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Geraghty Type Contractions in Fuzzy b-metric Spaces with Application to Integral Equations

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Abstract. In this paper, we study the notion of a fuzzy b-metric space and establish certain fixed point results for Geraghty-type contraction in the setting of G-complete fuzzy b-metric space. We furnish an example to illustrate our main result. Our results extend and generalize the existing results in the literature. An application related to our main result for the existence of solution of nonlinear integral equations is also presented.

1. Introduction

Banach [3] proved the well known fixed point theorem in 1922. After that a lot of contribution has been made in this field by many researchers. In 1989, Bakhtin [4] introduced the concept of b-metric spaces which was further investigated by Czerwik [8]. Recall that, a b-metric space is a nonempty set X together with a function $d_b : X \times X \to [0, \infty)$, where $b \geq 1$, such that for all $x_1, x_2, x_3 \in X$, we have $d_b(x_1, x_2) = d_b(x_2, x_1) = 0$ if and only if $x_1 = x_2$ and $d_b(x_1, x_3) \leq b[d_b(x_1, x_2) + d_b(x_2, x_3)]$. Many fixed point theorems have been proved by many researchers in the field of metric spaces, b-metric spaces and their various extensions. For instance, see [5–8, 13, 24, 25] and the references therein.

The concept of a fuzzy set was introduced by Zadeh [29] in 1965. By a fuzzy set A in X, we mean a function whose domain is the set X and range is the closed interval $[0, 1]$. Since then a substantial literature has been developed to investigate fuzzy sets and their applications. Kramosoil and Michálek [14] used the idea of fuzzy set to define the notion of fuzzy metric space and also proved some fixed point results. In 1988, Grabiec [12] introduced a weaker form of completeness of fuzzy metric spaces which is now known as G-completeness in literature and the Cauchy sequence in such spaces is called a G-Cauchy sequence and also proved the well known Banach fixed point theorem in the setting of fuzzy metric spaces. In 1994, George and Veermani [10] modified the definition of fuzzy metric space given by Kramosil and Michálek [14] and proved some fixed point results. In 2004, Dorel Mehit [17] defined the notion of triangular fuzzy metric space and proved some fixed point results. After that many fixed point results have been established by many researchers in fuzzy metric spaces. For instance see [18, 21, 26–28].

In 2015, Hussain et.al [20] gave the idea of fuzzy b-metric space and established a relation between the parametric b-metric and fuzzy b-metric. Later, in 2016, Nădăban [19] establsihed some interesting results

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for fuzzy b-metric spaces. Recently, Mehmood et al [15] generalized the notion of a fuzzy b-metric space by presenting the idea of an extended fuzzy b-metric on a nonempty set X and proved the famous Banach [3] fixed point result in this new more general setting.

The present article uses the idea of fuzzy b-metric space to prove some new fixed point results for Geraghty type [11] contraction in G-complete fuzzy b-metric spaces. Particularly, the extension of main result of Grabiec [12] is established in our first theorem. Second result is the extention of the main result of Faraji et.al [9] and other results are the generlization of the results of Alsulami et.al [2] in the setting of G-complete fuzzy b-metric space.

The rest of the article is organized as: In Section 2, we collected some basic definitions and results that are related to our main work. Section 3 consists of our main contributions and in Section 4, an application of the main result is presented to show the existence of a solution of a nonlinear integral equation.

2. Preliminaries

In this section, we recall some fundamental definitions which will help in understanding the rest of the sections.

A commuting binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ that is also associative is called t-norm (or a triangular norm) if $a*1=a$ for all $a\in[0,1]$ and if $a\le c$ and $b\le d$ then $a*b\le c*d\ \forall\ a,b,c,d\in[0,1]$. Three well known examples of such a triangular norm include the binary operations $\wedge$, $\cdot$ and $*_L$, respectively defined as $a_1\wedge a_2=\min\{a_1,a_2\}$, $a_1\cdot a_2=a_1a_2$ and $a_1 *_L a_2=\max\{a_1+a_2-1,0\}$. For details see [23].

Kramosoil and Michálek [14] introduced the notion of fuzzy metric spaces in 1975 by using the concept of a fuzzy set and t-norm as follows:

Definition 2.1. *[14] A nonempty set X together with a fuzzy set $M : X\times X\times[0,\infty)\to[0,1]$ and a continuous t-norm $*$, is said to be a fuzzy metric space if for all $x_1,x_2,x_3\in X$ and $t,s>0$, the metric M satisfies the following conditions:*

FM1: $M(x_1,x_2,0)=0$

FM2: $M(x_1,x_2,t)=1$ *if and only if* $x_1=x_2$

FM3: $M(x_1,x_2,t)=M(x_2,x_1,t)$

FM4: $M(x_1,x_3,t+s)\ge M(x_1,x_2,t)*M(x_2,x_3,s)$

FM5: $M(x_1,x_2,.)\colon[0,\infty)\to[0,1]$ *is left continuous.*

Later many authors have further studied and developed the fixed point theory for fuzzy metric spaces. Recently, Hussain et al. [20] and Nădăban [19] studied the concept of fuzzy b-metric space and developed some interesting results in these space.

