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A Boundary Schwarz Lemma for Pluriharmonic Mappings Between the Unit Polydiscs of Any Dimensions

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Abstract. In this paper, we present a boundary Schwarz lemma for pluriharmonic mappings between the unit polydiscs of any dimensions, which extends the classical Schwarz lemma for bounded harmonic functions to higher dimensions.

1. Introduction

The Schwarz lemma is regarded as one of the most important results in complex analysis. Let f be a holomorphic self-mapping of the unit disk D. The classical Schwarz lemma states that for holomorphic mapping f satisfying the condition $f(0) = 0$, the inequality $|f(z)| \leq |z|$ is true for any $z \in D$. This result is a potent tool to study several research fields in complex analysis. An increasing number of mathematicians thus focus attention on establishing various versions of the Schwarz lemma.

Schwarz lemma at the boundary is an active topic in complex analysis. Various interesting results associated with the boundary Schwarz lemma have been presented in recent years. For the convenience of representation, we introduce some notations and definitions.

Let $\mathbb{C}^n$ be the complex space of dimension n with the norm given by $\|z\| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$ for any $z = (z_1, z_2, \cdots, z_n)^T \in \mathbb{C}^n$. For any $z = (z_1, z_2, \cdots, z_n)^T, \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \in \mathbb{C}^n$, the inner product on $\mathbb{C}^n$ is defined by $\langle z, \omega\rangle = \sum_{i=1}^n z_i\overline{\omega_i}$, therefore $\langle z, z\rangle^{\frac{1}{2}} = \|z\|$ also represents the norm of z. Let $B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the unit ball in $\mathbb{C}^n$, and $\partial B^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$ be the unit sphere. Denote by D the unit disk with unit circle T in the complex plane $\mathbb{C}$, then the unit polydisc can be represented as $D^n = D \times \cdots \times D = \{z \in \mathbb{C}^n : |z_i| < 1, 1 \leq i \leq n\}$ which belongs to the complex space $\mathbb{C}^n$. Furthermore, denote $\|z\|_\infty = \max_{1\leq i\leq n}$, then we have $\partial D^n = \{z \in \mathbb{C}^n : \|z\|_\infty = 1\}$ and $T^n = T \times \cdots \times T = \{z \in \mathbb{C}^n : |z_i| = 1\, 1 \leq i \leq n\}$ which represent the topological boundary and the distinguished boundary ofD^n, respectively. If there are only r $(1 \leq r \leq n)$ components of z_0 whose modules equals to 1, then the set of all this kind of boundary points is denoted by E_r. It is obvious that $E_n = T^n$ and $\bigcup_{1\leq r\leq n} E_r = \partial D^n$ if taking all boundary points into consider.

Denote the set of all holomorphic mappings between the bounded domains of any dimensions as $H(\Omega_1, \Omega_2)$ where $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^N$. For any $f = (f_1, f_2, \cdots, f_N)^T \in H(\Omega_1, \Omega_2)$, the Jacobian matrix of f

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at $z \in \Omega_1$ is given by

$$Df(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{N\times n}.$$

Moreover, we use $\overline{D}f(z)$ to represent the $N \times n$ matrice $\left(\frac{\partial f_i}{\partial \overline{z_j}}(z)\right)_{N\times n}$. For the same function, denote by $J_f(z)$ the $2N \times 2n$ Jacobian matrix of f at z in terms of real coordinates. Let $C^{\alpha}(V)$ be the set of all functions f on the bounded domain V for which

$$\sup\left\{\frac{|f(z) - f(z')|}{|z - z'|^{\alpha}} : z, z' \in \overline{V}\right\}$$

is finite with $0 < \alpha < 1$. Then we denote $C^{k+\alpha}(V)$ as the set of all functions f on V whose k-th order partial derivatives exist and belong to $C^{\alpha}(V)$ for an integer k.

In [1], the classical boundary Schwarz lemma for holomorphic mappings is described as follows:

Theorem 1.1. [1] *Let $f \in H(D, D)$ be a holomorphic mapping. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f^{'}(1) \geq 1$. Moreover, the inequality is sharp.*

If we remove the condition $f(0) = 0$ in the above theorem and take the holomorphic mapping

$$g(z) = \frac{1 - \overline{f(0)}}{1 - f(0)} \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)},$$

we have the following estimate instead:

$$f'(1) \geq \frac{|1 - \overline{f(0)}|^2}{1 - |f(0)|^2} > 0. \tag{1}$$

Chelst[2] and Osserman[3] further studied the Schwarz lemma at the boundary of the unit disk, respectively. Ornek[4] explored some new expressions of Schwarz inequality at the boundary of the unit disk and acquired the sharpness of these inequalities.

