w < b\}$ *is a C-K chain of rank 2.* To prove the claim note that if $a < t < w < \tau < b$, and if $a < \alpha_n < t$ then,

$$M^{[t,w]}M^{[w,\tau]} = \begin{pmatrix} P(t,w)_{2\times 2} & u(t,w)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix}\begin{pmatrix} P(w,\tau)_{2\times 2} & u(w,\tau)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix} =$$

$$\begin{pmatrix} \Psi_n^{-1}(t)\Psi_n(w)_{2\times 2} & \Psi_n^{-1}(t)\varphi_n(\tau)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix}\begin{pmatrix} \Psi_n^{-1}(w)\Psi_n(\tau)_{2\times 2} & \Psi_n^{-1}(w)\varphi_n(\tau)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix} =$$

$$\begin{pmatrix} \Psi_n^{-1}(t)\Psi_n(w)\Psi_n^{-1}(w)\Psi_n(\tau) & \Psi_n^{-1}(t)\Psi_n(w)\Psi_n^{-1}(w)\varphi_n(\tau) \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix} =$$

$$\begin{pmatrix} \Psi_n^{-1}(t)\Psi_n(\tau) & \Psi_n^{-1}(t)\varphi_n(\tau) \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix} = \begin{pmatrix} P(t,\tau)_{2\times 2} & u(t,\tau)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix} = M^{[t,\tau]}.$$

Consequently, $M^{[t,w]}M^{[w,\tau]} = M^{[t,\tau]}$ which proves that $\mathcal{M}_{[a,b]}$ is a C-K chain of rank 2, as desired.

3.2. *Description of the C-K chains of rank 2*

Our next goal now is to prove that, if $\mathcal{M}_{[a,b]}$ is a C-K chain of rank 2 then, except in a marginal case that we will described below, the chain can be determined by means of

$$\cup_{n\in\mathbb{N}}\{M^{[\alpha_n,t]} : t > \alpha_n\},$$

where α_n is a strictly decreasing sequence such that $\alpha_n \to a$. In the aforementioned remaining marginal case (i.e., every decreasing sequence such that $\alpha_n \to a$ contains a critical point) the chain can be determined as long as we additionally know some other elements (the third row of $M^{[t_0,w]}$ when t_0 is a critical point) as we will see.

Assume that $\mathcal{M}_{[a,b]}$ is a given C-K chain of rank 2 and let us determine all the matrices in $\mathcal{M}_{[a,b]}$ by knowing only a few of these matrices. More precisely suppose that we know the matrices of $\mathcal{M}_{[a,b]}$ given by the set

$$\cup_{n\in\mathbb{N}}\{M^{[\alpha_n,t]} : t > \alpha_n\},$$

where α_n is a strictly decreasing sequence of (a,b) converging to a.

Claim 1.- It is not restrictive to assume that the two first rows of $M^{[\alpha_n,t]}$ are linearly independent. If this is not the case then, for these particular $n \in \mathbb{N}$, replace in $\mathcal{M}_{[a,b]}$ the matrices $M^{[\alpha_n,t]}$ and $M^{[\tau,\alpha_n]}$ by $\widetilde{M}^{[\alpha_n,t]}$ and $\widetilde{M}^{[\tau,\alpha_n]}$ where

$$\widetilde{M}^{[\alpha_n,t]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} M^{[\alpha_n,t]} \qquad \text{and} \qquad \widetilde{M}^{[\tau,\alpha_n]} = M^{[\tau,\alpha_n]}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore we obtain a C-K chain of rank 2, denoted by $\widetilde{\mathcal{M}}_{[a,b]}$. In fact, the Chapman-Kolmogorov equations are trivially satisfied, as

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, if we determine all the matrices of $\widetilde{\mathcal{M}}_{[a,b]}$ then all the matrices in $\mathcal{M}_{[a,b]}$ are also described since

$$M^{[\alpha_n,t]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \widetilde{M}^{[\alpha_n,t]} \qquad \text{and} \qquad M^{[\tau,\alpha_n]} = \widetilde{M}^{[\tau,\alpha_n]} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Claim2. It is not restrictive to assume that the 3rd row of $M^{[\alpha_n,t]}$ *is zero.* In fact, otherwise suppose that

$$M^{[\alpha_n,t]} = \begin{pmatrix} a_1(\alpha_n,t) & a_2(\alpha_n,t) & a_3(\alpha_n,t) \\ b_1(\alpha_n,t) & b_2(\alpha_n,t) & b_3(\alpha_n,t) \\ c_1(\alpha_n,t) & c_2(\alpha_n,t) & c_3(\alpha_n,t) \end{pmatrix},$$

with $rank M^{[\alpha_n,t]} = 2$ where, for constants $k(\alpha_n, t)$ and $\widetilde{k}(\alpha_n, t)$ that are not simultaneously zero, we have

$$\begin{aligned} &(c_1(\alpha_n,t), c_2(\alpha_n,t), c_3(\alpha_n,t)) = \\ = \;& k(\alpha_n,t)(a_1(\alpha_n,t), a_2(\alpha_n,t), a_3(\alpha_n,t)) + \widetilde{k}(\alpha_n,t)(b_1(\alpha_n,t), b_2(\alpha_n,t), b_3(\alpha_n,t)). \end{aligned}$$

Then we replace $M^{[\alpha_n,t]}$ and $M^{[\tau,\alpha_m]}$ by $\widetilde{M}^{[\alpha_n,t]}$ and $\widetilde{M}^{[\tau,\alpha_m]}$, respectively, where

$$\widetilde{M}^{[\alpha_n,t]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k(\alpha_n,t) & \widetilde{k}(\alpha_n,t) & -1 \end{pmatrix} M^{[\alpha_n,t]}, \tag{4}$$

$$\widetilde{M}^{[\tau,\alpha_m]} = M^{[\tau,\alpha_m]} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k(\alpha_m,t) & \widetilde{k}(\alpha_m,t) & -1 \end{pmatrix}. \tag{5}$$

From the fact that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k(\alpha_n,t) & \widetilde{k}(\alpha_n,t) & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k(\alpha_n,t) & \widetilde{k}(\alpha_n,t) & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the Chapman-Kolmogorov equation holds for the matrices of the new family $\widetilde{\mathcal{M}}_{[a,b]}$. Therefore we obtain a C-K chain of rank 2 such that

$$\widetilde{M}^{[\alpha_n,t]} = \begin{pmatrix} a_1(\alpha_n,t) & a_2(\alpha_n,t) & a_3(\alpha_n,t) \\ b_1(\alpha_n,t) & b_2(\alpha_n,t) & b_3(\alpha_n,t) \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that, if we are able to describe the matrices of this C-K chain $\widetilde{\mathcal{M}}_{[a,b]}$, then we will also describe those of the original chain $\mathcal{M}_{[a,b]}$ via the formulas (4) and (5).