Definition 2.2. *[19] Consider a nonempty set X and a real number $b\ge1$ and let $*$ be a continuous t-norm. A fuzzy set $M_b : X\times X\times[0,\infty)\to[0,1]$ is called fuzzy b-metric on X if the following conditions hold for all $x_1,x_2,x_3\in X$:*

FbM1: $M_b(x_1,x_2,0)=0$

FbM2: $M_b(x_1,x_2,t)=1,\forall\ t>0$ *if and only if* $x_1=x_2$

FbM3: $M_b(x_1,x_2,t)=M_b(x_2,x_1,t)$

FbM4: $M_b(x_1,x_3,b(t+s))\ge M_b(x_1,x_2,t)*M_b(x_2,x_3,s)\ \forall\ t,s\ge0$

FbM5: $M_b(x_1,x_2,.)\colon(0,\infty)\to[0,1]$ *is continuous and* $\lim_{t\to\infty}M_b(x_1,x_2,t)=1$.

Then the triplet $(X,M_b,*)$ is called a fuzzy b-metric space.

Remark 2.3. *The class of fuzzy b-metric spaces is the generalization of class of fuzzy metric spaces because for $b = 1$ in Definition 2.2, a fuzzy b-metric space becomes a fuzzy metric space.*

Example 2.4. *Let (X, d_b) be a b-metric space defined by $d_b(x_1, x_2) = (x_1 - x_2)^2$ with coefficient $b = 2$. Let $M_b\colon X \times X \times [0, \infty) \to [0, 1]$ defined by*

$$M_b(x_1, x_2, t) = e^{-\frac{(x_1 - x_2)^2}{t}}.$$

Then $(X, M_b, *)$ is a fuzzy b-metric space but it is not a fuzzy metric space, where the continuous t-norm is taken as $a * b = ab$.

Following Grabiec [12], the concepts of a convergent sequence and a Cauchy sequence in fuzzy b-metric spaces will be termed as a G-convergent sequence and a G-Cauchy sequence respectively.

Definition 2.5. *[19] A sequence $\{x_n\} \subset X$ of a fuzzy b-metric space $(X, M_b, *)$ is said to be G-convergent if there exits $x \in X$ such that $\lim_{n\to\infty} M_b(x_n, x, t) = 1,\ \forall\ t > 0$.*

Example 2.6. *Let $X = [0, 1]$ and a mapping $M_b\colon X \times X \times [0, \infty) \to [0, 1]$ defined by*

$$M_b(x_1, x_2, t) = \begin{cases} \dfrac{t}{t + (x_1 - x_2)^2} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then $(X, M_b, *)$ is a fuzzy b-metric space. Consider a sequence $\{x_n\}$ in X such that $x_n = \frac{1}{n}\ \forall\ n \in \mathbb{N}$ then clearly x_n G-converges to 0 as follows

$$\lim_{n\to\infty} M_b(x_n, 0, t) = \lim_{n\to\infty} \frac{t}{t + \left(\frac{1}{n} - 0\right)^2} = \lim_{n\to\infty} \frac{t}{t + \frac{1}{n^2}} = 1.$$

Definition 2.7. *[19] Let $(X, M_b, *)$ be a fuzzy b-metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be a G-Cauchy sequence if $\lim_{n\to\infty} M_b(x_n, x_{n+q}, t) = 1$ for all $t > 0$ and $q > 0$.*

If every G-Cauchy sequence is G-convergent in a fuzzy b-metric space, then the space is called a G-complete fuzzy b-metric space.

Example 2.8. *Consider the fuzzy b-metric space given in Example 2.6 and a sequence $x_n = \frac{1}{n}\ \forall\ n \in \mathbb{N}$. Now for all $q \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} M_b(x_n, x_{n+q}, t) = \lim_{n\to\infty} \frac{t}{t + \left(\frac{1}{n} - \frac{1}{n+q}\right)^2} = \frac{t}{t + 0} = 1.$$

Hence $\{x_n\}$ is a G-Cauchy sequence.

Lemma 2.9. *[18] Let $(X, M, *)$ be a complete fuzzy metric space and $M(x_1, x_2, kt) \geq M(x_1, x_2, t)$ for all $x_1, x_2 \in X$, $k \in (0, 1)$ and $t > 0$ then $x_1 = x_2$.*

In 1973, Banach contraction principle was generalized by Geraghty [11]. After that various authors proved many fixed point results by using Geraghty type contractive mappings. Recently, Faraji [9] proved some fixed point theorems in b-metric spaces by using Geraghty type contractions.

In this paper, we establish some fixed point results for Geraghty type contraction in fuzzy b-metric spaces.

