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Hyperbolicity of the Complement of Arrangements of Non Complex Lines

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Abstract. The goal of this paper is twofold. We study holomorphic curves $f : \mathbb{C} \longrightarrow \mathbb{C}^3$ avoiding four complex hyperplanes and a real subspace of real dimension five in $\mathbb{C}^3$ where we study the cases where the projection of f into the complex projective space $\mathbb{CP}^2$ is constant. On the other hand, we investigate the kobayashi hyperbolicity of the complement of five perturbed lines in $\mathbb{CP}^2$.

Introduction

The concept of Kobayashi and Brody hyperbolicity has been studied by several authors. A complex monifold M is said to be Brody hyperbolic if there is no non-constant holomorphic map $f : \mathbb{C} \to M$. Using his famous reparametrisation theorem (see [3]), Brody proved the following Theorem:

Theorem 0.1. *Let M be a compact complex monifold. Then M is Kobayashi hyperbolic if and only if M is Brody hyperbolic.*

We know many examples of compact complex manifolds M that are hyperbolic according to Kobayashi, and then, have the prorerty that each holomorphic curve $f : \mathbb{C} \to M$ is constant. Since Bloch and Cartan, the hyperbolicity of the complement of arrangements of projective lines in the complex projective plane has been the subject of numerous studies for many years. Several researchers obtained different results for some special cases, especially the next Theorem due to Borel, stated by Cartan in the following form (see [4] and also [2]):

Theorem 0.2 (Borel Theorem). *Let L_1, L_2, L_3 and L_4 be four projective lines in the general position in $\mathbb{CP}^2$, that is, such that the configuration $C = L_1 \cup L_2 \cup L_3 \cup L_4$ does not have a triple point. Let us note Δ the union of the three diagonals Δ_1, Δ_2 and Δ_3 of the configuration C, that is to say the projective lines passing each through double points of C, (there is six double points). Then*
Any non-constant entier curve with values in $\mathbb{CP}^2 \setminus C$ degenerates in Δ. (i.e There exists $i \in \{1, 2, 3\}$ such that $f(\mathbb{C}) \subset \Delta_i$.)

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Thus, via R. Brody's reparameterization Theorem (see [3]), M. Green was able to deduce from Borel's theorem the hyperbolically embedded character of the complement of five lines in the projective plane. He prove the next:

Theorem 0.3. *Let $H_1, H_2,..., H_{2n+1}$ be $2n+1$ hyperplanes in general position in $\mathbb{CP}^n$. Then any holomorphic curves $f : \mathbb{C} \to \mathbb{CP}^n \setminus \cup_{i=1}^{2n+1} H_i$ is constant.*

As a direct consequence of the Green Theorem, the canonical projection into the complex projective space $\mathbb{CP}^2$ of any holomorphic map $f : \mathbb{C} \to \mathbb{C}^3$ which avoids five complex hyperplanes in $\mathbb{C}^3$ is constant, since its image avoids the projections of the five complex hyperplanes, which are complex projective lines in general position in $\mathbb{CP}^2$ (see Lemma 2.2). Our first main goal is to study the projection into $\mathbb{CP}^2$ of a holomorphic curve $f : \mathbb{C} \to \mathbb{C}^3$ which avoids four complex hyperplanes in general position in $\mathbb{C}^3$ and a real subspace H of real dimension five (we change the nature of one complex hyperplane) and we check the case where the projection remains constant. We show the following

Theorem A. *Let H_1, H_2, H_3, H_4 be four complex hyperplanes in $\mathbb{C}^3$ and let H be a real subspace of $\mathbb{R}^6$ of real dimension five. Let $\tilde{H}$ be a complex hyperplane of $\mathbb{C}^3$ such that $\tilde{H} \subset H$. Then:*
(1) *If $(\tilde{H}, H_j, H_k)$ are in general position for all $j \neq k,\ \ j,k \in \{1, ..., 4\}$, then every holomorphic map $f : \mathbb{C} \to \mathbb{C}^3$ such that $f(\mathbb{C}) \cap (\cup_{i=1}^4 H_i \cup H) = \emptyset$ is constant.*
(2) *If there exists $H_j, H_k,\ j \neq k,\ j,k \in \{1, ..., 4\}$, such that $(\tilde{H}, H_j, H_k)$ are not in general position, then there exists $f : \mathbb{C} \to \mathbb{C}^3$, holomorphic, such that $f(\mathbb{C}) \cap \left(\cup_{i=1}^4 H_i \cup H \right) = \emptyset$ and $\pi(f)$ is non constant.*

Remark 0.4.
(*a*) *The existence and uniqueness of $\tilde{H} \subset H$ is explained in the proof of Theorem A.*
(*b*) *The condition "$(\tilde{H}, H_j, H_k)$ are not in general position" is equivalent to the condition "$\dim_{\mathbb{R}} Span_{\mathbb{R}}(\tilde{H}^{\perp}, H_j^{\perp}, H_k^{\perp}) = 4$".*
(*c*) *The fact of considering four complex hyperplanes is an optimal condition (see the end of section two for more details).*

Recently, J. Duval, following R. Debalme and S. Ivashkovich has obtained an almost complex version of the previous Green theorem, reducing the hyperbolicity of the complement of five almost complexs lines in $\mathbb{CP}^2$ to global geometrical properties. This makes sense only in real dimension 4: indeed any almost complex structure is integrable in real dimension 2, while one lacks $J-$holomorphic hypersurfaces in real dimension greater than 4. Let's recall the Duval's result [6] (see also [2]). We denote by J an almost complex structure on $\mathbb{CP}^2$.

