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Weighted Orlicz Algebras on Hypergroups

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Abstract. Let K be a hypergroup, w be a weight function and let (Φ, Ψ) be a complementary pair of Young functions. We consider the weighted Orlicz space $L^\Phi_w(K)$ and investigate some of its algebraic properties under convolution. We also study the existence of an approximate identity for the Banach algebra $L^\Phi_w(K)$. Further, we describe the maximal ideal space of the convolution algebra $L^\Phi_w(K)$ for a commutative hypergroup K.

1. Introduction

Orlicz spaces are vast generalizations of L^p spaces. They can also contain Sobolev spaces as subspaces. It is well known that for a locally compact group G, $L^p(G)$ $(1 < p < \infty)$ is an algebra under the convolution product precisely when G is compact [21] (see [1] and [9], which obtain a version of conditions for an Orlicz space L^Φ on a group and a hypergroup to be an algebra). However a weighted $L^p(G)$ space may be a Banach algebra for general locally compact groups as shown in [26] and [10]. Recently in [13] A. Osançlıol and S. Öztop considered weighted Orlicz algebras over locally compact groups and studied their properties, extending in part the results of [10]; see also [14, 15]. They found sufficient conditions under which a weighted Orlicz space becomes an algebra and studied its properties such as the existence of an approximate identity and the maximal ideal space of the algebra when the underlying group G is abelian.

Hypergroups are generalizations of locally compact groups. They are locally compact spaces whose regular complex Borel measures form an algebra which has properties similar to the convolution algebra $(M(G), *)$ of a locally compact group G. All locally compact groups are hypergroups. They have applications in analysis, probability and approximation theory [2]. The theory of hypergroups was initiated by C. F. Dunkl [4], R. I. Jewett [8] and R. Spector [22]. In the literature there is some variation in the precise definition of a hypergroup. We consider hypergroups in the sense of R. I. Jewett [8].

In this paper, we replace the locally compact group G with a hypergroup K and develop some algebraic properties of weighted Orlicz spaces $L^\Phi_w(K)$ with a weight function w. We transfer some results to hypergroups such as existence of an approximate identity and the description of the maximal ideal space. However, the method of proof for the generalization is significantly different from that of the locally compact group case.

As an application of our results, we also obtain certain weighted Banach function spaces on hypergroups generated by Young functions, such as Zygmund spaces and weighted L^p spaces.

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This paper is organized as follows. In Section 2 we introduce the notation and present some preliminaries regarding hypergroups and Orlicz spaces. In Section 3 we introduce weighted Orlicz spaces $L^\Phi_w(K)$ on a hypergroup K and prove that $L^\Phi_w(K)$ is a Banach algebra with respect to convolution under some conditions. In Section 4 we show that if $L^\Phi_w(K)$ is an algebra, then it has an approximate identity. In particular, we prove that for a discrete hypergroup K, it has an identity. In Section 5 we describe the maximal ideal space of $L^\Phi_w(K)$. Finally, we prove that a left invariant closed linear subspace of $L^\Phi_w(K)$ is a closed left ideal of it. On the other hand, our results may be considered for the unweighted case which are also new.

2. Preliminaries

In this section, we recall some definitions and facts that will be used in the rest of this paper.

2.1. Locally Compact Hypergroups

Let K be a locally compact Hausdorff space. We denote by $\mathcal{M}(K)$ the space of all complex Radon measures on K and by $\mathcal{M}^+(K)$ the set of all nonnegative measures in $\mathcal{M}(K)$. For a Borel measurable function $f : K \to \mathbb{C}$ and $\mu \in \mathcal{M}(K)$, $\mathrm{supp}(f)$ and $\mathrm{supp}(\mu)$ denote the support of f and μ, respectively. The Dirac measure at $x \in K$ is denoted by δ_x.