Moreover, in the case of several complex variables, Wu generalized the classical Schwarz lemma for holomorphic mappings to higher dimension [5]. Recently, Liu et al.[6] presented a version of the boundary Schwarz lemma for holomorphic mappings from the unit ball B^n to the unit ball B^N, which is not restricted by the condition $f(0) = 0$.

Theorem 1.2. [6] *Let $f \in H(B^n, B^N)$ for $n, N \geq 1$. If f is $C^{1+\alpha}$ at $z_0 \in \partial B^n$ with $f(z_0) = \omega_0 \in \partial B^N$, then there exists $\lambda \in \mathbb{R}$ such that*

$$\overline{Df(z_0)}^T \omega_0 = \lambda z_0$$

where $\lambda = \frac{|1-\overline{a}^T\omega_0|^2}{1-\|a\|^2} > 0$, $a = f(0)$.

Furthermore, in [7] Liu et al. presented the result of Schwarz lemma for holomorphic mappings from the unit polydisc D^n to the unit ball B^N at the boundary as follows.

Theorem 1.3. [7] *Let $f \in H(D^n, B^N)$ for $n, N \geq 1$. Given $z_0 \in \partial D^n$. Assume $z_0 \in E_r$ with the first r components at the boundary of D for some $1 \leq r \leq n$. If f is $C^{1+\alpha}$ at z_0 with $f(z_0) = \omega_0 \in \partial B^N$, then there exist a sequence of nonnegative real numbers $\gamma_1, \gamma_2, \cdots, \gamma_r$ satisfying $\sum_{j=1}^{r} \gamma_j \geq 1s$ and $\lambda \in \mathbb{R}$ such that*

$$\overline{Df(z_0)}^T \omega_0 = \lambda diag(\gamma_1, \cdots, \gamma_r, 0, \cdots, 0)z_0$$

where $\lambda = \frac{|1-\overline{a}^T\omega_0|^2}{1-\|a\|^2} > 0$, $a = f(0)$ and "diag" represents the diagonal matrix.

Harmonic mapping is a complex-valued harmonic function defined in the complex space, which is in touch with geometric functions and locally quasiconformal mappings. For the harmonic mappings, there are also some interesting analogues of the Schwarz lemma. For example, the Schwarz lemma for the harmonic self-mapping of the unit disk is stated as follows.

Theorem 1.4. [8] *Let f is a harmonic mapping of the unit disk D on itself, and $f(0) = 0$, then*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|, z \in D.$$

In [9], the boundary Schwarz lemma for the harmonic self-mapping of the unit disk is restated with a simple proof. Considering the several complex variables, Mateljevic offered the boundary Schwarz lemma for harmonic mappings between the unit balls with any dimensions in [10].

Note that the pluriharmonic mapping can be considered as a generalization of the harmonic function. A continuous complex-valued function f defined on a domain $\Omega \in \mathbb{C}^n$ is said to be pluriharmonic if for each fixed $z \in \Omega$ and $\theta \in \partial B^n$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta : \|\zeta\| < d\Omega(z)\}$, where $d\Omega(z)$ denotes the distance from z to the boundary $\partial\Omega$ of Ω. Therefore, it is a very natural task to obtain various versions of the Schwarz lemma for pluriharmonic mappings.

It is obtained in [11] that when Ω is a simply connected domain, then $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if f could be represented by $f = \eta + \overline{\zeta}$ where η and ζ are holomorphic in Ω. Hence, a holomorphic mapping can be regarded as a special pluriharmonic function. Furthermore, $f : \Omega \to \mathbb{C}^n$ is called a pluriharmonic mappings if all its components are pluriharmonic functions from Ω to $\mathbb{C}$. Similarly to $H(\Omega_1, \Omega_2)$, the set of pluriharmonic mappings between the bounded domains of any dimensions is denoted as $P(\Omega_1, \Omega_2)$ where $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^N$.

In [12], Mateljević introduced Kobayashi metrics and obtained the Kobayashi-Schwarz lemma for the holomorphic mappings on the bounded connected open subsets of complex Banach space. As an application of the lemma obtained, a boundary Schwarz lemma is established for pluriharmonic mappings defined on the unit ball B^2.

For the pluriharmonic mappings between unit balls with any dimensions, in [13], Liu et al. presented the boundary Schwarz lemma for pluriharmonic mappings defined on the unit ball.

Theorem 1.5. [13] *Let $f \in P(B^n, B^N)$ for $n, N \geq 1$. If f is $C^{1+\alpha}$ at $z_0 \in \partial B^n$ and $f(z_0) = \omega_0 \in \partial B^N$, then there exists a positive $\lambda \in \mathbb{R}$ such that*

$$Df(z_0')^T \omega_0' = \lambda z_0'$$

where z_0' and ω_0' are real versions of z_0 and ω_0, and $\lambda \geq \frac{1-\|f(0)\|}{2^{2n-1}} > 0$.