Theorem 0.5. *Let C be a configuration of five $J-$lines in general position (i.e, having no triple point) in the almost complex projective space $(\mathbb{CP}^2, J)$. Then:*
$(\mathbb{CP}^2 \setminus C, J)$ is Kobayashi hyperbolic.

Our second main result has for objective to perform a real small perturbation on a given arrangemants of complex lines in $\mathbb{CP}^2$, and to check the Kobayashi hyperbolicity of their complement in the complex projective space $\mathbb{CP}^2$. We shall prove the following theorem B.

Theorem B. *Let $\{L_i\}_{i=1}^5$ be five complex lines in general position in $\mathbb{CP}^2$ and let $\{\tilde{L}_i\}_{i=1}^5$ be a C^1-small real deformation of L_i, $i = 1, ..., 5$. Let J_{st} be the standard structure on $\mathbb{C}P^2$. Then the following hold.*

i) There exists an almost complex structure on $\mathbb{CP}^2$, denoted by J, such that $(\mathbb{CP}^2 \setminus \cup_{i=1}^5 \tilde{L}_i, J)$ is J-Hyperbolic in the sens of Kobayashi.

ii) $(\mathbb{CP}^2 \setminus \cup_{i=1}^5 \tilde{L}_i, J_{st})$ is Hyperbolic in the sens of Kobayashi for the standard structure.

The paper is organized as follows: In the first section we recall some generalities on the geometry of the complex projective plane. In section tow we prove our main result.

1. Preliminaries

1.1. Almost complex manifolds and pseudoholomorphic discs

Let X be a real smooth monifold of dimension $2n$. An almost complex structure on X is an automorphism J of TX such that $J^2 = -Id$. The pair (X, J) is called an *almost complex manifold* that is a C^∞ smooth real manifold equipped with a C^∞ smooth almost complex structure J. This almost complex manifold has $J-$holomorphic curves; that are, smooth real surfaces whose tangent plane at any point is a complex line for J. In particular, it has many non-constant $J-$disks (see J-C. Sikorav [1]). A J-disk is a smooth map $f : (D, i) \to (X, J)$ from the unit disk D of $\mathbb{C}$ to X, which is J-holomorphic: its differential $df : TD \to TX$ satisfies

$$df_z \circ i = J_{f(z)} \circ df.$$

Let J be an almost complex structure on $\mathbb{CP}^2$ positive with respect to the Fubini-Study form $\omega : \omega(., J.) > 0$. A $J-$line of the almost complex projective plane $(\mathbb{CP}^2, J)$ is a $J-$holomorphic curve in $\mathbb{CP}^2$, diffeomorphic to $\mathbb{CP}^1$ and of degree 1 in homology. According to M. Gromov [10] (see also J.-C. Sikorav [16]), the space of $J-$lines is diffeomorphic to $\mathbb{CP}^2$. In addition, by two distinct points passes a single $J-$line, two distinct $J-$lines intersect transversely on a single point, and $J-$lines passing through a point P form a pencil, diffeomorphic to $\mathbb{CP}^1$, and giving a Central projection:

$$\Pi : \mathbb{CP}^2 \setminus \{P\} \to \mathbb{CP}^1.$$

1.2. Kobayashi hyperbolicity of almost complex manifolds

Let (X, J) be a almost complex manifold. The existence, for any point $P \in X$ and any vector $v \in T_P X$, of a non-constant $J-$disk f passing through P tangentially to v (see the article of J-C. Sikorav in [1]), motivates the definition of a pseudometric of Kobayashi-Royden almost complex K_X^J (see [11]).
For every $p \in X$, there is a neighborhood $\mathcal{V}$ of 0 in T_pX such that for every $\xi \in \mathcal{V}$ there exists $f : \Delta \longrightarrow (X, J)$ a pseudoholomorphic curve satisfying $f(0) = p$ and $f'(0) = \xi$ (see [14]). This allows one to define the Kobayashi-Royden infinitesimal pseudometric K_X^J, where

$$K_X^J(p, \xi) = \inf\left\{\frac{1}{r}; f : \Delta_r \longrightarrow X,\ J-\text{holomorphic}\,; f(0) = p, f'(0) = \xi\right\}$$

Kruglikov [13] extended Royden's results [15] and proved that K_X^J is upper semicontinuous on the tangent bundle TX of X and that the integrated form of the Kobayashi-Royden metric coincides with the Kobayashi pseudo-distance d_X^J, that is,

$$d_X^J(p, q) = \inf_{\gamma\in\Gamma_{p,q}} \int_0^1 K_X^J(\gamma(t), \gamma'(t)).dt$$

