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On Automorphisms of Graded Quasi-Lie Algebras

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Abstract. Let $\mathbb{Z}$ be the ring of integers and let $K(\mathbb{Z}, 2n)$ denote the Eilenberg-MacLane space of type $(\mathbb{Z}, 2n)$ for $n \geq 1$. In this article, we prove that the graded group

$$A_m := \mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}, 2n))/\mathrm{torsions})$$

of automorphisms of the graded quasi-Lie algebras $\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}, 2n))$ modulo torsions that preserve the Whitehead products is a finite group for $m \leq 2$ and an infinite group for $m \geq 3$, and that the group $\mathrm{Aut}(\pi_*(\Sigma K(\mathbb{Z}, 2n))/\mathrm{torsions})$ is non-abelian. We extend and apply those results to techniques in localization (or rationalization) theory.

1. Introduction

1.1. The Functors Σ and Ω

As usual, we write Σ (resp. Ω) for the suspension (resp. loop) functor from the pointed homotopy category of pointed topological spaces and base point preserving continuous maps to itself. It is well known that the suspension and loop functors are examples of adjoint functors. Moreover, the co-Hopf spaces and the Hopf spaces are important topological objects of study in classical homotopy theory, equivariant homotopty theory and equivariant (co)homology theories, and they are the Eckmann-Hilton dual notions; see [2], [3], [4], [17], [15] and [16] for a survey of co-Hopf spaces and Hopf spaces.

1.2. The Group of Self-Homotopy Equivalences

Many of the sets that we encounter in classical homotopy classification problems have a natural group structure such as $[X, \Omega Y]$ or equivalently $[\Sigma X, Y]$. Other examples are furnished by the group $\mathrm{Aut}(X)$ as follows. For a pointed space X, we let $\mathrm{Aut}(X)$ be the set of homotopy classes of base point preserving self-maps of X that are self-homotopy equivalences as a geometric version of the group of automorphisms of a group. This set becomes a group, called the *group of self-homotopy equivalences* of X, whose operation is induced by the composition of maps. We note that $\mathrm{Aut}(X)$ is a homotopy invariant; that is, if $X \simeq Y$, then

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$\mathrm{Aut}(X) \cong \mathrm{Aut}(Y)$. However, $\mathrm{Aut}(X)$ does not have the functorial property since the maps $f : X \to Y$ do not induce homomorphisms between $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$. We refer to the works of Arkowitz [1] and Rutter [22] for a survey of $\mathrm{Aut}(X)$ and its related topics.

1.3. Motivation

McGibbon and Møller [19] constructed an example showing that even though two spaces X and Y have the same n-type for all n, it is possible that one group, $\mathrm{Aut}(X)$, is infinite, while the other, $\mathrm{Aut}(Y)$, is finite. They also showed in [20, Theorem 1] that the set of homotopy classes of same n-types of X is trivial if and only if the map

$$h : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\pi_{\leq n}(X)) \ (f \mapsto f_\sharp) \tag{1}$$

has a finite cokernel for each n, where X is a 1-connected space of finite type over the some subring of the rationals with rational homotopy type of a bouquet of spheres. We note that the previous results on the same n-types of CW-complexes connect the size of the cokernel of h on the groups $\mathrm{Aut}(\pi_{\leq n}(X))$ with the set of homotopy classes of same n-types; see [11], [13] and [14] for the same n-type structures of CW-spaces. It raises the following querry: How can we calculate the groups $\mathrm{Aut}(\pi_{\leq n}(X))$ for each n? In this paper, we are mostly interested in the size of the graded group $\mathrm{Aut}(\pi_{\leq n}(X))$ in itself instead of the cardinality of the cokernel of h when X is the suspension of the Eilenberg-MacLane spaces of type $(\mathbb{Z}, 2n)$ for $n \geq 1$ and their localizations (or rationalizations).