Definition 2.1. *Let K be a locally compact Hausdorff space. Suppose there exists a positive continuous mapping $(\mu, \nu) \mapsto \mu * \nu$ from $\mathcal{M}(K) \times \mathcal{M}(K)$ into $\mathcal{M}(K)$ such that*

1. *$(\mathcal{M}(K), +, *)$ is an algebra;*
2. *for all $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support;*
3. *there exists an element $e \in K$ such that $\delta_e * \delta_x = \delta_e = \delta_x * \delta_e$ for all $x \in K$;*
4. *there exists a topological involution $x \mapsto x^-$ from K onto K such that for all $x, y \in K$ and $f \in C_0(K)$, we have $(\delta_x * \delta_y)(f^-) = (\delta_{y^-} * \delta_{x^-})(f)$, where $f^-(t) := f(t^-)$ for each $t \in K$;*
5. *$e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = x^-$;*
6. *the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space $\boldsymbol{C}(K)$ of compact subsets of K is continuous, where $\boldsymbol{C}(K)$ is equipped with the topology whose subbasis is given by all*

$$\boldsymbol{C}_{U,V} := \{A \in \boldsymbol{C}(K) \ : \ A \cap U \neq \emptyset \text{ and } A \subseteq V\},$$

   in which U and V are open subsets of K.

Then $K = (K, *, ^-, e)$ is called a locally compact hypergroup *(or simply a* hypergroup*).*

A hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$.

A nonzero nonnegative regular measure m on K is called a (left) Haar measure if for all $x \in K$, $\delta_x * m = m$. It has not been proved that *every* hypergroup has a Haar measure. However, commutative hypergroups, compact hypergroups, discrete hypergroups, and double-coset hypergroups have Haar measures (see [8, 23]). Throughout this paper, we assume that K is a hypergroup with a Haar measure m. For each $1 \le p < \infty$ we write $L^p(K) = L^p(K, m)$. Let f and g be Borel functions on K. For any $x, y \in K$ we denote

$$f_x(y) := f(x * y) := \int_K f d(\delta_x * \delta_y) \quad \text{and} \quad (f * g)(x) := \int_K f(y) g(y^- * x) dm(y).$$

If $A, B \subseteq K$ we put

$$A^- := \{x^- : x \in A\} \quad \text{and} \quad A * B := \bigcup_{x \in A, y \in B} \mathrm{supp}(\delta_x * \delta_y).$$

Weight functions on hypergroups have been studied in several papers. F. Ghahramani and A. R. Medghalchi in [6, 7] studied compact multipliers on weighted hypergroup algebras and W. R. Bloom and P. Ressel in [3] obtained a Bochner representation for w-bounded positive definite functions on a commutative hypergroup, where w is a special weight function. See also [11, 16, 24].

Definition 2.2. *Let K be a hypergroup. A continuous function $w : K \to (0, \infty)$ is called a* weight function *if for all $x, y \in K$,*

$$w(x * y) \leq w(x)w(y). \tag{1}$$

Remark 2.3. *Let w be a weight function on a hypergroup K. The set of all complex-valued Borel measurable functions f on K such that $wf \in L^1(K)$ is denoted by $L^1_w(K)$. As usual, two functions $f, g \in L^1_w(K)$ are identified if and only if $f = g$ m-a.e. The norm of each $f \in L^1_w(K)$ is defined by*

$$\|f\|_{1,w} := \|wf\|_1 = \int_K w|f|\, dm.$$

Because of the relation (1)*, for all $f, g \in L^1_w(K)$ we have $\|f * g\|_{1,w} \leq \|f\|_{1,w}\|g\|_{1,w}$, and so $L^1_w(K)$ is a Banach algebra.*

2.2. Orlicz Spaces

In this subsection, we recall some basic definitions and facts about Orlicz spaces; see [17, 18] as two main monographs on this topic. A non-zero convex function $\Phi : [0, \infty) \to [0, \infty]$ is called a *Young function* if $\Phi(0) = \lim_{x\to 0^+} \Phi(x) = 0$ and $\lim_{x\to+\infty} \Phi(x) = +\infty$. For a Young function Φ, the *complementary function* Ψ of Φ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\} \qquad (y \geq 0).$$

It is easy to check that Ψ is also a Young function. Moreover, if Ψ is the complementary function of Φ, then Φ is the complementary of Ψ and (Φ, Ψ) is called *a complementary pair*. For complementary functions Φ and Ψ, we have the *Young inequality*

$$xy \leq \Phi(x) + \Psi(y)$$

for $x, y \geq 0$. We always consider a pair of complementary Young functions (Φ, Ψ) with both Φ and Ψ continuous and strictly increasing.