Where $\Gamma_{p,q}$ is the set of all C^1-paths $\gamma : [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.
We say that (X, J) is *Kobayashi hyperbolic* if d_X^J is a distance. Otherwise In the opposite case, there is a point through which pass arbitrarily large J-disks in a given direction. If the manifold X is compact, this is equivalent to the existence of a non constant J-holomorphic map $f : (\mathbb{C}, i) \to (X, J)$. This criterion, due to Brody [3] in the complex case and to B. Kruglikov and M. Overholt [11] in the almost complex case, follows from the next reparametrisation Theorem (see [3] in the complex case and [11] in the almost complex case):

Theorem 1.1. *(Brody's Reparameterization Theorem). Let (X, J) be a almost complex manifold. Let $f_n : D \to X$ a non-normal sequence of J-disks. There is a sequence of affine contractions r_n of $\mathbb{C}$ converging to a point of the unit disc, called an explosion point, such that $f_n \circ r_n$ converges uniformly on any compact in $\mathbb{C}$ to a Brody curve: a non-constant entier curve with uniformly bounded derivative.*

2. Proof of Theorem A.

Throughout this section we identify $\mathbb{R}^6$, endowed with its standard complex structure J_{st}, to $\mathbb{C}^3$.

Definition 2.1. *Let $n \geqslant 3$ and let $\mathcal{H} = (H_1, ..., H_n)$ be a family of real subspaces of $\mathbb{R}^6$ such that $codim_{\mathbb{R}} H_j = 2$ for $j = 1, ..., n$. Then $\mathcal{H}$ is said to be in general position if for every 3-tuple (i, j, k) of distinct integers $i, j, k \in \{1, ..., n\}$,*

$$Span_{\mathbb{R}}(H_i^{\perp}, H_j^{\perp}, H_k^{\perp}) = \mathbb{R}^6.$$

Here, if H is a real subspace in $\mathbb{R}^6$, then $H^{\perp}$ denotes the orthogonal complement of H with respect to the Euclidean metric.

We will need the following properties satisfied by the canonical projection in $\mathbb{CP}^2$ of a holomorphic curve $f : \mathbb{C} \to \mathbb{C}^3$. For H a real subspace of $\mathbb{R}^6$, we denote by $H^{\star}$ the set $H \setminus \{0\}$. Then, we have the following Lemma

Lemma 2.2. *Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$ be the canonical projection. Then:*

1. *If H is a complex hyperplane in $\mathbb{C}^3$, then $\pi(H^{\star})$ is a complex projective line in $\mathbb{CP}^2$.*
2. *If $f : \mathbb{C} \to \mathbb{C}^3$ is holomorphic and H is a complex hyperplane in $\mathbb{C}^3$, then*

$$f(\mathbb{C}) \cap H = \emptyset \Rightarrow \pi(f)(\mathbb{C}) \cap \pi(H^{\star}) = \emptyset.$$

3. *If H_1, H_2, H_3 are complex hyperplanes in general position in $\mathbb{C}^3$, then $\pi(H_1^{\star}), \pi(H_2^{\star}), \pi(H_3^{\star})$ are in general position in $\mathbb{CP}^2$.*

Notation: if $Z \in \mathbb{CP}^2$, we denote $[z_1 : z_2 : z_3]$ its homogeneous coordinates, where $(z_1, z_2, z_3) \in \mathbb{C}^3$.

Proof.
Point (1). We may assume that $H = \{(z_1, z_2, z_3) \in \mathbb{C}^3 / a_1 z_1 + a_2 z_2 + a_3 z_3 = 0\}$, with $a_1, a_2, a_3 \in \mathbb{C}$, $a_3 \neq 0$. Then

$$\pi(H^{\star}) = \{[1 : z_2 : z_3] \in \mathbb{CP}^2 / a_1 + a_2 z_2 + a_3 z_3 = 0\} \cup \{[0 : 1 : -\frac{a_2}{a_3}]\}$$

$$= \{[1 : z : -\frac{a_1 + a_2 z}{a_3}],\ z \in \mathbb{C}\} \cup \{[0 : 1 : -\frac{a_2}{a_3}]\}.$$

We notice that $[0 : 1 : -\frac{a_2}{a_3}]$ corresponds to $[\frac{1}{\infty} : 1 : -\frac{a_1 + a_2\infty}{a_3\infty}]$. Hence $\pi(H^{\star})$ is a projective complex line in $\mathbb{CP}^2$.
Point (2). We first notice that $\pi(f)$ is well defined since, by assumption $f(\mathbb{C}) \cap H = \emptyset$, which implies that $f(\mathbb{C}) \subset \mathbb{C}^3 \setminus \{0\}$. Assume now, to get a contradiction, that $\pi(f)(\mathbb{C}) \cap \pi(H^{\star}) \neq \emptyset$. Then there are two possibilities.
Case (α). There exists $z \in \mathbb{C}$ and there exists $\lambda \in \mathbb{C}$ such that