By applying Claim 1 and Claim 2 if needed, the problem that we are considering can be formulated as follows: *Describing all the C-K chains,* $\mathcal{M}_{[a,b]}$*, of rank 2, such that for a strictly decreasing sequence* $\alpha_n \to a$ *we have that*

$$M^{[\alpha_n,t]} = \begin{pmatrix} a_1(\alpha_n,t) & a_2(\alpha_n,t) & a_3(\alpha_n,t) \\ b_1(\alpha_n,t) & b_2(\alpha_n,t) & b_3(\alpha_n,t) \\ 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

for every $t > \alpha_n$. Let

$$M^{[t,w]} = \begin{pmatrix} P(t,w)_{2\times 2} & u(t,w)_{2\times 1} \\ v^T(t,w)_{1\times 2} & c(t,w)_{1\times 1} \end{pmatrix},$$

for every $a < t < w < b$. For $\alpha_n < t$, denote

$$\Psi_n(t) := P(\alpha_n, t) = \begin{pmatrix} a_1(\alpha_n, t) & a_2(\alpha_n, t) \\ b_1(\alpha_n, t) & b_2(\alpha_n, t) \end{pmatrix}; \ \varphi_n(t) := u(\alpha_n, t) = \begin{pmatrix} a_3(\alpha_n, t) \\ b_3(\alpha_n, t) \end{pmatrix}.$$

Then,

$$M^{[\alpha_n,t]} = \begin{pmatrix} P(\alpha_n, t)_{2\times2} & u(\alpha_n, t)_{2\times1} \\ 0_{1\times2} & 0_{1\times1} \end{pmatrix} = \begin{pmatrix} \Psi_n(t)_{2\times2} & \varphi_n(t)_{2\times1} \\ 0_{1\times2} & 0_{1\times1} \end{pmatrix}. \tag{7}$$

Since, for every $a < \alpha_n < t < w < b$, we have

$$M^{[\alpha_n,t]} M^{[t,w]} = M^{[\alpha_n,w]}, \tag{8}$$

it follows that

$$\begin{aligned} P(t, w) &= \Psi_n^{-1}(t)\Psi_n(w) - \Psi_n^{-1}(t)\varphi_n(t)v^T(t, w), \\ u(t, w) &= \Psi_n^{-1}(t)\varphi_n(w) - \Psi_n^{-1}(t)\varphi_n(t)c(t, w), \end{aligned} \tag{9}$$

so that the above functions do not depend on n, in the meaning that n can be replaced for any m such that $\alpha_m < t$. Because of this,

$$\begin{aligned} \Psi_n^{-1}(t)\Psi_n(w) - \Psi_m^{-1}(t)\Psi_m(w) &= (\Psi_n^{-1}(t)\varphi_n(t) - \Psi_m^{-1}(t)\varphi_m(t))v^T(t, w), \\ \Psi_n^{-1}(t)\varphi_n(w) - \Psi_m^{-1}(t)\varphi_m(w) &= (\Psi_n^{-1}(t)\varphi_n(t) - \Psi_m^{-1}(t)\varphi_m(t))c(t, w), \end{aligned} \tag{10}$$

for every $n, m \in \mathbb{N}$ such that $a < \alpha_m < \alpha_n < t$.

On the other hand, for every $a < t < w < b$ and $a < \alpha_n < t$, we have,

$$M^{[t,w]} = \Delta_n^{(t,w)} - S_n^{(t,w)} \tag{11}$$

where

$$\begin{aligned} \Delta_n^{(t,w)} &= \begin{pmatrix} \Psi_n^{-1}(t)\Psi_n(w)_{2\times2} & \Psi_n^{-1}(t)\varphi_n(w)_{2\times1} \\ 0_{1\times2} & 0_{1\times1} \end{pmatrix}, \\ S_n^{(t,w)} &= \begin{pmatrix} \Psi_n^{-1}(t)\varphi_n(w)v^T(t, w)_{2\times2} & \Psi_n^{-1}(t)\varphi_n(w)c(t, w)_{2\times1} \\ -v^T(t, w)_{1\times2} & -c(t, w)_{1\times1} \end{pmatrix}. \end{aligned}$$

It is easy to check that, for $a < s < t < w < b$,

$$\Delta_n^{(s,t)} S_n^{(t,w)} = 0_{3\times3}. \tag{12}$$

Moreover, for $K \in \mathcal{M}_{2\times1}(\mathbb{K})$,

$$S_n^{(s,t)} \begin{pmatrix} K_{2\times1} \\ -1_{1\times1} \end{pmatrix} = \tag{13}$$

$$\begin{aligned} &= \begin{pmatrix} \Psi_n^{-1}(s)\varphi_n(t)v^T(s, t)_{2\times2} & \Psi_n^{-1}(s)\varphi_n(t)c(s, t)_{2\times1} \\ -v^T(s, t)_{1\times2} & -c(s, t)_{1\times1} \end{pmatrix} \begin{pmatrix} K_{2\times1} \\ -1_{1\times1} \end{pmatrix} \\ &= \begin{pmatrix} \Psi_n^{-1}(s)\varphi_n(t)(v^T(s, t)_{2\times2}K_{2\times1} - c(s, t)_{2\times1}) \\ -v^T(s, t)_{1\times2}K_{2\times1} + c(s, t)_{1\times1} \end{pmatrix}. \end{aligned}$$

Thus, $S_n^{(s,t)} \begin{pmatrix} K_{2\times1} \\ -1_{1\times1} \end{pmatrix} = 0_{3\times1}$ if and only if $v^T(s, t)_{1\times2}K_{2\times1} = c(s, t)_{1\times1}$.

We say that t_0 is a **critical point** for the sequence α_n if there exists $K_{t_0} \in \mathcal{M}_{2\times 1}(\mathbb{K})$ such that $\Psi_n^{-1}(t_0)\varphi_n(t_0) = K_{t_0}$, for every $n \in \mathbb{N}$ with $\alpha_n < t_0$ and, for every $a < s < t_0$ with $c(s,t_0) \neq 0$, we have that at least one of the following two properties are satisfied:

(i) $v^T(s,t_0)K_{t_0} = c(s,t_0)$,

(ii) $\Delta_n^{(s,t_0)} = \Delta_m^{(s,t_0)}$ for every $n,m \in \mathbb{N}$ with $a < \alpha_m < \alpha_n < s$ (equivalently there exists $M_s \in \mathcal{M}_{3\times 3}(\mathbb{K})$ such that $\Delta_n^{(s,t_0)} = M_s$ for every $n \in \mathbb{N}$ with $\alpha_n < s$).

We claim that *if t_0 is not a critical point then, for every $t_0 < w < b$, the matrix $M^{[t_0,w]}$ can be determined from the set $\cup_{n\in\mathbb{N}}\{M^{[\alpha_n,t]} : a < t < b\}$.*

To prove the claim suppose that t_0 is not a critical point. Then we are in one of the following two cases:

(a) There exist $n,m \in \mathbb{N}$ with $a < \alpha_m < \alpha_n < t_0$ such that $\Psi_n^{-1}(t_0)\varphi_n(t_0) \neq \Psi_m^{-1}(t_0)\varphi_m(t_0)$. Then $v^T(t,w)$ and $c(t,w)$ can be obtained from (10), meanwhile $P(t,w)$ and $u(t,w)$ are getting from (9). Thus $M^{[t_0,w]}$ is determined.