3. Main Results

Following [20], for a real number $b > 1$, let F_b denotes the class of all functions $\beta\colon [0,\infty) \to [0, \frac{1}{b})$ satisfying the following condition:

$$\beta(t_n) \to \frac{1}{b} \text{ as } n \to \infty \quad \text{implies} \quad t_n \to 0 \text{ as } n \to \infty.$$

We now establish a fixed point result, analogue to [12, Theorem 1] in the setting of G-complete fuzzy b-metric spaces, as follows:

Theorem 3.1. *Let $(X, M_b, *)$ be a G-complete fuzzy b-metric space with $b \geq 1$. Let $g\colon X \to X$ be a mapping satisfying*

$$M_b(gx_1, gx_2, \beta(M_b(x_1, x_2, t))t) \geq M_b(x_1, x_2, t) \tag{1}$$

$\forall\ x_1, x_2 \in X$ and $\beta \in F_b$. Then g has a unique fixed point.

Proof. *Let $a_0 \in X$ and generate a sequence $\{a_n\}$; $(n \in \mathbb{N})$ by the iterative process $a_n = g^n a_0$. First, note that by successive application of the contractive condition* (1)*, we have for all $n \in \mathbb{N}$ and $t > 0$,*

$$\begin{aligned} M_b(a_n, a_{n+1}, t) &= M_b(ga_{n-1}, ga_n, t) \\ &\geq M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right) \\ &\geq M_b\left(a_{n-2}, a_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t)).\beta(M_b(a_{n-2}, a_{n-1}, t))}\right) \\ &\geq \ldots \geq M_b\left(a_0, a_1, \frac{t}{\beta(M_b(a_{n-1}, a_n, t)).\beta(M_b(a_{n-2}, a_{n-1}, t)).\ldots\beta(M_b(a_0, a_1, t))}\right) \end{aligned}$$

So, we have

$$M_b(a_n, a_{n+1}, t) \geq M_b\left(a_0, a_1, \frac{t}{\beta(M_b(a_{n-1}, a_n, t)).\beta(M_b(a_{n-2}, a_{n-1}, t))\ldots\beta(M_b(a_0, a_1, t))}\right) \tag{2}$$

For any $q \in \mathbb{N}$, writing $t = \frac{t}{q} + \frac{t}{q} + \ldots + \frac{t}{q}$ and using [*FbM*4] *repeatedly,*

$$M_b(a_n, a_{n+p}, t) \geq M_b\left(a_n, a_{n+1}, \frac{t}{qb}\right) * M_b\left(a_{n+1}, a_{n+2}, \frac{t}{qb^2}\right) * \ldots * M_b\left(a_{n+p-1}, a_{n+p}, \frac{t}{qb^{n+p}}\right)$$

Using (2)*, we get*

$$\begin{aligned} M_b(a_n, a_{n+p}, t) \geq\ & M_b\left(a_0, a_1, \frac{t}{qb\beta(M_b(a_{n-1}, a_n, t)).\beta(M_b(a_{n-2}, a_{n-1}, t))\ldots\beta(M_b(a_0, a_1, t))}\right) \\ & * M_b\left(a_0, a_1, \frac{t}{qb^2\beta(M_b(a_n, a_{n+1}, t)).\beta(M_b(a_{n-1}, a_n, t))\ldots\beta(M_b(a_0, a_1, t))}\right) * \ldots \\ & * M_b\left(a_0, a_1, \frac{t}{qb^{n+p}\beta(M_b(a_{n+q}, a_{n+q-1}, t))\ldots\beta(M_b(a_0, a_1, t))}\right) \end{aligned}$$

$$M_b(a_n, a_{n+p}, t) \geq M_b\left(a_0, a_1, \frac{b^{n-1}t}{q}\right) * M_b\left(a_0, a_1, \frac{b^{n-1}t}{q}\right) * \ldots * M_b\left(a_0, a_1, \frac{b^{n-1}t}{q}\right)$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M_b(a_n, a_{n+p}, t) = 1 * 1 * \ldots * 1 = 1$$

This shows that $\{a_n\}$ is a Cauchy sequence. So by G-completeness of the space $(X, M_b, *)$, there exists $z \in X$ such that

$$\lim_{n\to\infty} a_n = z.$$

To show that z is fixed point of g, we proceed as follows:

$$\begin{aligned} M_b(gz, z, t) &\geq M_b\left(gz, ga_n, \frac{t}{2b}\right) * M_b\left(ga_n, z, \frac{t}{2b}\right) \\ &\geq M_b\left(z, a_n, \frac{t}{2b\beta((M_b(z, a_n, t))}\right) * M_b\left(a_{n+1}, a_n, \frac{t}{2b}\right) \\ &\longrightarrow 1 * 1 = 1, \end{aligned}$$

which shows that $gz = z$ is a fixed point.
Uniqueness:
Assume $gw = w$ for some $w \in X$, then

$$\begin{aligned} M_b(w, z, t) &= M_b(gw, gz, t) \\ &\geq M_b\left(w, z, \frac{t}{\beta(M_b(w, z, t))}\right) = M_b\left(gw, gz, \frac{t}{\beta(M_b(w, z, t))}\right) \\ &\geq M_b\left(w, z, \frac{t}{\beta(M_b(w, z, t))^2}\right) \geq \ldots \geq M_b\left(w, z, \frac{t}{\beta(M_b(w, z, t))^n}\right) = M_b\left(w, z, b^n t\right) \\ &\longrightarrow 1 \, as \, n \to \infty \end{aligned}$$

Thus $z = w$. Hence fixed point is unique. □

The following example illustrates Theorem 3.1.