1.4. The Group of Automorphisms Based on the Eilenberg-MacLane Spaces

Let $\mathbb{Z}$ be the ring of integers and let $K(\mathbb{Z}, 2n)$ denote the Eilenberg-MacLane space of type $(\mathbb{Z}, 2n)$ for $n \geq 1$. As an algebraic counterpart of the group of self-homotopy equivalences, we let

$$A_m := \mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}, 2n))) \tag{2}$$

be the group of automorphisms of the graded $\mathbb{Z}$-module

$$\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}, 2n)) = \pi_*((\Sigma K(\mathbb{Z}, 2n))^{(2mn+1)}), \tag{3}$$

that preserves the Whitehead products on the CW-space $\Sigma K(\mathbb{Z}, 2n)$ for $n \geq 1$. Here, $X^{(j)}$ is the jth Postnikov approximation of X and m is a positive integer. It is well known that the infinite complex projective space $\mathbb{C}P^\infty$ is a nice example of the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ and that

$$BT^1 = BS^1 = BU(1). \tag{4}$$

The Eilenberg-MacLane spaces associated to torsion free groups are infinite dimensional and have a lot of torsions in their homology groups as the homotopy dimension increases (see [6, Théorème 2]), and after suspension, they become much more homotopically intractable spaces than the original spaces which we deal with in this paper.

1.5. The Main Goals

In this paper, we prove that the group

$$A_m := \mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}, 2n))/\mathrm{torsions}) \tag{5}$$

of automorphisms of the graded abelian group that preserves the Whitehead product pairings is a finite group for $m \leq 2$ and an infinite group for $m \geq 3$, and we also prove that the group $\mathrm{Aut}(\pi_*(\Sigma K(\mathbb{Z}, 2n))/\mathrm{torsions})$ is a non-abelian group. We extend and apply those results to techniques in localization (or rationalization) theory by considering the indecomposable and decomposable homotopy generators in the graded quasi-Lie algebras.

2. Self-Maps and Description of the Result

For notational convenience, we mostly write K for the Eilenberg-MacLane space $K(\mathbb{Z}, 2n)$ for $n \geq 1$. We let $\hat{\varphi}_1 : K \to \Omega\Sigma K$ and $\hat{\eta}_1 : S^{2n} \to \Omega\Sigma K$ be the canonical inclusion maps; see [21, Theorems 1.1 and 1.4] in the case of $\mathbb{C}P^\infty$, the infinite complex projective space.

Definition 2.1. By using induction on i, we define the maps $\hat{\varphi}_{i+1}$ and $\hat{\eta}_{i+1}$ as the compositions of maps as follows:

$$\hat{\varphi}_{i+1} : K \xrightarrow{\bar{\Delta}} K \wedge K \xrightarrow{\hat{\varphi}_1 \wedge \hat{\varphi}_i} \Omega\Sigma K \wedge \Omega\Sigma K \xrightarrow{\sharp} \Omega\Sigma K \tag{6}$$

and

$$\hat{\eta}_{i+1} : S^{2n+2in} = S^{2n} \wedge S^{2in} \xrightarrow{\hat{\eta}_1 \wedge \hat{\eta}_i} \Omega\Sigma K \wedge \Omega\Sigma K \xrightarrow{\sharp} \Omega\Sigma K. \tag{7}$$

Here,

- $\bar{\Delta}$ is the reduced diagonal map (that is, the composite of the diagonal $\Delta : K \to K \times K$ with the projection $p : K \times K \to K \wedge K$ onto the smash product); and
- $\sharp$ denotes the extension of the adjointness of the Hopf construction $H(\mu) : \Sigma(K \wedge K) \to \Sigma K$ assigning to the homotopically unique multiplication $\mu : K \times K \to K$.

Indeed, we can consider the above multiplication $\mu : K \times K \to K$ since $K = K(\mathbb{Z}, 2n)$ is a Hopf space, and then we have the extension

$$\sharp : \Omega\Sigma K \wedge \Omega\Sigma K \to \Omega\Sigma K \tag{8}$$

of the adjoint map $\widetilde{H(\mu)} : K \wedge K \to \Omega\Sigma K$ of $H(\mu)$.