(b) $\Psi_n^{-1}(t_0)\varphi_n(t_0) = \Psi_m^{-1}(t_0)\varphi_m(t_0)$ for every $n,m \in \mathbb{N}$ with $a < \alpha_m < \alpha_n < t_0$. This means that there exists $K_{t_0} \in \mathcal{M}_{2\times 1}(\mathbb{K})$ such that $\Psi_n^{-1}(t_0)\varphi_n(t_0) = K_{t_0}$, for every $n \in \mathbb{N}$ with $a < \alpha_n < t_0$. Since t_0 is not a critical point, there exists $a < s < t_0$ with $c(s,t_0) \neq 0$ such that $v^T(s,t_0)K_{t_0} \neq c(s,t_0)$ and $\Delta_n^{(s,t_0)} \neq \Delta_m^{(s,t_0)}$ for some $n,m \in \mathbb{N}$ with $\alpha_m < \alpha_n < s$. By (9) it follows that $\Psi_n^{-1}(s)\varphi_n(s) \neq \Psi_m^{-1}(s)\varphi_m(s)$. Then, $M^{[s,t_0]}$ as well as $M^{[s,w]}$ can be determined from (10) and (9). Moreover, by (13),

$$S_n^{(s,t_0)} \begin{pmatrix} K_{t_0} \\ -1 \end{pmatrix} \neq 0_{3\times 1}. \tag{14}$$

If $K_{t_0} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, if $v^T(t_0,w) = (c_1(w), c_2(w))$ and if $c(t_0,w) = c_3(w)$ then, since $\Psi_n^{-1}(t_0)\varphi_n(t_0) = K_{t_0}$ for every $n \in \mathbb{N}$ with $a < \alpha_n < t_0$, we have

$$S_n^{(t_0,w)} = \begin{pmatrix} k_1c_1(w) & k_1c_1(w) & k_1c_3(w) \\ k_2c_2(w) & k_2c_2(w) & k_2c_3(w) \\ -c_1(w) & -c_2(w) & -c_3(w) \end{pmatrix}.$$

Therefore, from (11), (12) and (13) it follows that

$$M^{(s,t_0)} \begin{pmatrix} K_{t_0} \\ -1 \end{pmatrix} = -S_n^{(s,t_0)} \begin{pmatrix} K_{t_0} \\ -1 \end{pmatrix} \neq 0_{3\times 1}.$$

Consequently, $S_n^{(t_0,w)}$ can be determined from the equality

$$M^{(s,t_0)}S_n^{(t_0,w)} = M^{(s,t_0)}\Delta_n^{(t_0,w)} - M^{(s,w)},$$

as the other matrices there are known. Hence, from (11), the matrix $M^{(t_0,w)}$ is determined for every $t_0 < w < b$.

We conclude that the C-K chain $\mathcal{M}_{[a,b]}$ is described from of the matrices

$$\cup_{n\in\mathbb{N}}\{M^{[\alpha_n,t]} : t > \alpha_n\} \tag{15}$$

whenever α_n has no critical points. Otherwise the C-K chain $\mathcal{M}_{[a,b]}$ is determined by the above set of matrices joint with the third row of $M^{[t_0,w]}$, for every critical point t_0 associated to α_n, and every $t_0 < w < b$. The reason is that there are many free choices of the third row of $M^{[t_0,w]}$ given rise to a "compatible" C-K chain $\mathcal{M}_{[a,b]}$ for the given matrices (15) (so that we cannot determine the particular $\mathcal{M}_{[a,b]}$ that we have chosen). The next example shows this situation.

Example 3.1. *Let $\alpha_n \to a$ be a strictly decreasing sequence, with $a < \alpha_n < b$ and, for every $a < \alpha_n < w$, let*

$$M^{[\alpha_n,w]} = \begin{pmatrix} \Psi_n(t)_{2\times 2} & \varphi_n(t)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix},$$

where $\Psi_n(t) \in inv(M_{2\times 2}(\mathbb{K}))$. Assume that, if $a < \alpha_m < \alpha_n < t < w < b$ then,

$$\begin{aligned} \Psi_n^{-1}(t)\Psi_n(w) &= \Psi_m^{-1}(t)\Psi_m(w), \\ \Psi_n^{-1}(t)\varphi_n(w) &= \Psi_m^{-1}(t)\varphi_m(w). \end{aligned} \tag{16}$$

Let $a < \alpha_1 < t_0 < b$ be such that $\varphi_n(t_0) = 0$ for every $n \in \mathbb{N}$. If $a < t < w < b$ is such that $t \neq t_0$ and $t \neq \alpha_n$ for every $n \in \mathbb{N}$, then define

$$M^{[t,w]} := \begin{pmatrix} \Psi_n^{-1}(t)\Psi_n(w)_{2\times 2} & \Psi_n^{-1}(t)\varphi_n(w)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix},$$

for some $n \in \mathbb{N}$ (from (16) this does not depend on the α_n chosen). Fix $\beta_{1\times 2} := (\beta_1, \beta_2) \in M_{1\times 2}(\mathbb{K})\backslash\{0\}$ and $n_0 \in \mathbb{N}$. For $t_0 < w < b$, define

$$M^{[t_0,w]} := \begin{pmatrix} \Psi_{n_0}^{-1}(t_0)\Psi_{n_0}(w)_{2\times 2} & \Psi_{n_0}^{-1}(t_0)\varphi_{n_0}(w)_{2x1} \\ \beta_{1\times 2}\Psi_{n_0}(w)_{2\times 2} & \beta_{1\times 2}\varphi_{n_0}(w)_{2\times 1} \end{pmatrix}.$$

From (16) we have, for every $n \in \mathbb{N}$,

$$M^{[t_0,w]} := \begin{pmatrix} \Psi_n^{-1}(t_0)\Psi_n(w)_{2\times 2} & \Psi_n^{-1}(t_0)\varphi_n(w)_{2\times 1} \\ \beta_{1\times 2}\Psi_{n_0}(w)_{2\times 2} & \beta_{1\times 2}\varphi_{n_0}(w)_{2\times 1} \end{pmatrix}. \tag{17}$$

We claim that $\mathcal{M}_{[a,b]} := \{M^{[t,w]} : a < t < w < b\}$ is a C-K chain of rank 2 such that t_0 is a critical point for α_n (indeed t_0 is the only critical point that α_n has). The claim follows from (i), (ii) and (iii) where:

(i) $M^{[\alpha_n,t]}M^{[t,w]} = M^{[\alpha_n,w]}$. Indeed, for $t \neq t_0$ it is obvious and for $t = t_0$ it is also easy to check as $\varphi_n(t_0) = 0$.