Let Φ be a Young function and $\mathcal{B}(K)$ denote the set of all equivalence classes of Borel measurable complex-valued functions on K with respect to the Haar measure m. Then the *Orlicz space* $L^\Phi(K)$ is defined by

$$L^\Phi(K) := \left\{ f \in \mathcal{B}(K) : \int_K \Phi(\lambda|f(x)|)\, dm(x) < +\infty \text{ for some } \lambda > 0 \right\},$$

and is a Banach space under the *Orlicz* norm defined by

$$\|f\|_\Phi := \sup\left\{ \int_K |fv|\, dm : \ v : K \to \mathbb{C} \text{ is measurable and } \int_K \Psi(|v|)\, dm \leq 1 \right\},$$

where Ψ is the complementary function of Φ [17, Chapter III, Proposition 11]. One can also define the *Luxemburg* norm on $L^\Phi(K)$ by

$$\|f\|^\circ_\Phi := \inf\left\{ \lambda > 0 : \int_K \Phi\Big(\frac{1}{\lambda}|f(x)|\Big)\, dm(x) \leq 1 \right\}.$$

These two norms are equivalent: $\|\cdot\|^\circ_\Phi \leq \|\cdot\|_\Phi \leq 2\|\cdot\|^\circ_\Phi$. Also, it is known that

$$\|f\|^\circ_\Phi \leq 1 \iff \int_K \Phi(|f(x)|)\, dm(x) \leq 1.$$

A Young function Φ satisfies the Δ_2-*condition* (and we write $\Phi \in \Delta_2$) if there exist constants $M > 0$ and $x_0 \geq 0$ such that $\Phi(2x) \leq M\Phi(x)$ for all $x \geq x_0$. For a complementary pair of Young functions (Φ, Ψ), if $\Phi \in \Delta_2$, then the dual space of $L^\Phi_w(K)$ is $L^\Psi_w(K)$.

If Φ satisfies in Δ_2-condition, then for each sequence $(f_n) \subseteq L^\Phi(K)$ and $f \in L^\Phi(K)$, f_n converges to f in the $\|\cdot\|_\Phi$ norm if and only if for some $\lambda > 0$,

$$\lim_{n\to\infty} \int_K \Phi\left(\lambda|f_n(t) - f(t)|\right)\, dm(t) = 0.$$

Definition 2.4. *Let Φ be a Young function and w be a weight function on K. Then the corresponding weighted Orlicz space $L^{\Phi}_w(K)$ is defined by*

$$L^{\Phi}_w(K) := \left\{ f : K \to \mathbb{C} : fw \in L^{\Phi}(K) \right\}.$$

For each $f \in L^{\Phi}_w(K)$ we put $\|f\|_{\Phi,w} := \|fw\|_{\Phi}$.

Our results can be applied to important Banach function spaces generated by various Young functions:

1. For $\Phi(x) = \frac{x^p}{p}$ with $1 < p < \infty$, we obtain weighted L^p spaces.
2. For $x \in (1, \infty)$, $\Phi(x) = x^p(\ln(e + x))^{\alpha}$ with $1 \le p < \infty$ and $\alpha \in \mathbb{R}$, we recover the weighted logarithmic Zygmund spaces $L^p(\ln L)^{\alpha}$.
3. For $x \in (1, \infty)$, $\Phi(x) = \exp(x^{\alpha})$ and $\alpha > 0$, we obtain weighted exponential Zygmund spaces $\exp L^{\alpha}$

These Zygmund spaces are studied in [12] for a finite measure.

In the following remark, we give weight functions for some special hypergroups.