Remark 2.2. We note that $\hat{\varphi}_i : K \to \Omega\Sigma K$ factors as $K \xrightarrow{p_i} K/K_{2in-1} \xrightarrow{\bar{\varphi}_i} \Omega\Sigma K$; that is, the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\hat{\varphi}_i} & \Omega\Sigma K \\ & {\scriptstyle p_i} \searrow \quad \nearrow {\scriptstyle \bar{\varphi}_i} & \\ & K/K_{2in-1} \simeq S^{2in} \cup (\text{other cells}) & \end{array} \tag{9}$$

is commutative up to homotopy such that the restriction of $\bar{\varphi}_i : K/K_{2in-1} \to \Omega\Sigma K$ to the bottom cell is equal to $\hat{\eta}_i : S^{2in} \to \Omega\Sigma K$, where

- K_{2in-1} is the $(2in-1)$th skeleton of K; and
- $p_i : K \to K/K_{2in-1}$ is the projection for each $i = 1, 2, 3, \ldots$.

Definition 2.3. We define the self-maps $\varphi_i : \Sigma K \to \Sigma K$ and maps $\eta_i : S^{2in+1} \to \Sigma K$ by the adjointness of $\hat{\varphi}_i : K \to \Omega\Sigma K$ and $\hat{\eta}_i : S^{2in} \to \Omega\Sigma K$, respectively, for $i = 1, 2, 3, \ldots$.

Let $[\varphi_{i_1}, \varphi_{i_2}] : \Sigma K \to \Sigma K$ be the commutator of φ_{i_1} and φ_{i_2}; that is

$$[\varphi_{i_1}, \varphi_{i_2}] = \varphi_{i_1} + \varphi_{i_2} - \varphi_{i_1} - \varphi_{i_2}, \tag{10}$$

where the addition and subtractions are derived from the suspension on ΣK.

Definition 2.4. We define the self-maps of ΣK by $1 + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \ldots, [\varphi_{i_1}, \varphi_{i_2}] \ldots]]$, where $1 : \Sigma K \to \Sigma K$ is the identity map, and the map $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \ldots, [\varphi_{i_1}, \varphi_{i_2}] \ldots]]$ is the iterated commutator map consisting of self-maps $\varphi_{i_s} : \Sigma K \to \Sigma K$ on ΣK for $s = 1, 2, \ldots, k$.

Since self-maps $1 + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \ldots, [\varphi_{i_1}, \varphi_{i_2}] \ldots]]$ of ΣK induce the identity morphism on the homotopy groups of ΣK, they are actually self-homotopy equivalences according to the classical Whitehead theorem. We observe that the total graded rational homotopy group

$$\hat{\mathcal{L}} := \pi_*(\Omega\Sigma K) \otimes \mathbb{Q} \tag{11}$$

of $\Omega\Sigma K$ comes with a graded Lie algebra over the rationals $\mathbb{Q}$ whose Lie bracket $< \ , \ >$ is given by the Samelson product. The total graded rational homotopy group is said to be the *rational homotopy Lie algebra* of ΣK; see [24, page 470] and [2].

We consider the subalgebra $\hat{\mathcal{L}}_{\leq m}$ of $\hat{\mathcal{L}}$ spanned by all of the free Lie algebra generators whose degrees are less than or equal to $2mn$, i.e.,

$$\hat{\mathcal{L}}_{\leq m} := \pi_{\leq 2mn}(\Omega\Sigma K) \otimes \mathbb{Q} \tag{12}$$

with rational generators $\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_m$. Here, $\hat{z}_i : S^{2in} \to \Omega\Sigma K_{\mathbb{Q}}$ is the composite $r \circ \hat{\eta}_i$ of the indecomposable homotopy class $\hat{\eta}_i : S^{2in} \to \Omega\Sigma K$ of $\pi_{2in}(\Omega\Sigma K)$, $i = 1, 2, \ldots, m$ with the rationalization $r : \Omega\Sigma K \to \Omega\Sigma K_{\mathbb{Q}}$; see [10, pages 24-26].