Example 3.2. *Let $X = [0, 1]$. Let $M_b \colon X \times X \times [0, \infty) \to [0, 1]$ defined by $M_b(x_1, x_2, t) = \frac{t}{t + (x_1 - x_2)^2}$. One can easily show that $(X, M_b, *)$ is a G-complete fuzzy b-metric space with $b = 2$. Consider the mapping $g \colon X \to X$ defined by $g(x) = \frac{x}{4(1 + x)}$. Moreover, set the map $\beta \colon [0, 1] \to [0, \frac{1}{2})$ as $\beta(t) = \frac{1}{4}$. Clearly $\beta \in F_2$. Now for all $x_1, x_2 \in X$ and $t > 0$, we have*

$$\begin{aligned} M_b(gx_1, gx_2, \beta(M_b(x_1, x_2, t))t) &= \frac{\beta(M_b(x_1, x_2, t))t}{\beta(M_b(x_1, x_2, t))t + \left(\frac{x_1}{4(1 + x_1)} - \frac{x_2}{4(1 + x_2)}\right)^2} \\ &= \frac{\frac{1}{4}t}{\frac{1}{4}t + \left(\frac{x_1}{4(1 + x_1)} - \frac{x_2}{4(1 + x_2)}\right)^2} \\ &= \frac{\frac{1}{4}t}{\frac{1}{4}t + \frac{1}{4}\left(\frac{x_1}{1 + x_1} - \frac{x_2}{1 + x_2}\right)^2} \\ &= \frac{t}{t + \frac{(x_1 - x_2)^2}{(1 + x_1)^2(1 + x_2)^2}}. \end{aligned}$$

Since

$$\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2} \le (x_1-x_2)^2$$
$$t+\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2} \le t+(x_1-x_2)^2$$
$$\frac{1}{t+\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}} \ge \frac{1}{t+(x_1-x_2)^2}$$
$$\frac{t}{t+\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}} \ge \frac{t}{t+(x_1-x_2)^2}.$$

This implies that

$$M_b(gx_1, gx_2, \beta(x_1, x_2, t)t) \ge M_b(x_1, x_2, t).$$

Note that $g0 = 0$*, so* 0 *is a unique fixed point of* g*.*

Some immediate consequences of Theorem 3.1 are stated below:

Remark 3.3.

1. *Taking* $\beta(a) = k \ \forall \ a \in [0,1]$ *for some* $k \in [0, \frac{1}{b})$*, then Theorem 3.1 becomes the well-known Banach contraction theorem for fuzzy b-metric spaces.*

2. *Similarly setting* $b = 1$ *we get the main result of Grabiec [12, Theorem 5].*

Now we extend the main result of Faraji at.el [9] in our setting as follows;

Theorem 3.4. *Let* $(X, M_b, *)$ *be a G-complete fuzzy b-metric space with* $b \ge 1$*. Let* $g\colon X \to X$ *be a mapping satisfying*

$$M_b(gx_1, gx_2, \beta(M_b(x_1, x_2, t))t) \ge \min\Big\{M_b(x_1, x_2, t), M_b(x_1, gx_1, t), M_b(x_2, gx_2, t), \big(M_b(x_1, gx_2, 2bt) * M_b(x_2, gx_1, 2bt)\big)\Big\} \quad (3)$$

$\forall \ x_1, x_2 \in X$ *and* $\beta \in F_b$*. Then T has a unique fixed point, where the continuous t-norm is taken as* $a * b = \min\{a, b\}$*.*

Proof. Starting the same way as in Theorem 3.1, we have

$$\begin{aligned} M_b(a_n, a_{n+1}, t) &= M_b(ga_{n-1}, ga_n, t) \\ &\ge \min\Bigg\{M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\quad M_b\left(a_{n-1}, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_n, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\quad \left(M_b\left(a_{n-1}, ga_n, \frac{2bt}{\beta(M_b(a_{n-1}, a_n, t))}\right) * M_b\left(a_n, ga_{n-1}, \frac{2bt}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right)\Bigg\} \end{aligned}$$

$$\geq \min\Big\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$
$$M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$
$$\left(M_b\left(a_{n-1},a_{n+1},\frac{2bt}{\beta(M_b(a_{n-1},a_n,t))}\right)*M_b\left(a_n,a_n,\frac{2bt}{\beta(M_b(a_{n-1},a_n,t))}\right)\right)\Big\}$$
$$\geq \min\Big\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$
$$M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),\left(M_b\left(a_{n-1},a_{n+1},\frac{2bt}{\beta(M_b(a_{n-1},a_n,t))}\right)*1\right)\Big\}$$
$$\geq \min\Big\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$
$$M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_{n+1},\frac{2bt}{\beta(M_b(a_{n-1},a_n,t))}\right)\Big\}$$
$$\geq \min\Big\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$
$$M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),\left(M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)*\right.$$
$$\left.M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right)\Big\}.$$