In this paper, we extend the boundary Schwarz lemma for planar harmonic mappings to higher dimensions, and establish a novel boundary Schwarz lemma for pluriharmonic mappings between the unit polydiscs of any dimensions.

Inspired by [13], we consider the real version of this problem. For $z = (z_1, z_2, \cdots, z_n)^T \in \mathbb{C}^n$ with $z_i = x_i + \mathbf{i}y_i$ where $1 \leq i \leq n$, denote z' as the real version of z and $z' = (x_1, y_1, \cdots, x_n, y_n)^T \in \mathbb{R}^{2n}$ only containing real elements. Therefore, D^n in $\mathbb{C}^n$ is equivalent to the unit polydisc $\mathrm{D}^{2n} \subset \mathbb{R}^{2n}$.

We first combine the Harnack's inequality with the minimum principle and establish a new inequality for the nonnegative harmonic function defined on the unit polydisc D^{2n} (see Lemma 2.1). This lemma provides an important technique support for estimating the lower bound of the function in the proof of the main results. Furthermore, we also present the Schwarz lemma for the pluriharmonic mapping $f \in P(D^n, D^N)$ (see Lemma 2.2), which generalizes the corresponding results in Theorem 1.4 to higher dimensions and plays a significant role in the proof of Theorem 1.6. Then we get the following boundary Schwarz lemma for pluriharmonic mappings in $P(D^n, D^N)$.

Theorem 1.6. *Let $f \in P(D^n, D^N)$ with $f(0) = 0$ for $n, N \geq 1$. Given $z_0 = (z_1, \cdots, z_r, z_{r+1}, \cdots, z_n)^T \in E_r \subset \partial D^n$. If f is $C^{1+\alpha}$ at z_0 and $f(z_0) = \omega_0 \in E_m \subset \partial D^N$, then there exist a sequence of nonnegative real numbers $\gamma_1, \gamma_2, \cdots, \gamma_r$ such that the following statements hold.*

1) *Suppose that z_0' and ω_0' are real versions of z_0 and ω_0, respectively. Then*

$$J_f(z_0')^T\omega_0' = diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)z_0'$$

where $\lambda_i = \sum_{j=1}^{m} \frac{\partial u_j}{\partial x_i}$ *with* $\frac{\partial u_j}{\partial x_i} \geq \frac{1-u_j(0)}{2^{2n-1}}$ *for* $1 \leq i \leq r$.
2) $tr(diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)) \geq \frac{\pi}{2}$ *for* $1 \leq r \leq n$.

2. Some lemmas

In this section, we exhibit some notations and present several basic lemmas, which play the significant roles in the proof of the main results.

Lemma 2.1. *Let f be a nonnegative function defined on the unit polydisc D^{2n} in $\mathbb{R}^{2n}$. If f is continuous on the unit polydisc and harmonic on its interior, then for any $z = (x_1, y_1, \cdots, x_n, y_n)^T \in \mathrm{D}^{2n}$ satisfying $\sqrt[2]{x_i^2 + y_i^2} = r_0 < 1 (1 \leq i \leq r < n)$ and $x_i = y_i = 0 (r+1 \leq i \leq n)$, the following inequation holds:*

$$f(z) \geq \frac{1-r_0}{(1+r_0)^{2n-1}} f(0).$$

Proof. Suppose that f is a nonnegative function defined on the unit ball B^{2n} in $\mathbb{R}^{2n}$. According to the description of [13], we know that if f is continuous on the unit ball and harmonic on its interior, then for any $z_0 \in \mathrm{B}^{2n}$ with $\| z_0 \| = r_0 < 1$ we have the Harnack's inequality

$$\frac{1-r_0}{(1+r_0)^{2n-1}} f(0) \leq f(z) \leq \frac{1+r_0}{(1-r_0)^{2n-1}} f(0).$$

Since the conditions that $z = (x_1, y_1, \cdots, x_n, y_n)^T \in \mathrm{D}^{2n}$ satisfies $|z_i| = \sqrt{x_i^2 + y_i^2} = r_0 < 1$ for $1 \leq i \leq r < n$ and $x_i = y_i = 0$ for $r+1 \leq i \leq n$, it is not difficult to derive that $\|z\| = (\sum_{i=1}^{n} |z_i|^2)^{\frac{1}{2}} = \sqrt{r} r_0 < \sqrt{r}$. Then we have

$$\frac{1-r_0}{(1+r_0)^{2n-1}} f(0) \leq f(\frac{z}{\sqrt{r}}) \leq \frac{1+r_0}{(1-r_0)^{2n-1}} f(0).$$