As adjointness, we also consider the graded homotopy group

$$\mathcal{L}_{\leq m} := \pi_{\leq 2mn+1}(\Sigma K) \otimes \mathbb{Q} \tag{13}$$

with generators $z_i : S^{2in+1} \to \Sigma K_{\mathbb{Q}}$ as the adjointness of $\hat{z}_i : S^{2in} \to \Omega\Sigma K_{\mathbb{Q}}$ for $i = 1, 2, \ldots, m$. Moreover, the graded homotopy group $\mathcal{L}_{\leq m}$ whose bracket is derived from the Whitehead product [,] satisfies the kinds of anticommutativity and the Jacobi identity up to signs. This graded homotopy group is said to be the *graded quasi-Lie algebra*. Indeed, it is not a graded Lie algebra because the bracket does not add degrees and the signs would be different; see [7] and [24, Chapter X, 7].

In classical homotopy theory, the automorphisms

$$\psi : \mathcal{L}_{\leq m} \to \mathcal{L}_{\leq m}, \tag{14}$$

of graded quasi-Lie algebras defined by

$$\psi(z_m) = z_m + \text{decomposables} \tag{15}$$

correspond exactly to our self-homotopy equivalences

$$1 + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \ldots, [\varphi_{i_1}, \varphi_{i_2}] \ldots]] : \Sigma K \to \Sigma K \tag{16}$$

inducing the identity maps on the homology groups with rational coefficients in the automorphism groups $\mathrm{Aut}_*(H_*(\Sigma K) \otimes \mathbb{Q})$; see [5, page 113].

Let

$$\mathcal{L}_{\mathbb{Z}} := \pi_*(\Sigma K(\mathbb{Z}, 2n))/\text{torsions} = \mathbb{Z}(\eta_1, \eta_2, \ldots, \eta_i, \ldots), \tag{17}$$

$$\mathcal{L}_{\mathbb{Z}_{(P)}} := \pi_*(\Sigma K(\mathbb{Z}_{(P)}, 2n))/p\text{-torsions} = \mathbb{Z}_{(P)}(y_1, y_2, \ldots, y_i, \ldots), \tag{18}$$

and

$$\mathcal{L}_{\mathbb{Q}} := \pi_*(\Sigma K(\mathbb{Q}, 2n)) = \mathbb{Q}(z_1, z_2, \ldots, z_i, \ldots) \tag{19}$$

denote the graded quasi-Lie algebras of $\Sigma K(\mathbb{Z}, 2n)$, $\Sigma K(\mathbb{Z}_{(P)}, 2)$ and $\Sigma K(\mathbb{Q}, 2)$, respectively. Here,

- $\eta_i : S^{2in+1} \to \Sigma K(\mathbb{Z}, 2n)$, $y_i : S^{2in+1} \to \Sigma K(\mathbb{Z}_{(P)}, 2n)$, and $z_i : S^{2in+1} \to \Sigma K(\mathbb{Q}, 2n)$ are the corresponding indecomposable generators in the homotopy groups in dimension $2in+1$, $i = 1, 2, 3, \ldots$ (see Section 3);
- the bracket operations are the Whitehead products; and
- P is a nonempty collection of primes with $p \in P$.

Note that $\mathcal{L}_{\mathbb{Q}}$ is equal to the minimal Lie model of $\Sigma K(\mathbb{Q}, 2n)$, and hence is the free Lie algebra (see [23] and [8, Example 1, page 331]) on generators $z_1, z_2, \ldots, z_i, \ldots$ in rational homotopy theory; that is,

$$\mathcal{L}_{\mathbb{Q}} = (\mathbb{L}(z_1, z_2, \ldots, z_i, \ldots), 0) \tag{20}$$

is the free Lie model of $\Sigma K(\mathbb{Q}, 2n)$, where the degree of z_i is $2in$ for $i = 1, 2, \ldots$, and the differential is zero.

We now describe the main result of this paper as follows.