So we have

$$M_b(a_n,a_{n+1},t)\geq \min\Big\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$
$$\left(M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)*M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right)\Big\}. \tag{4}$$

If

$$\min\left\{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}=M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

then (4) implies

$$M_b(a_n,a_{n+1},t)\geq M_b(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}),$$

so there is nothing to prove by Lemma 2.9.
If

$$\min\left\{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}=M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

then from (4) we have

$$M_b(a_n,a_{n+1},t)\geq M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right).$$

Continuing in this way, we get

$$M_b(a_n, a_{n+1}, t) \geq M_b\left(a_0, a_1, \frac{t}{\beta(M_b(a_{n-1}, a_n, t)).\beta(M_b(a_{n-2}, a_{n-1}, t)) \ldots \beta(M_b(a_0, a_1, t))}\right). \tag{5}$$

One can now complete the proof by following the same procedure as used after inequality (2) of Theorem 3.1.
Following is the immediate consequence of Theorem 3.4.

Corollary 3.5. *Consider a G-complete fuzzy metric space $(X, M, *)$. Let g be a self map on X satisfying*

$$M(gx_1, gx_2, \beta(M(x_1, x_2, t))t) \geq \min\Big\{M(x_1, x_2, t), M(x_1, gx_1, t), M(x_2, gx_2, t), \big(M(x_1, gx_2, 2t) * M(x_2, gx_1, 2t)\big)\Big\},$$

for all $x_1, x_2 \in X$ and $\beta \in F_1$. Then g has a unique fixed point, where the continuous t-norm is taken as $a*b = \min\{a, b\}$. □

Theorem 3.6. *Let $(X, M_b, *)$ be a G-complete fuzzy b-metric space with $b \geq 1$. Let $g\colon X \to X$ be a mapping satisfying the condition*

$$M_b(gx_1, gx_2, \beta(M_b(x_1, x_2, t))t) \geq \min\Big\{M_b(gx_1, gx_2, t), M_b(x_1, gx_1, t), M_b(x_2, gx_2, t), M_b(x_1, x_2, t)\Big\}, \tag{6}$$

for all $x_1, x_2 \in X$ and $\beta \in F_b$. Then g has a unique fixed point.

Proof. *Starting by the same way as in Theorem 3.1, we have*

$$\begin{aligned} M_b(a_n, a_{n+1}, t) &= M_b(ga_{n-1}, ga_n, t) \\ &\geq \min\Big\{M_b\left(ga_{n-1}, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad M_b\left(a_n, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\} \\ &\geq \min\Big\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\} \end{aligned}$$

$$M_b(a_n, a_{n+1}, t) \geq \min\Big\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\}. \tag{7}$$

By adopting the same procedure as in Theorem 3.4 after inequality (4)*, we can complete the proof.* □

Remark 3.7. *If we take*

$$\min\Big\{M_b(gx_1, gx_2, t), M_b(x_1, gx_1, t), M_b(x_2, gx_2, t), M_b(x_1, x_2, t)\Big\} = M_b(x_1, x_2, t)$$

in Theorem 3.6, then it reduces to Theorem 3.1.

Following is an immediate consequence of Theorem 3.6.

Corollary 3.8. *Consider a G-complete fuzzy metric space $(X, M, *)$. Let g be a self map on X satisfying the condition*

$$M(gx_1, gx_2, \beta(M(x_1, x_2, t))t) \geq \min\Big\{M(gx_1, gx_2, t), M(x_1, gx_1, t), M(x_2, gx_2, t), M(x_1, x_2, t)\Big\}$$

for all $x_1, x_2 \in X$ and $\beta \in F_1$. Then T has a unique fixed point.

Theorem 3.9. *Let $(X, M_b, *)$ be a G-complete fuzzy b-metric space with $b \geq 1$. Let $g\colon X \to X$ be a mapping satisfying the condition*

$$M_b(gx_1, gx_2, \beta(M_b(x_1, x_2, t))t) \geq \frac{\alpha(x_1, x_2, t)}{\max\{M_b(x_1, gx_1, t), M_b(x_2, gx_2, t)\}}, \tag{8}$$

where

$$\alpha(x_1, x_2, t) = \min\Big\{M_b(gx_1, gx_2, t).M_b(x_1, x_2, t), M_b(x_1, gx_1, t).M_b(x_2, gx_2, t)\Big\}$$

for all $x_1, x_2 \in X$ and $\beta \in F_b$. Then g has a unique fixed point.