Remark 2.5. *Let K be a hypergroup. A continuous function $\alpha : K \to (0, \infty)$ is called a positive* semicharacter *if $\alpha(x * y) = \alpha(x)\alpha(y)$ and $\alpha(x^-) = \alpha(x)$ for all $x, y \in K$. Easily, one can see that if α is a positive semicharacter of K and $c \ge 1$, then $c\alpha$ is a weight function on K. See [19, 25] for more discussion about positive semicharacters. For some classes of Sturm-Liouville hypergroups* $([0, \infty), *(A))$ *corresponding to a given Sturm-Liouville function A (see [2, Sec. 3.5]), if $k > \rho$, then the function α_k satisfying*

$$L_A \alpha_k = (\rho^2 - k^2)\alpha_k, \quad \alpha_k(0) = 1, \quad \alpha_k'(0) = 0$$

is a positive semicharacter of $([0, \infty), *(A))$ *[2, Page 223], where*

$$\rho := \frac{1}{2} \lim_{x \to \infty} \frac{A'}{A}(x) \quad \text{and} \quad L_A := -\frac{d^2}{dx^2} - \frac{A'}{A} \frac{d}{dx}.$$

For another example, fix a natural number n. The orthogonal group $O(n)$ naturally acts on the additive group $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, put $\tilde{x} := \{\gamma x : \gamma \in O(n)\}$. Then by [8, 8.3A] (see also [19, 2.1(2)]), the space $\mathcal{K} := \{\tilde{x} : x \in \mathbb{R}^n\}$ is an orbit hypergroup with the convolution

$$(\delta_{\tilde{x}} * \delta_{\tilde{y}})(f) := \int_{O(n)} f(\widetilde{\gamma x + y})\, d\sigma(\gamma) \quad (f \in C_0(K))$$

and the involution $\tilde{x}^- := \widetilde{-x}$, where $x, y \in \mathbb{R}^n$ and σ is the normalized left Haar measure of $O(n)$. By [19, 2.1(2)], for each $a \in \mathbb{R}^n$, the function $\alpha : \mathcal{K} \to (0, \infty)$ defined by

$$\alpha(\tilde{x}) := \int_{O(n)} \exp(\langle a, \gamma x \rangle)\, d\sigma(\gamma) \quad (x \in \mathbb{R}^n)$$

is a positive semicharacter of $\mathcal{K}$, and so as above, for each $c \ge 1$, $c\alpha$ is a weight function for $\mathcal{K}$.

3. Weighted Orlicz algebras

In the sequel, w is a weight function on a hypergroup K and m is a (left) Haar measure on K. If $\Phi \in \Delta_2$, then $C_c(K)$ is dense in $L^{\Phi}(K)$ [17], and similarly, $C_c(K)$ is dense in $L^{\Phi}_w(K)$. For each Borel measurable function $f : K \to \mathbb{C}$ and $x, y \in K$ we define $L_x f(y) := f(x^- * y) = \int_K f\, d(\delta_{x^-} * \delta_y)$.

Definition 3.1. *The center of a hypergroup K is defined by*

$$Z(K) := \{x \in K : \delta_x * \delta_{x^-} = \delta_e = \delta_{x^-} * \delta_x\}.$$

This concept was introduced by Jewett and Dunkl [4, 8] and then studied by Ross [20]. We note that if K is a locally compact group, then $Z(K) = K$.

We recall the following example from [20].

Example 3.2. *Let G be a locally compact group such that its inner automorphisms group, $Inn(G)$, is a compact group with a normalized Haar measure σ. For every $x \in G$, let $[x] := \{s(x) : s \in Inn(G)\}$ and $G_I := \{[x] : x \in G\}$. Then by [8, 8.3], G_I is a hypergroup with the convolution $(\delta_{[x]} * \delta_{[y]})(\phi) := \int_{Inn(G)} \phi([s(x)y])d\sigma(s)$ and the involution $[x]^- := [x^{-1}]$, where $\phi \in C_0(G_I)$. We have $Z(G_I) = \{[z] : z \in Z\}$, where $Z := \{x \in G : \text{for each } y \in G, xy = yx\}$.*