Theorem 2.5. *Let $A_m := Aut(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}, 2n))/torsion)$, and let Π be the collection of all primes. Then we have:*

1. *A_m is a finite abelian group for $m \leq 2$, and an infinite group for $m \geq 3$, and $Aut(\mathcal{L}_{\mathbb{Z}})$ is a non-abelian group.*
2. *If P is a finite collection of primes, then $Aut(\mathcal{L}_{\mathbb{Z}_{(P)}})$ is an infinite non-abelian group.*
3. *If P is an infinite collection of primes with $\Pi - P$ finite, then $Aut(\mathcal{L}_{\mathbb{Z}_{(P)}})$ is a finite non-abelian group.*
4. *If P is an infinite collection of primes with $\Pi - P$ infinite, then $Aut(\mathcal{L}_{\mathbb{Z}_{(P)}})$ is an infinite non-abelian group.*
5. *$Aut(\mathcal{L}_{\mathbb{Q}})$ is an infinite non-abelian group.*

More precisely, we describe the (in)finiteness and (non)abelianness of the above graded groups by providing the tables on a case by case approach in Section 4.

3. Proof of Theorem 2.5

More generally, we let X_j denote the jth skeleton of a pointed CW-space X. Then we obtain the following: If $\alpha, \beta : X \to \Omega Y$ are the morphism classes in the pointed homotopy category of CW-complexes satisfying both $\alpha|_{X_s} \simeq *$ and $\beta|_{X_t} \simeq *$, then the restriction map

$$[\alpha, \beta]\,|_{X_{s+t}} : X_{s+t} \to \Omega Y \tag{21}$$

to the $(s+t)$-skeleton X_{s+t} of the commutator map $[\alpha, \beta]$ is inessential; see [14, Lemma 3.3] for more details.

Proof. Considering the cofibration

$$K_{2in-1} \hookrightarrow K \to K/K_{2in-1} \tag{22}$$

and the construction of the maps

$$\hat{\varphi}_i : K \to \Omega\Sigma K \tag{23}$$

for $i = 1, 2, \ldots$, we see that the restriction of $\hat{\varphi}_i$ to the $(2in-1)$-skeleton of K is homotopic to the constant map for each $i = 1, 2, 3, \ldots$, and that by using adjointness the restriction

$$[\varphi_{i_1}, \varphi_{i_2}]|_{(\Sigma K)_{2(i_1+i_2)n}} : (\Sigma K)_{2(i_1+i_2)n} \to \Sigma K \tag{24}$$

to the skeleton is inessential for all positive integers i_1 and i_2.

We note that if X and Y are 1-connected CW-spaces, then they admit a P-localization (see [10] and [18]), and thus the above statements are still guaranteed for the localized version because of the functorial properties of the localizations of those spaces and maps.

We now develop an additive relation between the operations of the suspension structures and those of the homotopy groups as follows; see [12, Lemma 7] for more details. For a given iterated Whitehead product of the following type

$$[\eta_{i_k}, [\eta_{i_{k-1}}, \ldots, [\eta_{i_1}, \eta_{i_2}] \ldots]] \in \pi_*(\Sigma K), \tag{25}$$

we can always consider the homotopy class of the iterated commutator map

$$[\varphi_{i_k}, [\varphi_{i_{k-1}}, \dots, [\varphi_{i_1}, \varphi_{i_2}] \dots]] \in [\Sigma K, \Sigma K] \tag{26}$$

satisfying

$$(1 + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \dots, [\varphi_{i_1}, \varphi_{i_2}] \dots]])_\sharp(\eta_m) = \eta_m + r[\eta_{i_k}, [\eta_{i_{k-1}}, \dots, [\eta_{i_1}, \eta_{i_2}] \dots]]. \tag{27}$$

Here,

- $r \neq 0$;
- η_m and η_{i_j} are indecomposable homotopy classes for $j = 1, 2, \dots, k$;
- $m = i_1 + i_2 + \cdots + i_k$;
- the '+' on the left-hand side of the equation (27) is derived from the suspension on ΣK, while the '+' on the right-hand side denotes the homotopy addition on the homotopy groups; and
- $(1 + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \dots, [\varphi_{i_1}, \varphi_{i_2}] \dots]])_\sharp$ denotes the induced homomorphism between homotopy groups.