$$\pi(f)(z) = \left[1 : \lambda : -\frac{a_1 + a_2\lambda}{a_3}\right].$$

Then, there exists $c_z \in \mathbb{C}^*$ such that $f(z) = \left(c_z, \lambda c_z, -\frac{a_1 + a_2\lambda}{a_3} c_z\right)$. In particular $a_1 f_1(z) + a_2 f_2(z) + a_3 f_3(z) = 0$, where $f = (f_1, f_2, f_3)$. Hence, $f(z) \in H$. This is a contradiction.
Case (β). There exists $z \in \mathbb{C}$ such that

$$\pi(f)(z) = \left[0 : 1 : -\frac{a_2}{a_3}\right].$$

Then, there exists $c_z \in \mathbb{C}^*$ such that $f(z) = \left(0, c_z, -\frac{a_2}{a_3} c_z\right)$ and $a_1 f_1(z) + a_2 f_2(z) + a_3 f_3(z) = 0$. We obtain again that $f(z) \in H$: this is a contradiction.

Point (3). Since H_1, H_2, H_3 are complex hyperplanes in $\mathbb{C}^3$, then there is a linear change of coordinates such that the hyperplanes are defined by equations

$$\begin{aligned} H_1 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_1 = 0\}, \\ H_2 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_2 = 0\}, \\ H_3 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_3 = 0\}. \end{aligned}$$

Now by projection into $\mathbb{CP}^2$, we get

$$\begin{aligned} \pi(H_1^\star) = &\quad \{[0:1:z] \,|\, z \in \mathbb{C} \cup \{\infty\}\} \cup [0:0:1], \\ \pi(H_2^\star) = &\quad \{[1:0:z] \,|\, z \in \mathbb{C} \cup \{\infty\}\} \cup [0:0:1], \\ \pi(H_3^\star) = &\quad \{[1:z:0] \,|\, z \in \mathbb{C} \cup \{\infty\}\} \cup [0:1:0]. \end{aligned}$$

Hence $\pi(H_1^\star) \cap \pi(H_2^\star) \cap \pi(H_3^\star) = \emptyset$, meaning that $\pi(H_1^\star), \pi(H_2^\star), \pi(H_3^\star)$ are in general position since there is no triple point.

□

In the next lemma, we give the explicit form of all curves $f : \mathbb{C} \to \mathbb{C}^3$ which avoids five lines in general position, and those that avoid four lines in general position. This lemma will be used later in the proof of our main result.

Lemma 2.3. (i) *For all $n \geqslant 5$ and for all $H_1, ..., H_n$ complex hyperplanes in $\mathbb{C}^3$ in general position, there exists a non constant holomorphic $f : \mathbb{C} \to \mathbb{C}^3$ curve such that*

$$f(\mathbb{C}) \cap (\cup_{i=1}^n H_i) = \emptyset.$$

(ii) *Let H_1, H_2, H_3 and H_4 be four complex hyperplanes in $\mathbb{C}^3$. If there exists $f : \mathbb{C} \to \mathbb{C}^3$ holomorphic, such that $f(\mathbb{C}) \cap (\cup_{i=1}^4 H_i) = \emptyset$, then there exists two holomorphic curves $h, g : \mathbb{C} \to \mathbb{C}$ such that*

$$f = (e^h, -e^h, e^g)$$

Remark 2.4. *In case (i), according to the Green Theorem and to Lemma 2.2, $\pi(f)$ is constant. Here π denotes the canonical projection from $\mathbb{C}^3 \setminus \{0\}$ into $\mathbb{CP}^2$ and $\pi(f) := \pi \circ f$. Notice that $\pi(f)$ is well-defined since $f(\mathbb{C}) \subset \mathbb{C}^3 \setminus \{0\}$.*

Proof. **Point (i).** Consider first the case $n = 5$. By a linear change of coordinates, we take the hyperplanes H_1, H_2, H_3, H_4 and H_5 in standard form defined by the following equations

$$\begin{aligned} H_1 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_1 = 0\}, \\ H_2 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_2 = 0\}, \\ H_3 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_3 = 0\}, \\ H_4 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, z_1 + z_2 + z_3 = 0\}, \\ H_5 = &\quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\, a_1 z_1 + a_2 z_2 + a_3 z_3 = 0\}, \quad a_j \in \mathbb{C} \setminus \{0\} \quad \forall\, j = 1, 2, 3. \end{aligned}$$

Now, if we assume that $f(\mathbb{C}) \cap \left(\bigcup_{i=1}^5 H_i \right) = \emptyset$, then there exists $h_1, h_2, h_3 : \mathbb{C} \to \mathbb{C}$, holomorphic, such that

$$f = \left(e^{h_1}, e^{h_2}, e^{h_3} \right).$$

Moreover, since $\pi(f)(\mathbb{C})$ will omits $\pi(H_i)$ for $i = 1, ..., 5$ (see Lemma 2.2) and $\pi \circ f$ is constant by Green (see [9]), there exists $(\omega_1, \omega_2, \omega_3) \neq (0, 0, 0)$ such that for all $z \in \mathbb{C}$,

$$\left[e^{h_1(z)} : e^{h_2(z)} : e^{h_3(z)} \right] = [\omega_1 : \omega_2 : \omega_3].$$