Lemma 3.3. *Let w be a weight function on K.*

1. *For all $f \in L^\Phi_w(K)$ and $x \in Z(K)$, we have $L_x f \in L^\Phi_w(K)$ and $\|L_x f\|_{\Phi,w} \le w(x)\|f\|_{\Phi,w}$;*
2. *If $\Phi \in \Delta_2$, then for all $f \in C_c(K)$, the mapping $x \mapsto L_x f$ from K into $L^\Phi_w(K)$ is continuous at the identity e.*
3. *If $\Phi \in \Delta_2$, then for all $f \in L^\Phi_w(K)$, the mapping $x \mapsto L_x f$ from $Z(K)$ into $L^\Phi_w(K)$ is continuous at the identity e.*

Proof. (1) Let $f \in L^\Phi_w(K)$ and $x \in Z(K)$. Since m is a left Haar measure, for each function v with $\int_K \Psi(|v|)\, dm \le 1$ we have

$$\int_K \Psi(|L_{x^-} v|)\, dm = \int_K \Psi(|v|)\, dm \le 1,$$

and

$$\begin{aligned} \|vwL_x f\|_1 &= \int_K |f(x^- * y)|\, |v(y)| w(y)\, dm(y) \\ &= \int_K |f(y)|\, |v|(x * y) w(x * y)\, dm(y) \\ &\le w(x) \int_K |f(y)|\, |v|(x * y) w(y)\, dm(y) \\ &\le w(x) \int_K |f(y)|\, L_{x^-}|v|(y) w(y)\, dm(y) \\ &\le w(x) \int_K |f(y)|\, |L_{x^-} v|(y) w(y)\, dm(y). \end{aligned}$$

So

$$\begin{aligned} \|L_x f\|_{\Phi,w} &= \|wL_x f\|_\Phi \\ &= \sup\left\{\|vwL_x f\|_1 : \int_K \Psi(|v|)\, dm \le 1\right\} \\ &\le \sup\left\{w(x) \int_K |f(y)|\, |L_{x^-} v|(y) w(y)\, dm(y) : \int_K \Psi(|v|)\, dm \le 1\right\} \\ &\le w(x)\|f\|_{\Phi,w}. \end{aligned}$$

(2) First let $f \in C_c(K)$ and $E := \operatorname{supp} f$. We show that the mapping $x \mapsto L_x f$ is continuous at e. Let F be a compact neighborhood of e in K. For each $x \in F$, if $L_x f(y) = \int_K f\, d(\delta_{x^-} * \delta_y) \ne 0$, then there is some $t \in E \cap (\{x^-\} * \{y\})$ and so by [8, 4.1B], $y \in \{x\} * E$. This implies that $\operatorname{supp} L_x f \subseteq \{x\} * E \subseteq F * E$. Put $S := F \cup (F * E)$. Hence, by [8, 3.2B], $S \subseteq K$ is a compact set and we have $L^\Psi(S) \subseteq L^1(S)$, where Ψ is the complementary function of Φ. By the open mapping theorem, there is some $C > 0$ such that for all function v with $\int_K \Psi(|v|)\, dm \le 1$,

$$\|v\chi_S\|_1 \le C\|v\chi_S\|_\Psi \le 2C\|v\chi_S\|^\circ_\Psi \le 2C\|v\|^\circ_\Psi \le 2C.$$

By continuity of w and compactness of S, there exists some $B > 0$ such that $w \le B$ on S. Since $f \in C_c(K)$, by [8, 4.3B], for each $\epsilon > 0$, there is a symmetric neighborhood V of e such that $V \subseteq F$, and for all $x, y \in K$.