Considering the homotopy groups of ΣK, we see that the groups A_1 and A_2 are finite; see below. However, $A_m := \mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K)/\text{torsions})$ for $m \geq 3$, $\mathrm{Aut}(\mathcal{L}_{\mathbb{Z}})$, $\mathrm{Aut}(\mathcal{L}_{\mathbb{Z}_{(P)}})$ and $\mathrm{Aut}(\mathcal{L}_{\mathbb{Q}})$ have the different algebraic structures to which Theorem 2.5 refers.

We also note that ΣK is the rational homotopy equivalent to the infinitely many wedge product of spheres; that is,

$$\Sigma K \simeq_0 S^{2n+1} \vee S^{4n+1} \vee S^{6n+1} \vee S^{8n+1} \vee \cdots \vee S^{2in+1} \vee \cdots, \tag{28}$$

where $i = 1, 2, 3, \dots$. The Hilton's theorem [9] says that there exist many kinds of decomposable generators as the basic Whitehead products and only one type of rationally non-trivial indecomposable generators in each of its dimensions on the graded homotopy groups $\pi_*(\Sigma K)/\text{torsions}$, $\pi_*(\Sigma K) \otimes \mathbb{Z}_{(P)}$ and $\pi_*(\Sigma K) \otimes \mathbb{Q}$, as follows:

- η_1, y_1, z_1 in dimension $2n + 1$
- η_2, y_2, z_2 in dimension $4n + 1$
- $\eta_3, y_3, z_3, [\eta_1, \eta_2], [y_1, y_2], [z_1, z_2]$ in dimension $6n + 1$
- $\eta_4, y_4, z_4, [\eta_1, \eta_3], [\eta_1, [\eta_1, \eta_2]],$
  $[y_1, y_3], [y_1, [y_1, y_2]], [z_1, z_3], [z_1, [z_1, z_2]]$ in dimension $8n + 1$
- $\eta_5, y_5, z_5,$
  $[\eta_1, \eta_4], [\eta_1, [\eta_1, \eta_3]], [\eta_1, [\eta_1[\eta_1, \eta_2]]], [\eta_2, \eta_3], [\eta_2, [\eta_1, \eta_2]],$
  $[y_1, y_4], [y_1, [y_1, y_3]], [y_1, [y_1[y_1, y_2]]], [y_2, y_3], [y_2, [y_1, y_2]],$
  $[z_1, z_4], [z_1, [z_1, z_3]], [z_1, [z_1[z_1, z_2]]], [z_2, z_3], [z_2, [z_1, z_2]]$ in dimension $10n + 1$

and so on.

We also observe that all the basic Whitehead products described above are the rationally non-trivial homotopy classes.

(1) Since there exists only one indecomposable generator in dimension $2n + 1$ and $4n + 1$, we have

$$A_1 \cong \{-1, +1\} \tag{29}$$

and

$$A_2 \cong \{-1, +1\} \oplus \{-1, +1\}, \tag{30}$$

where $\{-1, +1\}$ is the group of units of $\mathbb{Z}$.

We now turn to the infiniteness of homomorphisms of the graded homotopy groups induced by the self-homotopy equivalences. If we consider the self-homotopy class $\Phi = I + [\varphi_1, \varphi_2]$ of ΣK, then the homomorphism $\Phi_\sharp = (I + [\varphi_1, \varphi_2])_\sharp \in A_3$ induced by the self-homotopy equivalences sends $\eta_i \mapsto \eta_i$ for $i = 1, 2$, and $\eta_3 \mapsto \eta_3 + r[\eta_1, \eta_2]$, where $r \neq 0$. Moreover, $\Phi_\sharp^2$ carries $\eta_i \mapsto \eta_i$ for $i = 1, 2$, and $\eta_3 \mapsto \eta_3 + r[\eta_1, \eta_2] + r[\eta_1, \eta_2] + 0 = \eta_3 + 2r[\eta_1, \eta_2]$. In general, the map $\Phi_\sharp^s$ carries $\eta_3 \mapsto \eta_3 + sr[\eta_1, \eta_2]$ for each $s = 1, 2, 3, \ldots$. Therefore, the map $\Phi_\sharp$ induced by the self-map

$$I + [\varphi_1, \varphi_2] : \Sigma K \to \Sigma K \tag{31}$$

has infinite order in A_m for $m \geq 3$.