(ii) $M^{[s,t]}M^{[t,w]} = M^{[s,w]}$ for $s \neq t_0$. In fact, if $t \neq t_0$ the equality is clear and otherwise, from (17),

$$\begin{pmatrix} \Psi_n^{-1}(s)\Psi_n(t_0)_{2\times 2} & 0_{2x1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix}\begin{pmatrix} \Psi_n^{-1}(t_0)\Psi_n(w)_{2\times 2} & \Psi_n^{-1}(t_0)\varphi_n(w)_{2x1} \\ \beta_{1\times 2}\Psi_{n_0}(w)_{2\times 2} & \beta_{1\times 2}\varphi_{n_0}(w)_{2\times 1} \end{pmatrix}$$

$$= \begin{pmatrix} \Psi_n^{-1}(s)\Psi_n(w)_{2\times 2} & \Psi_n^{-1}(s)\varphi_n(w)_{2x1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix}.$$

(iii) $M^{[t_0,w]}M^{[w,\tau]} = M^{[t_0,\tau]}$. This is also trivial since

$$\begin{pmatrix} \Psi_{n_0}^{-1}(t_0)\Psi_{n_0}(w)_{2\times 2} & \Psi_{n_0}^{-1}(t_0)\varphi_{n_0}(w)_{2x1} \\ \beta_{1\times 2}\Psi_{n_0}(w)_{2\times 2} & \beta_{1\times 2}\varphi_{n_0}(w)_{2\times 1} \end{pmatrix}\begin{pmatrix} \Psi_{n_0}^{-1}(w)\Psi_{n_0}(\tau)_{2\times 2} & \Psi_{n_0}^{-1}(w)\varphi_{n_0}(\tau)_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 1} \end{pmatrix}$$

$$= \begin{pmatrix} \Psi_{n_0}^{-1}(t_0)\Psi_{n_0}(\tau)_{2\times 2} & \Psi_{n_0}^{-1}(t_0)\varphi_{n_0}(\tau)_{2x1} \\ \beta_{1\times 2}\Psi_{n_0}(\tau)_{2\times 2} & \beta_{1\times 2}\varphi_{n_0}(\tau)_{2\times 1} \end{pmatrix}.$$

Consequently for every arbitrary $\beta_{1\times 2}$ that we fix we obtain a C-K chain $\mathcal{M}_{[a,b]}$ compatible with the matrices $M^{[\alpha_n,w]}$ originally given. Thus, we cannot determine a particular one of these C-K chains from $\cup_{n\in\mathbb{N}}\{M^{[\alpha_n,t]} : t > \alpha_n\}$ without knowing the third row of $M^{[t_0,w]}$ for the critical point t_0.

4. The Chapman-Kolmogorov chains of matrices 3 × 3 of rank 1

In the next subsection we will provide a method to obtain C-K chains of rank 1. Later, we will see that these examples cover the class of C-K chains of rank 1.

Note that if every matrix of a C-K chain $\mathcal{M}_{[a,b]}$ is non-zero, and if $M^{[s_0,t_0]} \in \mathcal{M}_{[a,b]}$ is such that $rankM^{[s_0,t_0]} = 1$ then, $rankM^{[s_0,t]} = 1$ for every $t > t_0$. This follows from the fact that

$$rankM^{[s_0,t]} \leq rankM^{[s_0,t_0]} = 1,$$

as $M^{[s_0,t]} = M^{[s_0,t_0]}M^{[t_0,t]}$.

4.1. Prototype of C-K chain of rank 1

Consider a strictly decreasing sequence α_n converging to a. For every $n \in \mathbb{N}$, let $f_n, g_n, h_n : (\alpha_n, b) \to \mathbb{R}$ be functions satisfying that, if $m > n$ then, there exists a constant $k_{n,m}$ such that for every $t > \alpha_n$ we have

$$\begin{aligned} f_n(t) &= k_{n,m} f_m(t), \\ g_n(t) &= k_{n,m} g_m(t), \\ h_n(t) &= k_{n,m} h_m(t). \end{aligned} \tag{18}$$

Let $\Psi_n : (\alpha_n, b) \to \mathbb{R}$ be the function given by

$$\Psi_n(t) = f_n(t) + g_n(t)\varphi_1(t) + h_n(t)\varphi_2(t),$$

where $\varphi_1, \varphi_2 : (a, b) \to \mathbb{R}$ are arbitrary functions such that $\Psi_n(t) \neq 0$, for every $t > \alpha_n$. Note that

$$\frac{\Psi_n(t)}{\Psi_m(t)} = k_{n,m}, \tag{19}$$

for every $m > n$. For $a < t < \tau < b$, define

$$a_1(t,\tau) = \frac{f_n(\tau)}{\Psi_n(t)}, \qquad a_2(t,\tau) = \frac{g_n(\tau)}{\Psi_n(t)}, \qquad a_3(t,\tau) = \frac{h_n(\tau)}{\Psi_n(t)}, \tag{20}$$

for some n such that $\alpha_n < t$. Note that this definition does not depend on the chosen n, as follows straightforwardly from (18) and (19).

We claim that, *if*

$$M^{[s,t]} = \begin{pmatrix} a_1(s,t) & a_2(s,t) & a_3(s,t) \\ \varphi_1(s)a_1(s,t) & \varphi_1(s)a_2(s,t) & \varphi_1(s)a_3(s,t) \\ \varphi_2(s)a_1(s,t) & \varphi_2(s)a_2(s,t) & \varphi_2(s)a_3(s,t) \end{pmatrix},$$

then, the family $M_{[a,b]} := \{M^{[s,t]} : a < s < t < b\}$ *defines a C-K chain of rank 1.* To prove the claim we need to check that

$$M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]},$$

for $a < s < t < \tau < b$. This means that

$$\begin{aligned} a_i(t,\tau)\left\langle (a_1(s,t), a_2(s,t), a_3(s,t)), (1, \varphi_1(t), \varphi_2(t)) \right\rangle &= a_i(s,\tau), \qquad i = 1,2,3. \\ \varphi_1(s)a_i(t,\tau)\left\langle (a_1(s,t), a_2(s,t), a_3(s,t)), (1, \varphi_1(t), \varphi_2(t)) \right\rangle &= \varphi_1(s)a_i(s,\tau), \qquad i = 1,2,3. \\ \varphi_2(s)a_i(t,\tau)\left\langle (a_1(s,t), a_2(s,t), a_3(s,t)), (1, \varphi_1(t), \varphi_2(t)) \right\rangle &= \varphi_2(s)a_i(s,\tau), \qquad i = 1,2,3. \end{aligned}$$

where, $< -, - >$ denotes the scalar product of two vectors.