Therefore

$$\left[1 : \frac{e^{h_2(z)}}{e^{h_1(z)}} : \frac{e^{h_3(z)}}{e^{h_1(z)}} \right] = \left[1 : \frac{\omega_2}{\omega_1} : \frac{\omega_3}{\omega_1} \right]$$

which implies that

$$\begin{cases} e^{h_2(z)-h_1(z)} &= \frac{\omega_2}{\omega_1} \\ \\ e^{h_3(z)-h_1(z)} &= \frac{\omega_3}{\omega_1} \end{cases} \Rightarrow \begin{cases} e^{h_2(z)} &= \frac{\omega_2}{\omega_1} e^{h_1(z)} \\ \\ e^{h_3(z)} &= \frac{\omega_3}{\omega_1} e^{h_1(z)} \end{cases}$$

Hence $f = (e^{h_1}, c_2 e^{h_1}, c_3 e^{h_1})$, with $1 + c_2 + c_3 \neq 0$, and f is not constant.
Essentially the same type of argument works in general. Let $H_1, ..., H_n$, $\quad n \geqslant 5$, be n hyperplanes defined by:

$$H_k := \left\{ Z \in \mathbb{C}^3 | \sum_{i=1}^{3} \alpha_i^k z_i = 0, \;\; \alpha_i^k \in \mathbb{C}, \;\; 1 \leqslant k \leqslant n \right\}.$$

Pose $f = (e^h, c_2 e^h, c_3 e^h)$ which is not constant, where h is holomorphic from $\mathbb{C}$ to $\mathbb{C}$. Hence, in order that f avoids $H_1, ..., H_n$, it is sufficient to choose $c_2, c_3 \in \mathbb{C}$ such that for every $k = 1, ..., n$

$$\alpha_1^k + \alpha_2^k c_2 + \alpha_3^k c_3 \neq 0.$$

We point out that what preceeds proves more generally that given a countable set of complex hyperplanes in $\mathbb{C}^3$ passing through the origin, there exists $f : \mathbb{C} \to \mathbb{C}^3$ not constant and avoiding each hyperplane. This proves Point (i).
Point (ii). Let H_1, H_2, H_3 and H_4 be four complex hyperplanes in general position in $\mathbb{C}^3$. We know that there is a linear change of coordinate such that H_1, H_2, H_3 and H_4 are defined in standard form by :

$$\begin{aligned} H_1 = &\ \{(z_1, z_2, z_3) \in \mathbb{C}^3 | \, z_1 = 0\}, \\ H_2 = &\ \{(z_1, z_2, z_3) \in \mathbb{C}^3 | \, z_2 = 0\}, \\ H_3 = &\ \{(z_1, z_2, z_3) \in \mathbb{C}^3 | \, z_3 = 0\}, \\ H_4 = &\ \{(z_1, z_2, z_3) \in \mathbb{C}^3 | \, z_1 + z_2 + z_3 = 0\}, \end{aligned}$$

Now the fact that $f(\mathbb{C}) \cap (\cup_{i=1}^4 H_i) = \emptyset$ is equivalent to the existence of holomorphic functions $f_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2, 3$ such that

$$f = (e^{f_1}, e^{f_2}, e^{f_3}).$$

Then, by Lemma 2.2 (2), $g := \pi(f)$ satisfies $g(\mathbb{C}) \subset \mathbb{CP}^2 \setminus \bigcup_{j=1}^4 \pi(H_j^\star)$. Hence g has the following form

$$g = [1 : e^{g_2} : e^{g_3}], \tag{1}$$

where $g_2 = f_2 - f_1$ and $g_3 = f_3 - f_1$. According to Theorem 0.2, there exists three diagonals $\Delta_{12,34}, \Delta_{13,24}, \Delta_{14,23}$ such that $g(\mathbb{C}) = \pi(f(\mathbb{C}))$ is contained in one of these diagonals, where $\Delta_{ij,kl}$ is the diagonal line passing through $\left(\pi(H_i^\star) \cap \pi(H_j^\star)\right)$ and $\left(\pi(H_k^\star) \cap \pi(H_l^\star)\right)$.
We recall that

$$\begin{cases} \pi(H_i^\star) &= \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_i = 0\} \; For \; j = 1, 2, 3 \\ \;\; and & \\ \pi(H_4^\star) &= \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_1 + z_2 + z_3 = 0\}. \end{cases}$$

Hence $\Delta_{12,34}, \Delta_{13,24}, \Delta_{14,23}$ are given by

$$\begin{aligned} \Delta_{12,34} &= \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_1 + z_2 = 0\}, \\ \Delta_{13,24} &= \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_2 + z_3 = 0\}, \\ \Delta_{14,23} &= \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_1 + z_3 = 0\}. \end{aligned} \tag{2}$$