Therefore, it follows from the minimum principle for harmonic function that

$$f(z) \geq \frac{1-r_0}{(1+r_0)^{2n-1}} f(0).$$

This completes the proof. □

Lemma 2.2. *Let $f = (f_1, \cdots, f_n)^T \in P(D^n, D^n)$ with $n, N \geq 1$ and $f(0) = 0$, then*

$$\|f(z)\|_\infty \leq \frac{4}{\pi} \arctan \|z\|_\infty.$$

Proof. For any fixed $z' \in D^n \setminus \{0\}$ and any $\zeta \in D$, let

$$F_i(\zeta) = \left\langle f_i\left(\frac{\zeta z'}{\|z'\|_\infty}\right), \frac{f_i(z')}{\|f_i(z')\|} \right\rangle$$

for any $1 \leq i \leq n$. Applying Theorem 1.4 to the complex-valued harmonic mapping F_i, we have the inequality

$$|f_i(z')| = |F_i(\|z\|_\infty)| \leq \frac{4}{\pi} \arctan \|z\|_\infty$$

for any $1 \le i \le n$. Thus, the inequality

$$\|f(z)\|_\infty \le \frac{4}{\pi} \arctan \|z\|_\infty$$

holds for any $z \in D^n$.

The proof of the lemma is complete. □

Remark 2.3. *When $n = N = 1$, Lemma2.2 reduces to Theorem 1.4, which extends the boundary Schwarz lemma to high dimensions.*

3. Proof of Theorem 1.6

In the following, we will prove Theorem 1.6.

Proof. *1)* The first proof is divided into six steps for reader's convenience.

Step 1. Assume $z_0 \in \partial D^n$ and $f = (f_1, \cdots, f_N)^T$ is $C^{1+\alpha}$ in a neighborhood V of z_0. Without loss of generality, we let $z_0 = \sum_{i=1}^r e_i^n$ where e_i^n represents the i-th column of the identity matrix I_n. Since $f_j = u_j + \mathbf{i}v_j$ for $1 \le j \le N$ is defined on the unit polydisc, it is obtained from f that $u_k(\sum_{i=1}^r e_i^n) = 1$ for $1 \le k \le m \le N$. Moreover, $1 - u_k \ge 0$ is harmonic on the unit polydisc. By using Lemma 2.1 and considering $x = (x_1, 0, x_2, 0, \cdots, x_r, 0, \cdots, 0)^T \in \mathbb{R}^{2n}$ for $x_i = r_0$ near 1 ($1 \le i \le r \le n$), we have

$$1 - u_k(x) \ge \frac{1 - x_i}{(1 + x_i)^{2n-1}} \left(1 - u_k(0)\right),$$

which gives that

$$\frac{1 - u_k(x)}{1 - x_i} \ge \frac{1 - u_k(0)}{(1 + x_i)^{2n-1}}.$$

Letting $x_i \to 1^-$, we can derive

$$\frac{\partial u_k(\sum_{i=1}^r e_i^n)}{\partial x_i} = \lim_{x_i \to 1^-} \frac{(1 - u_k(x)) - (1 - u_k(\sum_{i=1}^r e_i^n))}{1 - x_i} \ge \frac{1 - u_k(0)}{2^{2n-1}} \tag{2}$$

for $1 \le i \le r$ and $1 \le k \le m$.

Step 2. Let $p = z_0, q_l = -\sum_{i=1}^r e_i^n + \mathbf{i}ke_l^n$ for $1 \le l \le r$ and $k \in \mathbb{R}$. It is clear that $p + tq_l = (1 - t)z_0 + \mathbf{i}kte_l^n$ for $t \in \mathbb{R}$, so we have $\|p + tq_l\|_\infty < 1 \Leftrightarrow |1 - t + \mathbf{i}kt| < 1$ and $|1 - t| < 1 \Leftrightarrow 0 < t < \frac{2}{1+k^2}$. The equivalence relationship means that for a given k when $t \to 0^+, p + tq_l \in D^n \bigcap V$. For $t \to 0^+$, taking the Taylor expansion of $f_j(p + tq_l)$ at $t = 0$, we have

$$f_j(p + tq_l) = (\omega_0)_j + Df_j(z_0)q_l t + \overline{D}f_j(z_0)\overline{q_l} t + O(t^{1+\alpha})$$

where $(\omega_0)_j$ denotes the j-th element of ω_0. By Lemma 2.2,

$$\|f(p + tq_l)\|_\infty = \max_{1 \le j \le N} |f_j(p + tq_l)| \le \frac{4}{\pi} \arctan \|p + tq_l\|_\infty. \tag{3}$$