Proof. Starting in same way as in Theorem 3.1, we have

$$\begin{aligned} M_b(a_n, a_{n+1}, t) &= M_b(ga_{n-1}, ga_n, t) \\ &\geq \frac{\alpha\left(a_n, a_{n+1}, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)}{\max\left\{M_b\left(a_{n-1}, ga_{n-1}, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_n, ga_n, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\}}. \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} \alpha\Big(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) &= \min\Big\{M_b\left(ga_{n-1}, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad M_b\left(a_{n-1}, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_n, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\} \\ &= \min\Big\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\} \end{aligned}$$

$$\alpha\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right) = M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_n, a_{n+1}, \frac{t}{\beta(a_{n-1}, a_n, t)}\right) \tag{10}$$

Using (10) in (9), we get

$$M_b(a_n, a_{n+1}, t) \geq \frac{M_b\left(a_n, a_{n+1}, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_n, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)}{\max\left\{M_b\left(a_{n-1}, a_n, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_n, a_{n+1}, \dfrac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\}} \tag{11}$$

If

$$\max\left\{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}=M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

then (11) implies

$$M_b(a_n,a_{n+1},t)\geq M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

so there is nothing to prove by Lemma 2.9.
If

$$\max\left\{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}=M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

then from (11), we have

$$\begin{aligned}M_b(a_n,a_{n+1},t)&\geq M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\\&\geq\ldots\geq M_b\left(a_0,a_1,\frac{t}{\beta(M_b(a_{n-1},a_n,t)).\beta(M_b(a_{n-2},a_{n-1},t))\ldots\beta(M_b(a_0,a_1,t))}\right).\end{aligned}$$

By following the same procedure as in Theorem 3.1 after inequality (2), we can complete the proof. □

Following is the immediate consequence of Theorem 3.9.

Corollary 3.10. *Consider a G-complete fuzzy metric space $(X,M,*)$. Let g be a self map on X satisfying the condition*

$$M(gx_1,gx_2,\beta(M(x_1,x_2,t))t)\geq\frac{\alpha(x_1,x_2,t)}{\max\left\{M(x_1,gx_1,t),M(x_2,gx_2,t)\right\}},$$

where

$$\alpha(x_1,x_2,t)=\min\left\{M(gx_1,gx_2,t).M(x_1,x_2,t),M(x_1,gx_1,t).M(x_2,gx_2,t)\right\},$$

for all $x_1,x_2\in X$ and $\beta\in F_1$. Then g has a unique fixed point.

Following result is the generalization of Theorem 2.3 of Alsulami et.al. [2] in the setting of fuzzy b-metric spaces.

Theorem 3.11. *Let $(X,M_b,*)$ be a G-complete fuzzy b-metric space with $b\geq 1$. Let $g\colon X\to X$ mapping satisfying the condition*

$$M_b(gx_1,gx_2,\beta(M_b(x_1,x_2,t))t)\geq\lambda(x_1,x_2,t)*\gamma(x_1,x_2,t),\tag{12}$$

where,

$$\left\{\begin{aligned}\lambda(x_1,x_2,t)&=\min\left\{M_b(gx_1,gx_2,t),M_b(x_1,gx_1,t),M_b(x_2,gx_2,t),M_b(x_1,x_2,t)\right\}\\\gamma(x_1,x_2,t)&=\max\left\{M_b(x_1,gx_2,t),M_b(gx_1,x_2,t)\right\}\end{aligned}\right\}\tag{13}$$

for all $x_1,x_2\in X$ and $\beta\in F_b$. Then g has a unique fixed point, where $a*b=\min(a,b)$.

Proof. *Starting in same way as in Theorem 3.1, we have*

$$M_b(a_n, a_{n+1}, t) = M_b(ga_{n-1}, ga_n, t)$$
$$\geq \lambda\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right) * \gamma\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right). \tag{14}$$

Now

$$\lambda\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) = \min\Big\{M_b\left(ga_{n-1}, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right),$$
$$M_b\left(a_n, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\}$$
$$= \min\Big\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t)})\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right),$$
$$M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\}$$

$$\lambda\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) = \min\left\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\}. \tag{15}$$

Also

$$\gamma\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) = \max\left\{M_b\left(a_{n-1}, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(ga_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\}$$
$$= \max\left\{M_b\left(a_{n-1}, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_n, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\}$$
$$= \max\left\{M_b\left(a_{n-1}, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), 1\right\}$$
$$\gamma\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t)}\Big) = 1 \tag{16}$$

Using (15) and (16) in (14), we have

$$M_b(a_n, a_{n+1}, t) \geq \min\left\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\} * 1$$
$$M_b(a_n, a_{n+1}, t) \geq \min\left\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\right\} \tag{17}$$

By using the the same procedure as in Theorem 3.4 after inequality (4), we can complete the proof. □

Following is the immediate consequence of Theorem 3.11.