Since, from (20), we have that

$$\begin{aligned} a_1(t,\tau)\Psi_n(t) &= f_n(\tau), \\ a_2(t,\tau)\Psi_n(t) &= g_n(\tau), \\ a_3(t,\tau)\Psi_n(t) &= h_n(\tau), \end{aligned}$$

we deduce that $a_i(t,\tau)\frac{\Psi_n(t)}{\Psi_n(s)} = a_i(s,\tau)$. Therefore, for $i = 1,2,3$,

$$\begin{aligned} & a_i(t,\tau)\left\langle (a_1(s,t), a_2(s,t), a_3(s,t)), (1, \varphi_1(t), \varphi_2(t)) \right\rangle \\ = \; & a_i(t,\tau)\left\langle \left(\frac{f_n(t)}{\Psi_n(s)}, \frac{g_n(t)}{\Psi_n(s)}, \frac{h_n(t)}{\Psi_n(s)} \right), (1, \varphi_1(t), \varphi_2(t)) \right\rangle \\ = \; & a_i(t,\tau)\frac{\Psi_n(t)}{\Psi_n(s)} = a_i(s,\tau), \end{aligned}$$

and the claim follows.

Next, we show that all the C-K chains of rank 1 fit in this pattern.

4.2. Description of the C-K chains of rank 1

Along this subsection $\mathcal{M}_{[a,b]}$, with $0 \leq a < b \leq \infty$, will denote a C-K chain of rank 1 associated to a 3 dimensional CEA, $\mathcal{E}_{[a,b]}$. Note that every $M^{[s,t]} \in \mathcal{M}_{[a,b]}$ is a non-zero matrix such that its 3 rows are linearly dependent.

Claim 1.- *It is not restrictive to assume that the first row of every* $M^{[s,t]} \in \mathcal{M}_{[a,b]}$ *is non-zero.*

The claim follows from the following two propositions. In the next one, we work with the set of matrices $M^{[s,t]}$ in $\mathcal{M}_{[a,b]}$ having their first row equal to zero and the second one non-zero. Note that in some C-K chains this set might be empty.

Proposition 4.1. *Let* $\mathcal{M}_{[a,b]} = \{\mathcal{M}^{[s,t]} : a < s < t < b\}$ *be a family of* 3×3 *matrices. Suppose that the set*

$$\mathcal{S} = \{s \in (a,b) : \exists\, t \ \textit{with}\ (1,0,0)M^{[s,t]} = (0,0,0) \ \textit{and}\ \ (0,1,0)M^{[s,t]} \neq (0,0,0)\,\}$$

is non-empty, and define

$$\widetilde{M}^{[s,t]} = \begin{cases} P_{213}M^{[s,t]} & if\ \ s \in \mathcal{S}, t \notin \mathcal{S}, \\ P_{213}M^{[s,t]}P_{213} & if\ \ s \in \mathcal{S}, t \in \mathcal{S}, \\ M^{[s,t]}P_{213} & if\ \ s \notin \mathcal{S}, t \in \mathcal{S}, \\ M^{[s,t]} & if\ \ s \notin \mathcal{S}, t \notin \mathcal{S}. \end{cases} \tag{21}$$

where

$$P_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, for every $a < s < t < \tau < b$*, we have that*

$$M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]} \ \textit{if and only if}\ \ \widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = \widetilde{M}^{[s,\tau]}.$$

Proof. Note that $P_{213}P_{213} = I$.

If $M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]}$, for every $a < s < t < \tau < b$ then, $\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = \widetilde{M}^{[s,\tau]}$ as it can be deduced directly from the following cases.

Case 1.- If $s, t, \tau \in S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} \ = \ P_{213}M^{[s,t]}P_{213}P_{213}M^{[t,\tau]}P_{213} = P_{213}M^{[s,t]}M^{[t,\tau]}P_{213} = P_{213}M^{[s,\tau]}P_{213} = \widetilde{M}^{[s,\tau]}.$$

Case 2.- If $s, t \in S$ and $\tau \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = P_{213}M^{[s,t]}P_{213}P_{213}M^{[t,\tau]} = P_{213}M^{[s,t]}M^{[t,\tau]} = P_{213}M^{[s,\tau]} = \widetilde{M}^{[s,\tau]}.$$

Case 3.- If $s, \tau \in S$ and $t \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = P_{213}M^{[s,t]}M^{[t,\tau]}P_{213} = P_{213}M^{[s,\tau]}P_{213} = \widetilde{M}^{[s,\tau]}.$$

Case 4.- If $s \in S$ and $t, \tau \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = P_{213}M^{[s,t]}M^{[t,\tau]} = P_{213}M^{[s,\tau]} = \widetilde{M}^{[s,\tau]}.$$

Case 5.- If $t, \tau \in S$ and $s \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = M^{[s,t]}P_{213}P_{213}M^{[t,\tau]}P_{213} = M^{[s,t]}M^{[t,\tau]}P_{213} = M^{[s,\tau]}P_{213} = \widetilde{M}^{[s,\tau]}.$$

Case 6.- If $t \in S$ and $s, \tau \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = M^{[s,t]}P_{213}P_{213}M^{[t,\tau]} = M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]} = \widetilde{M}^{[s,\tau]}.$$

Case 7.- If $\tau \in S$ and $s, t \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = M^{[s,t]}M^{[t,\tau]}P_{213} = M^{[s,\tau]}P_{213} = \widetilde{M}^{[s,\tau]}.$$

Case 8.- If $s, t, \tau \notin S$ then,

$$\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]} = \widetilde{M}^{[s,\tau]}.$$

This proves that $\widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = \widetilde{M}^{[s,\tau]}$, for every $a < s < t < \tau < b$. On the other hand, note that in (21) the matrices $\widetilde{M}^{[s,\tau]}$ and $M^{[s,\tau]}$ can be exchanged, so the result follows. □

Next, we work with the set of matrices $M^{[s,t]}$ in $\mathcal{M}_{[a,b]}$ having the first row equal to zero and the third one being non-zero.

Proposition 4.2. *Let* $\mathcal{M}_{[a,b]} = \{M^{[s,t]} : a < s < t < b\}$ *be a family of* 3×3 *matrices. Suppose that the set*

$$\mathcal{S} = \{s \in (a,b) : \exists t \ \ with \ \ (1,0,0)M^{[s,t]} = (0,0,0) \ and \ (0,0,1)M^{[s,t]} \neq (0,0,0)\}$$

is non-empty, and define

$$\widetilde{M}^{[s,t]} = \begin{cases} P_{321}M^{[s,t]} & if \ \ s \in \mathcal{S}, t \notin \mathcal{S}, \\ P_{321}M^{[s,t]}P_{321} & if \ \ s \in \mathcal{S}, t \in \mathcal{S}, \\ M^{[s,t]}P_{321} & if \ \ s \notin \mathcal{S}, t \in \mathcal{S}, \\ M^{[s,t]} & if \ \ s \notin \mathcal{S}, t \notin \mathcal{S}. \end{cases} \tag{22}$$

where

$$P_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then, for every $a < s < t < \tau < b$, *we have that*

$$M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]} \text{ if and only if } \widetilde{M}^{[s,t]}\widetilde{M}^{[t,\tau]} = \widetilde{M}^{[s,\tau]}.$$