Considering that

$$\|p + tq_l\|_\infty = |1 - t + \mathbf{i}kt| or |1 - t|,$$

it is easy to derive

$$\arctan |1 - t| = \frac{\pi}{4} - \frac{1}{4}t - \frac{1}{4}t + O(t^{1+\alpha}) = \frac{\pi}{4} - \frac{1}{2}t + O(t^{1+\alpha})$$

and

$$\arctan|1+t(-1+\mathbf{i}k)| = \frac{\pi}{4}+\frac{1}{4}(-1+\mathbf{i}k)t+\frac{1}{4}(-1-\mathbf{i}k)t+O(t^{1+\alpha}) = \frac{\pi}{4}-\frac{1}{2}t+O(t^{1+\alpha}).$$

Thus (3) is equivalent to

$$\max_{1\le j\le N}\left|(\omega_0)_j+Df_j(z_0)q_lt+\overline{D}f_j(z_0)\overline{q_l}t+O(t^{1+\alpha})\right|\le 1-\frac{2}{\pi}t+O(t^{1+\alpha}).$$

Substituting $\omega_0=\sum_{j=1}^{m}e_j^N$, we have

$$\max_{1\le j\le N}\left\{1+2Re(Df(z_0)q_l+\overline{D}f(z_0)\overline{q_l})t+O(t^{1+\alpha})\right\}\le 1-\frac{4}{\pi}t+O(t^{1+\alpha}).$$

Letting $t\to 0^+$, we deduce that

$$\max_{1\le j\le N}\left\{Re(Df(z_0)q_l+\overline{D}f(z_0)\overline{q_l})\right\}\le -\frac{2}{\pi}. \tag{4}$$

Substituting $q_l=-\sum_{i=1}^{r}e_i^n+\mathbf{i}ke_l^n$, we have

$$\max_{1\le j\le N}\left\{Re\left(Df(z_0)\left(-\sum_{i=1}^{r}e_i^n+\mathbf{i}ke_l^n\right)+\overline{D}f(z_0)\left(-\sum_{i=1}^{r}e_i^n-\mathbf{i}ke_l^n\right)\right)\right\}\le -\frac{2}{\pi}$$

which equals to

$$\max_{1\le j\le N}\left\{Re\left(-\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial z_i}+\mathbf{i}k\frac{\partial f_j(z_0)}{\partial z_l}-\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial \overline{z}_i}-\mathbf{i}k\frac{\partial f_j(z_0)}{\partial \overline{z}_l}\right)\right\}\le -\frac{2}{\pi}.$$

Let $\frac{\partial f_j(z_0)}{\partial z_i}=Re\frac{\partial f_j(z_0)}{\partial z_i}+\mathbf{i}Im\frac{\partial f_j(z_0)}{\partial z_i}$ and $\frac{\partial f_j(z_0)}{\partial \overline{z}_i}=Re\frac{\partial f_j(z_0)}{\partial \overline{z}_i}+\mathbf{i}Im\frac{\partial f_j(z_0)}{\partial \overline{z}_i}$. From the above inequality, we have

$$\max_{1\le j\le N}\left\{-Re\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial z_i}-kIm\frac{\partial f_j(z_0)}{\partial z_l}-Re\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial \overline{z}_i}+kIm\frac{\partial f_j(z_0)}{\partial \overline{z}_l}\right\}\le -\frac{2}{\pi},$$

i.e.

$$\min_{1\le j\le N}\left\{Re\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial z_i}+Re\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial \overline{z}_i}+k\left(Im\frac{\partial f_j(z_0)}{\partial z_l}-Im\frac{\partial f_j(z_0)}{\partial \overline{z}_l}\right)\right\}\ge \frac{2}{\pi}. \tag{5}$$

Since (5) is valid for any $k\in\mathbb{R}$, so that

$$Im\frac{\partial f_j(z_0)}{\partial z_l}-Im\frac{\partial f_j(z_0)}{\partial \overline{z}_l}=0, 1\le l\le r, \tag{6}$$

which gives

$$\min_{1\le j\le N}\left\{Re\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial z_i}+Re\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial \overline{z}_i}+\right\}\ge \frac{2}{\pi}, \tag{7}$$

and

$$Re\frac{\partial f_j(z_0)}{\partial z_l}-Re\frac{\partial f_j(z_0)}{\partial \overline{z}_l}=\frac{\partial f_j(z_0)}{\partial z_l}-\frac{\partial f_j(z_0)}{\partial \overline{z}_l}, 1\le l\le r. \tag{8}$$