Corollary 3.12. *Consider a G-complete fuzzy metric space* $(X, M, *)$*. Let* g *be a self map on* X *satisfying the condition*

$$M(gx_1, gx_2, \beta(M(x_1, x_2, t))t) \geq \lambda(x_1, x_2, t) * \gamma(x_1, x_2, t),$$

where

$$\begin{cases} \lambda(x_1, x_2, t) &= \min\left\{M(gx_1, gx_2, t), M(x_1, gx_1, t), M(x_2, gx_2, t), M(x_1, x_2, t)\right\} \\ \gamma(x_1, x_2, t) &= \max\left\{M(x_1, gx_2, t), M(gx_1, x_2, t)\right\} \end{cases}$$

for all $x_1, x_2 \in X$ *and* $\beta \in F_1$*. Then* g *has a unique fixed point, where* $a * b = \min(a, b)$*.*

Now we establish Theorem 2.10 of Alsulami et.al. [2] in the setting of fuzzy b-metric spaces.

Theorem 3.13. *Let $(X, M_b, *)$ be a G-complete fuzzy b-metric space with $b \geq 1$. Let $g\colon X \to X$ be a mapping satisfying the condition*

$$M_b(gx_1, gx_2, \beta(M_b(x_1, x_2, t))t) \geq \frac{\lambda(x_1, x_2, t) * \gamma(x_1, x_2, t)}{\alpha(x_1, x_2, t)}, \tag{18}$$

where

$$\left.\begin{cases} \lambda(x_1, x_2, t) &= \min\left\{M_b(gx_1, gx_2, t).M_b(x_1, x_2, t), M_b(x_1, gx_1, t).M_b(x_2, gx_2, t)\right\} \\ \gamma(x_1, x_2, t) &= \max\left\{M_b(x_1, gx_1, t).M_b(x_1, gx_2, t), (M_b(x_2, gx_1, t))^2\right\} \\ \alpha(x_1, x_2, t) &= \max\left\{M_b(x_1, gx_1, t), M_b(x_2, gx_2, t)\right\} \end{cases}\right\} \tag{19}$$

$\forall\ x_1, x_2 \in X$ and $\beta \in F_b$. Then g has a unique fixed point.

Proof. By the same way as in Theorem 3.1, we have

$$\begin{aligned} M_b(a_n, a_{n+1}, t) &= M_b(ga_{n-1}, ga_n, t) \\ &\geq \frac{\lambda\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right) * \gamma\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)}{\alpha\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)}. \end{aligned} \tag{20}$$

Now

$$\begin{aligned} \lambda\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) &= \min\Big\{M_b\left(ga_{n-1}, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad M_b\left(a_{n-1}, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_n, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\} \\ &= \min\Big\{M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right)\Big\} \end{aligned}$$

$$\lambda\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) = M_b\left(a_n, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right) \tag{21}$$

$$\begin{aligned} \gamma\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) &= \max\Big\{M_b\left(a_{n-1}, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, ga_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad \Big(M_b\Big(a_n, ga_{n-1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big)\Big)^2\Big\} \\ &= \max\Big\{M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), \\ &\qquad \Big(M_b\Big(a_n, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big)\Big)^2\Big\} \\ &= \max\left\{M_b\left(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right).M_b\left(a_{n-1}, a_{n+1}, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\right), 1\right\} \end{aligned}$$

$$\gamma\Big(a_{n-1}, a_n, \frac{t}{\beta(M_b(a_{n-1}, a_n, t))}\Big) = 1 \tag{22}$$

$$\alpha\Big(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\Big) = \max\left\{M_b\left(a_{n-1},ga_{n-1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_n,ga_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}$$
$$= \max\left\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}. \tag{23}$$

Using (21), (22) and (23) in (20), we have

$$M_b(a_n,a_{n+1},t) \geq \frac{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right).M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)}{\max\left\{M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\}}. \tag{24}$$

If

$$\max\left\{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\} = M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

then (24) implies

$$M_b(a_n,a_{n+1},t) \geq M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

there is nothing to prove by Lemma 2.9.
If

$$\max\left\{M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)\right\} = M_b\left(a_n,a_{n+1},\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right),$$

then from (24), we have

$$M_b(a_n,a_{n+1},t) \geq M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right).$$

Continuing in this way, we get

$$M_b(a_n,a_{n+1},t) \geq M_b\left(a_{n-1},a_n,\frac{t}{\beta(M_b(a_{n-1},a_n,t))}\right)$$
$$\geq \ldots \geq M_b\left(a_0,a_1,\frac{t}{\beta(M_b(a_{n-1},a_n,t)).\beta(M_b(a_{n-2},a_{n-1},t))\ldots\beta(M_b(a_0,a_1,t))}\right).$$

Continuing the same way as in Theorem 3.1 after inequality (2), we can complete the proof. □

Following is an immediate consequence of Theorem 3.13.