Suppose that $g(\mathbb{C})$ is contained in $\Delta_{12,34}$, the cases $g(\mathbb{C}) \subset \Delta_{13,24}$ or $g(\mathbb{C}) \subset \Delta_{14,23}$ being similar. Then $e^{g_2} + 1 = 0 \Rightarrow e^{g_2} = -1 \Rightarrow g = [1 : -1 : e^{g_3}]$, where $g_3 = f_3 - f_1$. Hence

$$f = (e^{f_1}, -e^{f_1}, e^{f_3}). \tag{3}$$

□

Now we may show Theorem A,
Let H be a real subspace of $\mathbb{C}^3$ such that $dim_{\mathbb{R}}H = 5$, then H contains a unique complex hyperplane $\tilde{H}$ of $\mathbb{C}^3$. Indeed, there exists $(a_1, b_1, a_2, b_2, a_3, b_3) \in \mathbb{R}^6 \setminus \{0\}$ such that

$$\begin{aligned} H &= \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6 | \ \sum_{j=1}^{3}(a_j x_j + b_j y_j) = 0\right\} \\ &= \left\{z \in \mathbb{C}^3 | \ \Re\Big(\sum_{j=1}^{3}(a_j - ib_j)z_j\Big) = 0\right\}. \end{aligned}$$

Hence $\tilde{H} := \left\{z \in \mathbb{C}^3 | \ \sum_{j=1}^{3}(a_j - ib_j)z_j = 0\right\}$ is a complex hyperplane in $\mathbb{C}^3$, contained in H.
Point (1). Assume that $(\tilde{H}, H_j, H_k)$ are in general position for some $j \neq k$, $j, k \in \{1, ..., 4\}$. Since $\tilde{H} \subset H$, where $\tilde{H}$ is a complex hyperplane of $\mathbb{C}^3$, and

$$f(\mathbb{C}) \cap (\cup_{i=1}^{4} H_i \cup H) = \emptyset \Rightarrow f(\mathbb{C}) \cap (\cup_{i=1}^{4} H_i \cup \tilde{H}) = \emptyset,$$

then it follows from Lemma 2.3 (i) that there is $(c_1, c_2) \in (\mathbb{C}^*)^2$ which satisfies $1 + c_2 + c_3 \neq 0$ and there exists $h : \mathbb{C} \to \mathbb{C}$ holomorphic such that

$$f(z) = (e^h, c_2 e^h, c_3 e^h).$$

On another hand $H := \{(x_1, y_1, ..., x_3, y_3) \in \mathbb{R}^6 | \ \sum_{j=1}^{3}(a_i x_i + b_i y_i) = 0\}$. By hypothesis $f(\mathbb{C}) \cap H = \emptyset$ then for every $z \in \mathbb{C}$ we have,

$$\begin{aligned} &a_1 \Re(e^{h(z)}) + a_2 \Re(c_2 e^{h(z)}) + a_3 \Re(c_3 e^{h(z)}) \\ &+ b_1 \Im(e^{h(z)}) + b_2 \Im(c_2 e^{h(z)}) + b_3 \Im(c_3 e^{h(z)}) \quad \neq 0. \end{aligned}$$

Thus, for every $z \in \mathbb{C}$

$$\begin{aligned} &\Re(e^{h(z)})\Big[a_1 + a_2\Re(c_2) + a_3\Re(c_3) + b_2\Im(c_2) + b_3\Im(c_3)\Big] \\ &+ \Im(e^{h(z)})\Big[b_1 + b_2\Re(c_2) + b_3\Re(c_3) - a_1\Im(c_2) - a_3\Im(c_3)\Big] \quad \neq 0. \end{aligned}$$

We denote

$$\begin{aligned} a &:= \Big[a_1 + a_2\Re(c_2) + a_3\Re(c_3) + b_2\Im(c_2) + b_3\Im(c_3)\Big] \\ b &:= \Big[b_1 + b_2\Re(c_2) + b_3\Re(c_3) - a_1\Im(c_2) - a_3\Im(c_3)\Big] \end{aligned}$$

then

$$f(\mathbb{C}) \cap H = \emptyset \Leftrightarrow e^{h(\mathbb{C})} \cap \{(x, y) \in \mathbb{R}^2 \ / \ ax + by = 0\} = \emptyset.$$

However $\{(x, y) \in \mathbb{R}^2 \mid ax + by = 0\}$ is either a real line or $\mathbb{R}^2$, depending on the values of a and b. Then by the little Picard Theorem e^h is constant because it avoids an infinite number of points. Hence h is constant and f is then constant. We point out that the projection of f into $\mathbb{CP}^2$ is also constant.
Point (2). Suppose there exists $j \neq k$, $j, k \in \{1, ..., 4\}$, such that
$dim_{\mathbb{R}}Span_{\mathbb{R}}(\tilde{H}^\perp, H_j^\perp, H_k^\perp) = 4$. Then:

$$\tilde{H}^\perp \subset Span_{\mathbb{R}}(H_j^\perp, H_k^\perp).$$

In fact for all $i = 1, ..., 4$, $dim_{\mathbb{R}}H_i^\perp = 2$, then $dim_{\mathbb{R}}Span_{\mathbb{R}}(H_i^\perp, H_j^\perp) = 4$.
Suppose $\tilde{H}^\perp \subset Span_{\mathbb{R}}(H_1^\perp, H_2^\perp)$ then there exists $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$\tilde{H} = \{\alpha_1 z_1 + \alpha_2 z_2 = 0\}.$$