Proof. This proof is analogous to that of Proposition 4.1. □

We conclude that applying Proposition 4.1 and then Proposition 4.2, if necessary (each of them), it is not restrictive to assume that $\mathcal{M}_{[a,b]}$ (that is the C-K chain of rank 1 associated to the 3 dimensional CEA $\mathcal{E}_{[a,b]}$ we are considering) is such that all the matrices $M^{[s,t]} \in \mathcal{M}_{[a,b]}$ have their first row being non-zero. In fact, if this is not the case, then Proposition 4.1 and Proposition 4.2 provide a new C-K chain $\widetilde{\mathcal{M}}_{[a,b]}$ of rank 1 which has this property, and it turns out that the matrices of $\widetilde{\mathcal{M}}_{[a,b]}$ can be described if and only if the matrices of $\mathcal{M}_{[a,b]}$ can be described (as the changes of chain provided in Proposition 4.1 and in Proposition 4.2 are involutive).

Therefore, our aim now is *to describe a C-K chain of rank* 1, *say* $\mathcal{M}_{[a,b]}$, *satisfying that the first row of every* $M^{[s,t]} \in \mathcal{M}_{[a,b]}$ *is non-zero*. Let

$$M^{[s,t]} = \begin{pmatrix} a_1(s,t) & a_2(s,t) & a_3(s,t) \\ k(s,t)a_1(s,t) & k(s,t)a_2(s,t) & k(s,t)a_3(s,t) \\ \widetilde{k}(s,t)a_1(s,t) & \widetilde{k}(s,t)a_2(s,t) & \widetilde{k}(s,t)a_3(s,t) \end{pmatrix},$$

where $\mathbf{a}(s,t) = (a_1(s,t), a_2(s,t), a_3(s,t))$ is a non-zero vector of $\mathbb{K}^3$. Note that $\mathbf{k}(s,t) = (1, k(s,t), \widetilde{k}(s,t))$ is another vector of $\mathbb{K}^3$ that, joint with $\mathbf{a}(s,t)$, determines $M^{[s,t]}$. The Chapman-Kolmogorov equation, $M^{[s,t]}M^{[t,\tau]} = M^{[s,\tau]}$, read by columns have the following view:

$$\begin{aligned} \left\langle \mathbf{a}(s,t), a_j(t,\tau)\mathbf{k}(t,\tau)\right\rangle &= a_j(s,\tau), \text{ for } j=1,2,3. \\ k(s,t)\left\langle \mathbf{a}(s,t), a_j(t,\tau)\mathbf{k}(t,\tau)\right\rangle &= k(s,\tau)a_j(s,\tau), \text{ for } j=1,2,3. \\ \widetilde{k}(s,t)\left\langle \mathbf{a}(s,t), a_j(t,\tau)\mathbf{k}(t,\tau)\right\rangle &= \widetilde{k}(s,\tau)a_j(s,\tau), \text{ for } j=1,2,3. \end{aligned} \tag{23}$$

Since $\mathbf{a}(s,\tau)$ is non-zero, it follows that $a_j(s,\tau)\neq 0$ for some $j\in\{1,2,3\}$ so that, from (23), we deduce that $\left\langle \mathbf{a}(s,t), a_j(t,\tau)\mathbf{k}(t,\tau)\right\rangle \neq 0$ and hence,

$$w(s,t,\tau) := \langle \mathbf{a}(s,t), \mathbf{k}(t,\tau)\rangle \neq 0.$$

From the first equality in (23) we have that $a_j(t,\tau)w(s,t,\tau) = a_j(s,\tau)$, and from the two last ones we obtain that $k(s,t)=k(s,\tau)$ and $\widetilde{k}(s,t)=\widetilde{k}(s,\tau)$, for every $a<s<t<\tau<b$. This means that $k(s,t)$ and $\widetilde{k}(s,t)$ only depend on its first variable. Thus we obtain functions $\varphi_1,\varphi_2 : (a,b)\to\mathbb{R}$ such that $k(s,t)=\varphi_1(s)$ and $\widetilde{k}(s,t)=\varphi_2(s)$, for every $a<s<t<b$. Therefore,

$$\mathbf{k}(t,\tau) = (1, k(t,\tau), \widetilde{k}(t,\tau)) = (1,\varphi_1(t),\varphi_2(t))$$

does not depend on τ. Consequently, (23) can be written as follows:

$$\begin{aligned} a_j(t,\tau)w(s,t,\tau) &= a_j(s,\tau), \text{ for } j=1,2,3. \\ \varphi_1(s)a_j(t,\tau)w(s,t,\tau) &= \varphi_1(s)a_j(s,\tau), \text{ for } j=1,2,3. \\ \varphi_2(s)a_j(t,\tau)w(s,t,\tau) &= \varphi_2(s)a_j(s,\tau), \text{ for } j=1,2,3. \end{aligned} \tag{24}$$

On the other hand, from (24), we obtain that $a_j(t,\tau)=0$ if and only if $a_j(s,\tau)=0$.

Finally, let α_n be a strictly decreasing sequence such that $\alpha_n\to a$. Let f_n, g_n, $h_n : (\alpha_n,b)\to\mathbb{K}$ be the functions given by

$$f_n(t) = a_1(\alpha_n,t), \qquad g_n(t) = a_2(\alpha_n,t), \qquad h_n(t) = a_3(\alpha_n,t),$$

for every $t>\alpha_n$ and every $n\in\mathbb{N}$. From the fact that $w(\alpha_n,t,\tau)\neq 0$ and that $a_j(t,\tau)w(\alpha_n,t,\tau) = a_j(\alpha_n,\tau)$, it follows that

$$\begin{aligned} a_1(t,\tau) &= \frac{f_n(\tau)}{f_n(t)+\varphi_1(t)g_n(t)+\varphi_2(t)h_n(t)}, \\ a_2(t,\tau) &= \frac{g_n(\tau)}{f_n(t)+\varphi_1(t)g_n(t)+\varphi_2(t)h_n(t)}, \\ a_3(t,\tau) &= \frac{h_n(\tau)}{f_n(t)+\varphi_1(t)g_n(t)+\varphi_2(t)h_n(t)}, \end{aligned} \tag{25}$$

for every n such that $\alpha_n<t<\tau$.

Since, for $j=1,2,3$, the equality $a_j(t,\tau)w(s,t,\tau)=a_j(s,\tau)$ holds for every $a<s<t$, it follows that in (25) we can replace n by some m such that $\alpha_m<t$.

Anyway,

$$M^{[s,t]} = \frac{1}{\Phi_n(s)}\begin{pmatrix} f_n(t) & g_n(t) & h_n(t) \\ \varphi_1(s)f_n(t) & \varphi_1(s)g_n(t) & \varphi_1(s)h_n(t) \\ \varphi_2(s)f_n(t) & \varphi_2(s)g_n(t) & \varphi_2(s)h_n(t) \end{pmatrix},$$

as desired, where $\Phi_n(s) = f_n(s)+\varphi_1(s)g_n(s)+\varphi_2(s)h_n(s)$.