Step 3. Consider $p = z_0, q_l = -\sum_{i=1}^{r} e_i^n + ke_l^n$ for $1 \le l \le r$ and $k \le 0$. Then $p + tq_l = (1-t)z_0 + kte_l^n$ for $t \in \mathbb{R}$. Hence $\|p + tq_l\|_\infty < 1 \Leftrightarrow |1 - t + kt| < 1$ and $|1 - t| < 1 \Leftrightarrow 0 < t < \frac{2}{1-k}$. The equivalence relationship implies that for any given $k \le 0$ there is $t \to 0^+$ such that $p + tq_l \in D^n \bigcap V$. Taking the Taylor expansion of $f_j(p + tq_l)$ at $t = 0$ and applying Lemma 2.2, we get

$$\max_{1\le j\le N} \left|(\omega_0)_j + Df_j(z_0)q_l t + \overline{D}f_j(z_0)\overline{q_l}t + O(t^{1+\alpha})\right| \le \frac{4}{\pi}\arctan \|p + tq_l\|_\infty.$$

Same to Step 2, it is not difficult to obtain

$$\max_{1\le j\le N} \left\{1 + 2Re(Df(z_0)q_l + \overline{D}f(z_0)\overline{q_l})t + O(t^{1+\alpha})\right\} \le 1 - \frac{4}{\pi}t + O(t^{1+\alpha}).$$

We also substitute $q_l = -\sum_{i=1}^{r} e_i^n + ke_l^n$ and let $t \to 0^+$, then

$$\max_{1\le j\le N} \left\{Re\left(Df(z_0)\left(-\sum_{i=1}^{r} e_i^n + ke_l^n\right) + \overline{D}f(z_0)\left(-\sum_{i=1}^{r} e_i^n - ke_l^n\right)\right)\right\} \le -\frac{2}{\pi}.$$

A straightforward computation shows that

$$\max_{1\le j\le N} \left\{Re\left(-\sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial z_i} + k\frac{\partial f_j(z_0)}{\partial z_l} - \sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial \overline{z}_i} - k\frac{\partial f_j(z_0)}{\partial \overline{z}_l}\right)\right\} \le -\frac{2}{\pi},$$

which is equivalent to

$$\min_{1\le j\le N} \left\{Re\sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial z_i} + Re\sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial \overline{z}_i} - k\left(Re\frac{\partial f_j(z_0)}{\partial z_l} - Re\frac{\partial f_j(z_0)}{\partial \overline{z}_l}\right)\right\} \ge \frac{2}{\pi}.$$

Since (7) and (8) is valid for $k \in \mathbb{R}$, we get

$$\min_{1\le j\le N} -k\left(\frac{\partial f_j(z_0)}{\partial z_l} - \frac{\partial f_j(z_0)}{\partial \overline{z}_l}\right) \ge 0$$

for $k \le 0$. We further derive

$$\frac{\partial f_j(z_0)}{\partial z_l} \ge \frac{\partial f_j(z_0)}{\partial \overline{z}_l}, 1 \le l \le r, 1 \le j \le N \tag{9}$$

since $k \le 0$ is arbitrary.

Step 4. Let $p = z_0, q_l = -\sum_{i=1}^{r} e_i^n + \mathbf{i}ke_l^n$ for $r + 1 \le l \le n$ and $k \ne 0 \in \mathbb{R}$. Then $p + tq_l = (1 - t)\sum_{i=1}^{r} e_i^n + \mathbf{i}kte_l^n$ for $t \in \mathbb{R}$. It is not difficult to verify that $\|p + tq_l\|_\infty < 1 \Leftrightarrow |1 - t| < 1$ and $|\mathbf{i}kt|^2 < 1 \Leftrightarrow 0 < t < \min\{\frac{1}{k^2}, 2\}$. Therefore, given a $k \ne 0 \in \mathbb{R}$, there exists $t \to 0^+$ such that $p + tq_l \in D^n \bigcap V$. Then taking the Taylor expansion of $f_j(p + tq_l)$ at $t = 0$, we can derive

$$\max_{1\le j\le N} \left|(\omega_0)_j + Df_j(z_0)q_l t + \overline{D}f_j(z_0)\overline{q_l}t + O(t^{1+\alpha})\right| \le \frac{4}{\pi}\arctan \|p + tq_l\|_\infty,$$

from which it is obvious that

$$\max_{1\le j\le N} \left\{1 + 2Re(Df(z_0)q_l + \overline{D}f(z_0)\overline{q_l})t + O(t^{1+\alpha})\right\} \le 1 - \frac{4}{\pi}t + O(t^{1+\alpha}).$$