Now by lemma 2.3 (*ii*), let pose

$$f = (e^{f_1}, -e^{f_1}, e^{f_3}).$$

We take $f_1 = c,\ c \in \mathbb{C} \setminus \{0\}$, such that $\Re(\alpha_1 e^c - \alpha_2 e^c) \neq 0$ and f_3 not constant. Then

$$f = (C, -C, e^{f_3})$$

avoids $\cup_{i=1}^{4} H_i \cup H$, and $\pi(f)$ is not constant.
This concludes the proof of Theorem A. $\square$

Proposition 2.5. *The consideration of four complex hyperplanes in Theorem A, is an optimal condition.*

Proof. Let H_1, H_2 and H_3 be three complex hyperplanes in $\mathbb{C}^3$, then there exists H a real hyperplane in $\mathbb{R}^6$ and a complex hyperplane $\tilde{H}$ contained in H such that $\left(H_1, H_2, H_3, \tilde{H}\right)$ are in general position, and there exists $f : \mathbb{C} \to \mathbb{C}^3$, holomorphic, such that $f(\mathbb{C}) \cap \left(\cup_{j=1}^{3} H_j \cup H\right) = \emptyset$ and $\pi \circ f$ is not constant. Indeed, we pose $H = \{x_1 + x_2 + x_3 = 0\}$ and $\tilde{H} = \{z_1 + z_2 + z_3 = 0\}$, which is clearly contained in H. Since $f(\mathbb{C}) \cap \left(\cup_{j=1}^{3} H_j \cup H\right) = \emptyset$, then $f(\mathbb{C}) \cap \left(\cup_{j=1}^{3} H_j \cup \tilde{H}\right) = \emptyset$ and $f = (e^{f_1}, e^{f_2}, e^{f_3})$. Hence

$$g := \pi(f) = [1 : e^{g_2} : e^{g_3}],$$

where $g_2 = f_2 - f_1$ and $g_3 = f_3 - f_1$. By the Borel Theorem 0.2, $g := \pi(f)$ is contained in one of diagonals $\Delta_{12,34}, \Delta_{13,24}, \Delta_{14,23}$ (see 2). Suppose $\pi(f)(\mathbb{C}) \subset \Delta_{13,24}$, then

$$\pi(f) = [1, e^{g_2}, -e^{g_2}],$$

Hence $f = (1, e^{g_2}, -e^{g_2})$ avoids $\left(\cup_{j=1}^{3} H_j \cup H\right)$ and $\pi(f)$ is not constant. $\square$

3. Proof of Theorem B.

In this section we perform a C^1−small deformation on the five avoided complex lines in $\mathbb{CP}^2$ and we show that there exists a almost complex structure denoted by J, such that the deformed lines become J−lines and their complement in $\mathbb{CP}^2$ is Kobayashi hyperbolic. Furthermore we show that this complement remains Kobayashi hyperbolic with the standard structure in $\mathbb{CP}^2$.
To show our main result we need the following lemmas.

Lemma 3.1. *Let $(L_i)_{i=1,...,5}$ be five lines in general position on $\mathbb{CP}^2$. Let $\tilde{L_i}$ be a sufficiently C^1−small deformation of L_i, $i = 1, ..., 5$. Then there exists a diffeomorphism $\phi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $\phi(L_i) = \tilde{L_i}$ are J−lines, where $J = \phi^\star(J_{st})$.*

Proof. The diffeomorphism is seen when we deform a single line (say L_1):
Let Z be the zero section of $\mathcal{N}_{L_1}$ (the normal bundle of L_1 which is the orthogonal to TL_1). let V be a neighborhood, with boundary, of Z. Let U be a tubular neighborhood of L_1. Let $\Phi : U \to V$ be a diffeomorphism and π be the projection of the normal bundle on its base. Now for all $i \neq 0$, L_i are transverse to L_1, then we can assume, without modifying Φ, that for all $i \neq 0$, $\Phi(L_i \cap U)$ is in a fiber F of π. Since $\tilde{L_1}$ is a sufficiently C^1−small deformation of L_1, then $\Phi(\tilde{L_1})$ is the graph of a C^1−small section of F, noted by "s".
Let χ be a tray function defined by:

$$\begin{cases} \chi(x) &= 1, \quad \forall x \in Z \\ \chi(x) &= 0, \quad \forall x \in \partial V \end{cases}$$

Let $\psi : V \to V$ be a map defined by

$$\psi(x) = x + \chi(x)(s \circ \pi)(x).$$

We can see that ψ is a diffeomorphism of V (C^1-small perturbation of the identity). It is equal to the identity at the boundary of V, preserves the fibers of π and sends the zero section to the graph of s. in fact

$$\begin{cases} x \in Z & \Rightarrow \psi(x) = & x + s \circ \pi(x) \in \Phi(\tilde{L_1}) \\ x \in \partial V & \Rightarrow \psi(x) = & x \\ y \in \pi^{-1}(z),\ z \in L_1 & \Rightarrow \psi(y) = & y + \chi(y)s(z) \in \pi^{-1}(z) \end{cases}$$