Remark 4.3. *Note that the Chapman-Kolmogorov chains of matrices 3×3 of rank 0 are trivial (they are families consisting of matrices equal to zero).*

5. The general case

Let $\mathcal{E} = \{E^{[s,t]} : s, t \in \mathbb{R}, 0 < s < t\}$ be a CEA of real evolution algebras with dimension 3, and let $\mathcal{M}_{\mathcal{E}} = \{M^{[s,t]} : 0 < s < t\}$ be the associated C-K chain. The aim of this section is to describe $\mathcal{E}$ locally, by showing that $\mathbb{R}_0^+$ can be written as a non-overlapped union of intervals (this means that two of these intervals cannot share any interior point) where, if $[a, b]$ is one of these intervals then $\mathcal{M}_{[a,b]} := \{M^{[s,t]} : a < s < t < b\}$ is a C-K chain with (fixed) rank r equal to either $0, 1, 2$, or 3. (These C-K chains were described in the previous sections of this paper).

For every $[s_0, t_0]$ with $rank M^{[s_0,t_0]} = 3$, we define its associated 3−rank chain as follows. Let

$$\beta(t_0) := \sup\{t : t \geq t_0 \text{ and } rank M^{[s_0,t]} = 3\}.$$

Note that $\mathcal{M}_{[s_0,\beta(t_0)]}$ is a C-K chain of rank 3 because if $s_0 < u < v < \beta(t_0)$ then,

$$3 = rank\, M^{[s_0,v]} = rank\, (M^{[s_0,u]} M^{[u,v]}) \leq rank\, M^{[u,v]}.$$

Similarly, we define

$$\alpha(s_0) := \inf\{s : s \leq s_0 \text{ and } rank M^{[s,s_0]} = 3\},$$

and it follows that, as above, if $\alpha(s_0) \leq s_0$ then, $\mathcal{M}_{[\alpha(t_0),s_0]}$ is a C-K chain of rank 3. Moreover, since the product of two non-singular matrices is a non-singular matrix we obtain that $\mathcal{M}_{[\alpha(s_0),\beta(t_0)]}$ is a C-K chain of rank 3, and we say that $\mathcal{M}_{[\alpha(t_0),\beta(t_0)]}$ is the **maximal chain of rank 3 associated** to $M^{[s_0,t_0]}$.

Therefore, the maximal chain of rank 3 associated to a matrix $M^{[s_0,t_0]} \in \mathcal{M}_{\mathcal{E}}$ with $rank M^{[s_0,t_0]} = 3$ determines an interval $[\alpha(s_0), \beta(t_0)]$ in $\mathbb{R}_0^+$. Consider the union of all these intervals by defining,

$$I_3 := \cup\{[\alpha(s_0), \beta(t_0)] : M^{[s_0,t_0]} \in \mathcal{M}_{\mathcal{E}} \text{ with } rank M^{[s_0,t_0]} = 3\}. \tag{26}$$

Thus, I_3 can be expressed as a disjoint union of intervals.

Consider now $\mathbb{R}_0^+ \backslash I_3$. This set also is a disjoint union of intervals with the property that if $[s, t] \subseteq \mathbb{R}_0^+ \backslash I_3$, with $s < t$, then $rank M^{[s,t]} \leq 2$. It follows that, if $rank M^{[s,t]} = 2$ then, $rank M^{[u,v]} = 2$ for $s \leq u < v \leq t$. In fact, if $s = u$ or $v = t$ it is clear and otherwise $M^{[s,t]} = M^{[s,u]} M^{[u,v]} M^{[v,t]}$ so that, $2 = rank M^{[s,t]} \leq rank M^{[u,v]} \leq 2$.

For every $[s_0, t_0] \subseteq \mathbb{R}_0^+ \backslash I_3$ such that $rank M^{[s_0,t_0]} = 2$, we define the **maximal chain of rank 2 associated to** $M^{[s_0,t_0]}$ as $\mathcal{M}_{[\alpha(t_0),\beta(t_0)]}$ where,

$$\beta(t_0) := \sup\{t : t \geq t_0, \text{ with } [s_0, t] \subseteq \mathbb{R}_0^+ \backslash I_3 \text{ and } rank M^{[s_0,t]} = 2\},$$

and

$$\alpha(s_0) := \inf\{s : [s, t_0] \subseteq \mathbb{R}_0^+ \backslash I_3 \text{ with } s \leq s_0 \text{ and } rank M^{[s,s_0]} = 2\}.$$

Since there exists a unique interval in the disjoint union $\mathbb{R}_0^+ \backslash I_3$ containing $[s_0, t_0]$, we can obtain the corresponding $\alpha(s_0)$ and $\beta(t_0)$ just working in this interval (as $[s_0, t]$ and $[s, s_0]$ are intervals contained in $\mathbb{R}_0^+ \backslash I_3$).

Now we consider the union of all the intervals associated to the maximal chains of rank 2, and define

$$I_2 := \cup\{[\alpha(s_0), \beta(t_0)] : [s_0, t_0] \subseteq \mathbb{R}_0^+ \backslash I_3 \text{ with } rank M^{[s_0,t_0]} = 2\}. \tag{27}$$

Consider $(\mathbb{R}_0^+ \backslash I_3) \backslash I_2$. As above, this set can be written as a disjoint union of intervals and, if $[s_0, t_0] \subseteq (\mathbb{R}_0^+ \backslash I_3) \backslash I_2$ then, it turns out that $rank\, M^{[s_0,t_0]} \leq 1$. If $rank\, M^{[s_0,t_0]} = 1$ then, as before, we obtain **the maximal chain of rank** 1 **associated to** $M^{[s_0,t_0]}$, say $\mathcal{M}_{[\alpha(t_0),\beta(t_0)]}$, and we consider the union of all these associated intervals to define I_1. Note that if $[s_0, t_0] \subseteq (\mathbb{R}_0^+ \backslash I_3) \backslash I_2 \backslash I_1$ then, the rank of $M^{[s_0,t_0]}$ is zero, and we can obtain **the maximal chain of rank** 0 **associated to** $M^{[s_0,t_0]}$. Therefore,

$\mathbb{R}_0^+ = I_3 \cup I_2 \cup I_1 \cup I_0.$

This shows how the description of the C-K chains of rank $r = 0, 1, 2, 3$ provided in the previous sections of this paper is very helpful to give a local description of general 3-dimensional CEAs.