Corollary 3.14. *Consider a G-complete fuzzy metric space $(X,M,*)$. Let g be a self map on X satisfying the condition*

$$M(gx_1,gx_2,\beta(M(x_1,x_2,t))t) \geq \frac{\lambda(x_1,x_2,t)*\gamma(x_1,x_2,t)}{\alpha(x_1,x_2,t)},$$

where

$$\left\{\begin{aligned}\lambda(x_1,x_2,t) &= \min\left\{M(gx_1,gx_2,t).M(x_1,x_2,t),M(x_1,gx_1,t).M(x_2,gx_2,t)\right\}\\ \gamma(x_1,x_2,t) &= \max\left\{M(x_1,gx_1,t).M(x_1,gx_2,t),(M(x_2,gx_1,t))^2\right\}\\ \alpha(x_1,x_2,t) &= \max\left\{M(x_1,gx_1,t),M(x_2,gx_2,t)\right\}\end{aligned}\right\}$$

$\forall\ x_1,x_2 \in X$ and $\beta \in F_1$. Then g has a unique fixed point.

4. Applications:

Fixed point theory turns out to be an important tool for studying the existence and uniqueness problems for the soloution of various types of of integral and differential equations, for instance see [1, 9, 16]. In this section, a particular non-linear integral equation has been studied for the existence of the solution as an application of fixed point results proved in the previous section.

Consider $X = C[0, I]$, the set of real valued continuous functions on $[0, I]$ and define a G-complete fuzzy b-metric $M_b\colon X \times X \times [0, \infty) \to [0, 1]$ by

$$M_b(x, y, t) = \begin{cases} e^{-\frac{\sup_{s\in[0,I]} |x(s) - y(s)|^2}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Consider the integral equation

$$x(t) = f(t) + \int_0^I h(t, s)F(t, s, x(s))ds, \tag{25}$$

where $I > 0$ and $f\colon [0, I] \to \mathbb{R}$, $h\colon [0, I] \times [0, I] \to \mathbb{R}$ and $F\colon [0, I] \times [0, I] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 4.1. *Suppose that the following conditions holds:*

(i) for all $t, s \in [0, I]$, $x, y \in X$ and $\beta \in F_b$, we have

$$|F(t, s, x(s)) - F(t, s, y(s))| < \sqrt{\beta(M_b(x, y, t))}|x(s) - y(s)|$$

(ii) for all $t, s \in [0, I]$, $\sup_{s\in[0,I]} \int_0^I (h(t, s))^2 ds \le \frac{1}{I}$.

Then the integral equation (25) *has a solution in X.*

Proof. Let $g : X \to X$ be the integral operator defined by

$$gx(t) = f(t) + \int_0^I h(t, s)F(t, s, x(s))ds, \quad x \in X, \text{ and } \ t, s \in [0, I].$$

For all $x, y \in X$ and by using Conditions (i) and (ii), we have

$$\begin{aligned} M_b(gx, gy, \beta(M_b(x, y, t))t) &= e^{-\frac{\sup_{s\in[0,I]} |gx(t) - gy(t)|^2}{\beta(M_b(x, y, t))t}} \\ &= e^{-\frac{\sup_{s\in[0,I]} |\int_0^I h(t, s)F(t, s, x(s))ds - \int_0^I h(t, s)F(t, s, y(s))ds|^2}{\beta(M_b(x, y, t))t}} \\ &= e^{-\frac{\sup_{s\in[0,I]} |\int_0^I h(t, s)\{F(t, s, x(s)) - F(t, s, y(s)\}ds|^2}{\beta(M_b(x, y, t))t}} \\ &\ge e^{-\frac{\sup_{s\in[0,I]} |\int_0^I (h(t, s))^2 ds| \int_0^I |F(t, s, x(s)) - F(t, s, y(s)|^2 ds}{\beta(M_b(x, y, t))t}} \end{aligned}$$

$$\geq e^{-\frac{\sup_{s\in[0,I]}\frac{1}{I}\int_0^I\{\sqrt{\beta(M_b(x,y,t))}|x(s)-y(s)|\}^2 ds}{\beta(M_b(x,y,t))t}}$$

$$\geq e^{-\frac{\sup_{s\in[0,I]}\beta(M_b(x,y,t))|x(s)-y(s)|^2}{\beta(M_b(x,y,t))t}} = e^{-\frac{\sup_{s\in[0,I]}|x(s)-y(s)|^2}{t}}$$

$$\Rightarrow \quad M_b(gx, gy, \beta(M_b(x,y,t))t) \geq M_b(x,y,t).$$

Since all the conditions of Theorem 3.1 are satisfied and hence g has a fixed point. That is the integral equation (25) has a solution. □

5. Conclusion

In this paper, by using Geraphty type contraction we have generalized the main result of Grabeic [12] for G-complete fuzzy b-metric space. The result is illustrated by an example. Moreover, we have also established analogue of the main results of Faraji et al [9] and Alsulami et al [2] in the setting of G-complete fuzzy b-metric space. The existence problem for the solution of a nonlinear integral equation is also presented as an application of our main result. Further, the presented corollaries indicate that the theorems established in this work generalize many existing results in the literature.

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