Let $\Psi_{|U} = \Phi^{-1} \circ \psi \circ \Phi$ and

$$\phi : \mathbb{CP}^2 \to \mathbb{CP}^2;\ \ Z \mapsto \Psi(Z)$$

Now The general case is obtained by composing such diffeomorphisms. Indeed more precisly, let $(\phi_i)_{i=1,...,5}$: $\mathbb{CP}^2 :\to \mathbb{CP}^2$ be five C^∞ diffeomorphisms such that, by what precedes we have,

$$\begin{cases} \phi_1(L_1) = \tilde{L_1} & , & \phi_1(L_i) = L_i,\ i = 2...5 \\ \phi_2(L_2) = \tilde{L_2} & , & \phi_2(L_i) = L_i,\ i \geqslant 3, \phi_2(\tilde{L_1}) = \tilde{L_1}. \\ \phi_3(L_3) = \tilde{L_3} & , & \phi_3(L_i) = L_i,\ i \geqslant 4,\ \ \phi_3(\tilde{L_j}) = \tilde{L_j},\ j \leqslant 2. \\ \phi_4(L_4) = \tilde{L_4} & , & \phi_4(L_5) = L_5,\ \ \phi_4(\tilde{L_j}) = \tilde{L_j},\ j \leqslant 3. \\ \phi_5(L_5) = \tilde{L_5} & , & \phi_5(\tilde{L_j}) = \tilde{L_j},\ j \leqslant 4. \end{cases}$$

Let $\phi = \phi_1 \circ \phi_2 \circ \phi_3 \circ \phi_4 \circ \phi_5$ and $J = \phi^* J_{st}$. Hence This global diffeomorphism of the projective plane turns the initial situation into the perturbed situation and these perturbations are J−lines for the new structure obtained by transporting, by the diffeomorphism, the standard structur, then $\tilde{L_1}$, $\tilde{L_2}$, $\tilde{L_3}$, $\tilde{L_4}$ and $\tilde{L_5}$ are J-lines of the projective space. Hence, by Duval's theorem [6], $(\mathbb{CP}^2 \setminus \bigcup_{i=1}^5 \tilde{H}_i, J)$ is J-Hyperbolic in the sens of Kobayashi. □

Lemma 3.2. *Let L_i be five complex lines in general position on $\mathbb{CP}^2$. There exists $\varepsilon > 0$ such that if φ is a diffeomorphism of $\mathbb{CP}^2$ with $|\varphi - id| < \varepsilon$ then, $(\mathbb{CP}^2 \setminus \bigcup \varphi(L_i), J_{st})$ is Kobayashi hyperbolic.*

Proof. Suppose that $\mathbb{CP}^2 \setminus \bigcup \varphi(L_i)$ is not hyperbolic. Then there is a sequence of diffeomorphisms φ_n tending to the identity with a divergent sequence of holomorphic disks $f_n : D \to \mathbb{CP}^2 \setminus \bigcup \varphi(L_i)$. By the Brody lemma, we can reparametrize these disks by a sequence of contractions r_n so that $f_n \circ r_n$ converges to a non constant entire curve $f : \mathbb{C} \to \mathbb{CP}^2$ (up to extracting a subsequence). This entire curve must cut $\cup L_i$ by Green's theorem. If it is contained in one of the L_i, it cuts some other L_j by Picard's theorem. So in any case it cuts one of the lines L_j without being contained in it. Locally its homological intersection with L_j is strictly positive. This remains true just before passing to the limit, that is $f_n(D) \cap \varphi_n(L_i) \neq \emptyset$. This contradicts the hypothesis that $f_n(D)$ avoids $\cup\varphi_n(L_i)$. □

Proof. **[Proof of Theorem B]**
The proof of Theorem B is a consequence of the two previous Lemmas. □

Particular case

In the case where we perturb a single line we can easily prove the folowing

Proposition 3.3. *Let $L_1, ..., L_5$ be five complex lines in general position in $\mathbb{CP}^2$. Let $\tilde{L_1}$ be a small deformation of L_1, $(\tilde{L_1} \simeq \mathbb{CP}^1)$ and $\omega_{FS}(L_1, \tilde{L_1}) < 1$.*
Then any holomorphic curve $f : \mathbb{C} \to \mathbb{CP}^2 \setminus \cup_{i=2}^5 H_i \cup \tilde{H}_1$ is constant.

Proof. By Borel's Theorem 0.2, a holomorphic curve of $\mathbb{C}P^2$ which avoids four complex lines in general position is contained in one of the diagonal lines $(\Delta_i)_{i=1,2,3}$ of the configuration (the lines joining two double points corresponding to two disjoint pairs of the initial lines). Now the fifth line after any small (real) deformation still cuts the three diagonals apart from the double points. In fact the fifth line is transverse to the diagonals so if we disturb it a little there remains an intersection close to the previous one (stability of transverse intersections). So finally the holomorphic curve is contained in $\mathbb{C}P^1$ (the diagonal) minus three points. It is constant by Picard's theorem (see [8]). □

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