This approach does not always provide a global description of an arbitrary CEA with dimension 3 because the intervals associated to maximal C-K chains of rank i can be distributed along $\mathbb{R}_0^+$, for instance, in a fractal way. Just to give a flavour, suppose that I_3 is the Cantor set in $[0,1]$ and that $I_2 = [0,1]\backslash I_3$. Then, no problem to obtain $M^{[1/6,1/4]}$ because this matrix belong to $\mathcal{M}_{[0,1/3]}$ that is a C-K chain of rank 3 (and therefore it is described). Nevertheless, to obtain $M^{[1/4,3/4]}$ by means of the Chapman-Kolmogorov equations we would need to consider an infinite product of matrices (because between 1/4 and 3/4 we can find an infinite number of - disjoint - intervals of I_3 as well as of I_2). Because of this, the description of the CEAs that we obtain here is local.

However, in the case that I_1, I_2 and I_3 consist of a finite union of intervals $[\alpha(s_0), \beta(t_0)]$ corresponding to C-K chains $\mathcal{M}_{[\alpha(t_0),\beta(t_0)]}$ of fixed rank then, we obtain a complete description of the CEA $\mathcal{E}$.

This also shows a procedure to obtain a big variety of CEAs containing matrices of different ranks. Particularly, examples of CEAs containing matrices of rank r where $r = 0, 1, 2, 3$ (or where r runs in some predetermined subset of $\{0, 1, 2, 3\}$) are easily obtained from this approach.

Example 5.1. *Let* $[0, +\infty) = J_0 \cup J_1 \cup J_2 \cup J_3 \cup J_4$ *where* $J_0 = [0, 1]$, $J_1 = [1, 2]$, $J_2 = [2, 3]$, $J_3 = [3, 4]$, *and* $J_4 = [4, +\infty)$. *Let*

$$A_0 = (0)_{3\times3},\ \ A_1 = I_{3\times3},\ \ A_2 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},\ \ A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_4 = (0)_{3\times3}.$$

Define $M^{[s,t]} = \frac{t}{s} A_i A_j$ *if* $s \in J_i$ *and* $t \in J_j$, *for every* $0 < s < t$. *We claim that* $M^{[s,t]} M^{[t,w]} = M^{[s,w]}$ *for every* $0 < s < t < w$. *In fact, if* $s \in J_i$, $t \in J_j$ *and* $w \in J_k$ *for some* $i, j, k \in \{0, 1, 2, 3, 4\}$ *then* $i \le j \le k$ *and*

$$M^{[s,t]} M^{[t,w]} = (\frac{t}{s} A_i A_j)(\frac{w}{t} A_j A_k) = \frac{w}{s} A_i A_j A_k.$$

Indeed, if $j = i$ *or* $j = k$ *then* $M^{[s,t]} M^{[t,w]} = \frac{w}{s} A_i A_k = M^{[s,w]}$ *since* $A_i^2 = A_i$, *for every* $i = 0, 1, 2, 3, 4$. *Otherwise* $i < j < k$. *If* $i = 0$ *or* $k = 4$ *then* $M^{[s,t]} M^{[t,w]} = (0)_{3\times3} = \frac{w}{s} A_i A_k = M^{[s,w]}$. *Similarly, if* $i \neq 0$ *and* $k \neq 4$ *then* $1 \le i < j < k \le 3$ *so that* $i = 1$, $j = 2$ *and* $k = 3$. *Since* $A_2 A_3 = A_3$ *it follows that* $A_1 A_2 A_3 = A_1 A_3$ *and therefore* $M^{[s,t]} M^{[t,w]} = \frac{w}{s} A_1 A_2 A_3 = \frac{w}{s} A_1 A_3 = M^{[s,w]}$. *This proves the claim.*

Thus, we obtain a CEA, say $\mathcal{E}_{[0,+\infty)}$, *whose C-K chain associated*

$$\mathcal{M}_{\mathcal{E}_{[0,+\infty)}} = \{M^{[s,t]} : 0 < s < t\}$$

is such that $[0, +\infty) = J_0 \cup J_1 \cup J_2 \cup J_3 \cup J_4$ *(a non-overlapped union of intervals) where* $\mathcal{M}_{[0,1]}$ *and* $\mathcal{M}_{[4,+\infty)}$ *are C-K chains of rank zero, and* $\mathcal{M}_{[1,2]}$, $\mathcal{M}_{[2,3]}$ *and* $\mathcal{M}_{[3,4]}$ *are C-K chains of rank* 3, 2, *and* 1 *respectively.*

References

[1] J.M. Casas, M. Ladra, U.A. Rozikov, A chain of evolution algebras, Linear Algebra Appl. 435 (2011) no.4, 852-870.

[2] J.M. Casas, M. Ladra, U.A. Rozikov, Markov processes of cubic stochastic matrices: Quadratic stochastic processes, Linear Algebra Appl. 575 (2019), 273-298.

[3] V.P. Dmitriev, Quantum mechanics as a theory of a non-Markov random process, Dokl. Akad. Nauk SSSR, 292(5) (1987), 1101-1105 (in Russian).

[4] P. H*ä*nggi, H. Thomas, Time evolution, correlations, and linear response of non-Markov processes, Z. Phys. B. 26 (1977), 85-92.

[5] A.N.Imomkulov, Classification of a family of three dimensional real evolution algebras, TWMS J. Pure Appl. Math., 10(2) (2019), 225-238.

[6] A.N.Imomkulov, U.A.Rozikov, Approximation of an algebra by evolution algebras, arXiv preprint arXiv:1910.03708.

[7] N.G. van Kampen, Remarks on non-Markov processes, Braz. J. Phys. 28(2) (1998), 90-96.
[8] M. Ladra, U.A. Rozikov, Flow of finite-dimensional algebras, J. Algebra, 470 (2017), 263-288.
[9] Y.I. Lyubich, Mathematical structures in population genetics, Springer-Verlag, Berlin, 1992.
[10] V.M. Maksimov, Cubic stochastic matrices and their probability interpretations, Theory Probab. Appl. 41(1) (1996), 55-69.
[11] SH.Murodov, Classification dynamics of two-dimensional chains of evolution algebras, Internat. J. Math., 25(02).(2014), 1450012-1-23.
[12] B.A.Omirov, U.A.Rozikov, K.M.Tulenbayev, On real chains of evolution algebras, Linear Multilinear A., 63(1) (2015), 586-600.
[13] U.A. Rozikov, Sh.N. Murodov, Chain of evolution algebras of "chicken' population, Linear Algebra Appl. 450 (2014), 186-201.
[14] U.A.Rozikov, Sh.N.Murodov, Dynamics of two-dimensional evolution algebras, Lobachevskii Jour. Math. 34(4) (2013), 344-358.
[15] U.A. Rozikov, J. P. Tian, Evolution algebras generated by Gibbs measures, Lobachevskii Jour. Math. 32(4) (2011), 270-277.
[16] Y. Suhov, M. Kelbert, Probability and Statistics by Example, vol. II, Markov Chains: a Primer in Random Processes and Their Applications, Cambridge Univ. Press, Cambridge, 2008.
[17] J. P. Tian, Evolution algebras and their applications, Lecture Notes in Mathematics, 1921, Springer-Verlag, Berlin, 2008.