Substituting $q_l = -\sum_{i=1}^{r} e_i^n + \mathbf{i}ke_l^n$ and letting $t \to 0^+$, we get

$$\max_{1\le j\le N} \left\{Re\left(Df(z_0)\left(-\sum_{i=1}^{r} e_i^n + \mathbf{i}ke_l^n\right) + \overline{D}f(z_0)\left(-\sum_{i=1}^{r} e_i^n - \mathbf{i}ke_l^n\right)\right)\right\} \le -\frac{2}{\pi},$$

i.e.

$$\max_{1\le j\le N}\left\{Re\left(-\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial z_i}+\mathbf{i}k\frac{\partial f_j(z_0)}{\partial z_l}-\sum_{i=1}^{r}\frac{\partial f_j(z_0)}{\partial \bar{z}_i}-\mathbf{i}k\frac{\partial f_j(z_0)}{\partial \bar{z}_l}\right)\right\}\le -\frac{2}{\pi}.$$

Reviewing that $\frac{\partial f_j(z_0)}{\partial z_i}=Re\frac{\partial f_j(z_0)}{\partial z_i}+\mathbf{i}Im\frac{\partial f_j(z_0)}{\partial z_i}$ and $\frac{\partial f_j(z_0)}{\partial \bar{z}_i}=Re\frac{\partial f_j(z_0)}{\partial \bar{z}_i}+\mathbf{i}Im\frac{\partial f_j(z_0)}{\partial \bar{z}_i}$, and exploiting (7), it is not difficult to obtain

$$\max_{1\le j\le N}k\left(Im\frac{\partial f_j(z_0)}{\partial \bar{z}_l}-Im\frac{\partial f_j(z_0)}{\partial z_l}\right)\le 0, r+1\le l\le n.$$

Since the above equality is valid for $k\neq 0\in\mathbb{R}$, with the similar argument to Step 2, we have

$$Im\frac{\partial f_j(z_0)}{\partial \bar{z}_l}-Im\frac{\partial f_j(z_0)}{\partial z_l}=0, r+1\le l\le n. \tag{10}$$

Step 5. Let $p=z_0, q_l=-\sum_{i=1}^{r}e_i^n+ke_l^n$ for $r+1\le l\le n$ and any $k\neq 0\in\mathbb{R}$. Then $p+tq_l=(1-t)\sum_{i=1}^{r}e_i^n+kte_l^n$ for $t\in\mathbb{R}$. It is not difficult to verify that $\|p+tq_l\|_\infty<1\Leftrightarrow|1-t|<1$ and $|kt|^2<1\Leftrightarrow 0<t<\min\{\frac{1}{k^2},2\}$. Therefore, given a $k\neq 0\in\mathbb{R}$, there exists $t\to 0^+$ such that $p+tq_l\in D^n\bigcap V$. With the similar argument to Step 4, it is not difficult to obtain

$$Re\sum_{i=1}^{m}\frac{\partial f_j(z_0)}{\partial \bar{z}_l}+Re\sum_{i=1}^{m}\frac{\partial f_j(z_0)}{\partial z_l}=0, r+1\le l\le n. \tag{11}$$

Review the formulas $f_j=u_j+\mathbf{i}v_j$ and $z_i=x_i+\mathbf{i}y_i$ for $1\le i\le n, 1\le j\le N$. Considering that $\frac{\partial}{\partial z_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-\mathbf{i}\frac{\partial}{\partial y_i}\right)$ and $\frac{\partial}{\partial \bar{z}_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}+\mathbf{i}\frac{\partial}{\partial y_i}\right)$, we can derive the following results for any $1\le i\le n$, $1\le j\le N$:

$$\frac{\partial f_j}{\partial z_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-\mathbf{i}\frac{\partial}{\partial y_i}\right)\left(u_j+\mathbf{i}v_j\right)=\frac{1}{2}\left[\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial v_j}{\partial y_i}\right)+\mathbf{i}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial u_j}{\partial y_i}\right)\right],$$

$$\frac{\partial f_j}{\partial \bar{z}_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}+\mathbf{i}\frac{\partial}{\partial y_i}\right)\left(u_j+\mathbf{i}v_j\right)=\frac{1}{2}\left[\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial v_j}{\partial y_i}\right)+\mathbf{i}\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial u_j}{\partial y_i}\right)\right].$$

In view of (2) and (6), it is obvious that for any $1\le j\le m$ we have

$$\frac{\partial u_j}{\partial y_i}=0, \frac{\partial u_j}{\partial x_i}\ge\frac{1-u_j(0)}{2^{2n-1}}, 1\le i\le r. \tag{12}$$

Similarly, it follows from (10) and (11) that

$$\frac{\partial u_j}{\partial y_i}=0, \frac{\partial u_j}{\partial x_i}=0, r+1\le i\le n. \tag{13}$$