To show that $\mathrm{Aut}(L_{\mathbb{Z}})$ is a non-abelian group, we consider the following maps

$$\Gamma_\sharp = \begin{cases} (1 + [\varphi_1, \varphi_{m-1}])_\sharp & \text{for } \dim \leq 2mn + 1, \\ 1_\sharp & \text{otherwise} \end{cases} \tag{32}$$

and

$$\Lambda_\sharp = \begin{cases} (1 + [\varphi_1, \varphi_m])_\sharp & \text{for } \dim \leq 2(m+1)n + 1, \\ 1_\sharp & \text{otherwise} \end{cases} \tag{33}$$

in the graded quasi-Lie algebras induced by the self-maps Γ and Λ of ΣK, respectively. Then we have

$$\Gamma_\sharp(\eta_i) = \begin{cases} \eta_i & \text{for } i \neq m, \\ \eta_m + r[\eta_1, \eta_{m-1}] & \text{for } i = m \end{cases} \tag{34}$$

and

$$\Lambda_\sharp(\eta_i) = \begin{cases} \eta_i & \text{for } i \neq m+1, \\ \eta_{m+1} + t[\eta_1, \eta_m] & \text{for } i = m+1, \end{cases} \tag{35}$$

where r and t are both nonzero. We observe that if $m \geq 3$, then the generators η_{m+2} above already vanished in

$$A_{m+1} = \mathrm{Aut}(\pi_{\leq 2(m+1)n+1}(\Sigma K)/\text{torsions}) = \mathrm{Aut}(\pi_*(\Sigma K^{(2(m+1)n+1)})/\text{torsions}), \tag{36}$$

where $X^{(j)}$ is the jth Postnikov section of X. Therefore, we obtain

$$\Lambda_\sharp \circ \Gamma_\sharp(\eta_{m+1}) = \Lambda_\sharp(\eta_{m+1}) = \eta_{m+1} + t[\eta_1, \eta_m] \tag{37}$$

and

$$\begin{aligned} \Gamma_\sharp \circ \Lambda_\sharp(\eta_{m+1}) &= \Gamma_\sharp(\eta_{m+1} + t[\eta_1, \eta_m]) \\ &= \Gamma_\sharp(\eta_{m+1}) + t\Gamma_\sharp([\eta_1, \eta_m]) \\ &= \Gamma_\sharp(\eta_{m+1}) + t[\Gamma_\sharp(\eta_1), \Gamma_\sharp(\eta_m)]) \\ &= \eta_{m+1} + t[\eta_1, \eta_m + r[\eta_1, \eta_{m-1}]] \\ &= \eta_{m+1} + t[\eta_1, \eta_m] + rt[\eta_1, [\eta_1, \eta_{m-1}]]. \end{aligned} \tag{38}$$

We note that the Whitehead products above are all basic Whitehead products which are rationally non-trivial. Therefore, by the equations (37) and (38), we have

$$\Gamma_\sharp \circ \Lambda_\sharp \neq \Lambda_\sharp \circ \Gamma_\sharp, \tag{39}$$

as required.

In a similar vein, it can be shown that $\mathrm{Aut}(\mathcal{L}_{\mathbb{Z}_{(P)}})$ and $\mathrm{Aut}(\mathcal{L}_{\mathbb{Q}})$ are non-abelian by substituting the generators η_i by y_i and z_i, respectively.

(2) We calculate the group $\mathrm{Aut}(\mathcal{L}_{\mathbb{Z}_{(P)}})$ of automorphisms as follows:

- $\mathrm{Aut}(\pi_{\leq 2n+1}(\Sigma K(\mathbb{Z}_{(P)}, 2))/\mathrm{torsion}) \cong \mathfrak{U}$;
- $\mathrm{Aut}(\pi_{\leq 4n+1}(\Sigma K(\mathbb{Z}_{(P)}, 2))/\mathrm{torsion}) \cong \mathfrak{U} \oplus \mathfrak{U}$;
- $\mathrm{Aut}(\pi_{\leq 6n+1}(\Sigma K(\mathbb{Z}_{(P)}, 2))/\mathrm{torsion}) \cong \mathfrak{U} \oplus \mathfrak{U} \oplus \mathrm{GL}(2, \mathbb{Z}_{(P)})$;

$\vdots$

- $\mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}_{(P)}, 2))/\mathrm{torsion})$ is an infinite non-abelian group for all $m \geq 3$,

where $\mathfrak{U}$ denotes the group of multiplicative units in $\mathbb{Z}_{(P)}$. Therefore, if P is a finite collection of primes, then it can be shown that $\mathrm{Aut}(\mathcal{L}_{\mathbb{Z}_{(P)}})$ is an infinite non-abelian group.

(3) and (4) Similarly, they can be obtained from (2).

(5) We finally have

- $\mathrm{Aut}(\pi_{\leq 2n+1}(\Sigma K(\mathbb{Q}, 2))) \cong (\mathbb{Q} - \{0\})$;
- $\mathrm{Aut}(\pi_{\leq 4n+1}(\Sigma K(\mathbb{Q}, 2))) \cong (\mathbb{Q} - \{0\}) \oplus (\mathbb{Q} - \{0\})$;
- $\mathrm{Aut}(\pi_{\leq 6n+1}(\Sigma K(\mathbb{Q}, 2))) \cong (\mathbb{Q} - \{0\}) \oplus (\mathbb{Q} - \{0\}) \oplus \mathrm{GL}(2, \mathbb{Q})$;

$\vdots$

- $\mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Q}, 2))/\mathrm{torsion})$ is an infinite non-abelian group for all $m \geq 3$,

as required. □

4. Summary

More precisely, we summarize our main results as follows:

Let

$$A_{m,\mathbb{Z}_{(P)}} := \mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Z}_{(P)}, 2n))/\mathrm{torsion}) \tag{40}$$

and

$$A_{m,\mathbb{Q}} := \mathrm{Aut}(\pi_{\leq 2mn+1}(\Sigma K(\mathbb{Q}, 2n))). \tag{41}$$

Then we have the following tables:

m	(in)finiteness	(non)abelianness
1	finite	abelian
2	finite	abelian
$m \geq 3$	infinite	non-abelian

Table 1: In the case of A_m

m	(in)finiteness	(non)abelianness
1	infinite	abelian
2	infinite	abelian
$m \geq 3$	infinite	non-abelian

Table 2: In the case of $A_{m,\mathbb{Z}_{(P)}}$ with P finite

m	(in)finiteness	(non)abelianness
1	finite	abelian
2	finite	abelian
$m \geq 3$	infinite	non-abelian

Table 3: In the case of $A_{m,\mathbb{Z}_{(P)}}$ with $\Pi - P$ finite

m	(in)finiteness	(non)abelianness
1	infinite	abelian
2	infinite	abelian
$m \geq 3$	infinite	non-abelian

Table 4: In the case of $A_{m,\mathbb{Z}_{(P)}}$ with $\Pi - P$ infinite

m	(in)finiteness	(non)abelianness
1	infinite	abelian
2	infinite	abelian
$m \geq 3$	infinite	non-abelian

Table 5: In the case of $A_{m,\mathbb{Q}}$

5. Final Remarks

We only considered the automorphisms of the graded quasi-Lie algebras based on the suspension of the Eilenberg-MacLane spaces of type $(\mathbb{Z}, 2n)$ for $n \geq 1$. This raises the following query: What will happen in the case of $\Sigma K(\mathbb{Z}, 2n+1)$, the suspension of the Eilenberg-MacLane space of type $(\mathbb{Z}, 2n+1)$, for $n \geq 1$? It is not difficult to obtain the corresponding results because

$$\Sigma K(\mathbb{Z}, 2n+1) \simeq_0 S^{2n+2}; \tag{42}$$

that is, the two CW-spaces have the same rational homotopy type. Thus, there exists exactly one indecomposable homotopy generator in the graded homotopy group modulo torsions in dimension $2n + 2$.

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