Rewrite $z=(z_1,\cdots,z_n)^T\in\mathbb{C}^n$ by $z'=(x_1,y_1,\cdots,x_n,y_n)^T\in\mathbb{R}^{2n}$, then $z_0'=(e_1^n)'+(e_3^n)'+\cdots+(e_{2r-1}^n)'=(1,0,\cdots,1,0,\cdots,0)^T\in\mathbb{R}^{2n}$ where $(e_3^n)'$ represents the i-th column of identity matrix I_{2n}. According to (12) and (13), it is concluded that

$$J_f(z_0')^T\omega_0'=diag(\lambda_1,0,\cdots,\lambda_r,0,\cdots,0)z_0'$$

where $\omega_0'=(e_1^N)'+(e_3^N)'+\cdots+(e_{2m-1}^N)'$ and $\lambda_i=\sum_{j=1}^{m}\frac{\partial u_j}{\partial x_i}$ with $\frac{\partial u_j}{\partial x_i}\ge\frac{1-u_j(0)}{2^{2n-1}}$ for $1\le i\le r$.

Step 6. Let z_0' be any given point at $\partial D^{2n} \subset \mathbb{R}^{2n}$. That is, z_0' is not necessary $(e_1^n)' + (e_3^n)' + \cdots + (e_{2r-1}^n)'$. Then there exists a special kind of real-valued diagonal unitary matrix $U_{z_0'}$ such that $U_{z_0'}(z_0') = (e_1^n)' + (e_3^n)' + \cdots + (e_{2r-1}^n)' = z'_{nor}$ referring to [7]. Assume f' is the real version of f, and $f'(z_0') = \omega_0'$ where ω_0' is not necessary $(e_1^N)' + (e_3^N)' + \cdots + (e_{2m-1}^N)'$ at ∂D^{2n}. In the same way, there exists a real-valued diagonal unitary matrix $U_{\omega_0'}$ such that $U_{\omega_0'}(\omega_0') = (e_1^N)' + (e_3^N)' + \cdots + (e_{2m-1}^N)' = \omega'_{nor}$. Denote

$$g'(z') = U_{\omega_0'} \circ f' \circ U_{z_0'}^T(z'), z' \in D^{2n}$$

and

$$g(z) = U_{\omega_0} \circ f \circ U_{z_0}^T(z), z \in D^n$$

where U_{z_0} and U_{ω_0} represent complex unitary matrices corresponding to $U_{z_0'}$ and $U_{\omega_0'}$ such that $U_{z_0}(z_0) = \sum_{i=1}^r e_i^n$ and $U_{\omega_0}(\omega_0) = \sum_{j=1}^m e_j^N$. It is easy to verify that g' is the real version of g and $g(\sum_{i=1}^r e_i^n) = \sum_{j=1}^m e_j^N$. Furthermore, the Jacobian matrix of g could be denoted as

$$J_g(z') = U_{\omega_0'} J_f\left(U_{z_0'}^T(z')\right) U_{z_0'}^T(z'), z' \in D^{2n}. \tag{14}$$

According to Step 5, we have

$$J_g\left(z'_{nor}\right)^T \omega'_{nor} = diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0) z'_{nor}$$

where $\lambda_i = \sum_{j=1}^m \frac{\partial u_j}{\partial x_i}$ with $\frac{\partial u_j}{\partial x_i} \geq \frac{1-u_j(0)}{2^{2n-1}}$ for $1 \leq i \leq r$. As a result, we obtain

$$\left(U_{\omega_0'} J_f\left(U_{z_0'}^T(z'_{nor})\right) U_{z_0'}^T\right)^T \omega'_{nor} = diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0) z'_{nor}$$

which equals to

$$U_{z_0'} J_f\left(z_0'\right)^T U_{\omega_0'}^T \omega'_{nor} = diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0) z'_{nor}.$$

Multiplying $U_{z_0'}^T$ at both sides of the above equation gives

$$J_f^T\left(z_0'\right) \omega_0' = diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0) z_0'$$

where $\lambda_i = \sum_{j=1}^m \frac{\partial u_j}{\partial x_i}$ with $\frac{\partial u_j}{\partial x_i} \geq \frac{1-u_j(0)}{2^{2n-1}}$ for $1 \leq i \leq r$.

2) According to (7), it is not difficult to obtain

$$\sum_{i=1}^r \sum_{j=1}^m \frac{\partial u_j}{\partial x_i} \geq \frac{2}{\pi}.$$

Since $tr\left(diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)\right) = \sum_{i=1}^r \lambda_i$ and $\lambda_i = \sum_{j=1}^m \frac{\partial u_j}{\partial x_i}$, it is obvious that

$$tr\left(diag(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)\right) \geq \frac{2}{\pi}.$$

The proof of Theorem 1.6 is finished. □

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