If $(\delta_x * \delta_{y^-})(V) > 0$, then $|f(x) - f(y)| < \epsilon/2CB$. So, if $(\{x\} * \{y^-\}) \cap V \neq \emptyset$ (and equivalently if $x \in V * \{y\}$), then $|f(x) - f(y)| < \epsilon/2CB$. Hence, for each $x \in V$ and $y \in K$, if $t \in \{x^-\} * \{y\}$, then $t \in V * \{y\}$, and so $|f(t) - f(y)| < \epsilon/2CB$ and we have

$$|L_x f(y) - f(y)| \leq \int_S |f(t) - f(y)|\, d(\delta_{x^-} * \delta_y)(t) \leq \frac{\epsilon(\delta_{x^-} * \delta_y)(S)}{2CB} \leq \frac{\epsilon}{2CB}$$

since $\delta_{x^-} * \delta_y$ is a probability measure. Then for each $x \in V$, we have

$$\begin{aligned}
\|L_x f - f\|_{\Phi,w} &= \sup\left\{\|(L_x f - f)vw\|_1 : \int_K \Psi(|v|)\, dm \leq 1\right\} \\
&\leq B\|L_x f - f\|_\infty \sup\left\{\int_S |v|\, dm : \int_K \Psi(|v|)\, dm \leq 1\right\} \\
&< \epsilon.
\end{aligned}$$

(3) follows from (1) and (2). □

Let α be a positive semicharacter of a commutative hypergroup $(K, *, \cdot^-, e)$. Then, by [25, Theorem 2.2], the space K equipped with a convolution defined by

$$\delta_x \bullet \delta_y := \frac{1}{\alpha(x)\alpha(y)} \alpha\,(\delta_x * \delta_y), \quad (x, y \in K)$$

and $.^-$ as involution, is a commutative hypergroup called *deformation* of $(K, *)$. In the following result, we give a sufficient condition, related to the above convolution $\bullet$, for $L^\Phi_\alpha(K, *)$ to be translation-invariant. In this statement the set of all functions v on K with $\int_K \Psi(v)\, dm \leq 1$ is denoted by Γ_Ψ.

Proposition 3.4. *Let α be a positive semicharacter on a commutative hypergroup $(K, *)$, and the convolution $\bullet$ be defined as above. Let for each $v \in \Gamma_\Psi$ and $x \in K$ we have $L^\bullet_x v \in \Gamma_\Psi$, where $L^\bullet_x v(y) := v(x^- \bullet y)$ for all $y \in K$. Then, for each $f \in L^\Phi_\alpha(K, *)$ and $x \in K$, $\|L_x f\|_{\Phi,\alpha} \leq \alpha(x)\, \|f\|_{\Phi,\alpha}$.*

Proof. Let $f \in L^\Phi_\alpha(K, *)$ and $x \in K$. Then, we have

$$\begin{aligned}
\|L_x f\|_{\Phi,\alpha} &= \sup\left\{\int_K |L_x f\ v|\alpha\, dm : v \in \Gamma_\Psi\right\} \\
&= \sup\left\{\int_K |f(y)|\, |v\alpha|(x * y)\, dm(y) : v \in \Gamma_\Psi\right\} \\
&= \sup\left\{\int_K |f(y)|\, |v\alpha|(x * y)\, dm(y) : v \in \Gamma_\Psi\right\} \\
&= \sup\left\{\int_K |f(y)|\, |v|(x \bullet y)\alpha(x)\alpha(y)\, dm(y) : v \in \Gamma_\Psi\right\} \\
&= \alpha(x) \sup\left\{\int_K |f(y)|\, (L_{x^-,\bullet}|v|)(y)\alpha(y)\, dm(y) : v \in \Gamma_\Psi\right\} \\
&\leq \alpha(x) \sup\left\{\int_K |f(y)|\, |v(y)|\alpha(y)\, dm(y) : v \in \Gamma_\Psi\right\} \\
&= \alpha(x)\, \|f\|_{\Phi,\alpha},
\end{aligned}$$

because Γ_Ψ is translation-invariant under the convolution $\bullet$. □

The next theorem gives a condition for $L^\Phi_w(K)$ to be a left $L^1_w(K)$-module

Theorem 3.5. *Suppose $L^\Phi_w(K) \subseteq L^1_w(K)$ and for each $y \in K$ and $f \in L^\Phi_w(K)$, $\|L_y f\|_{\Phi,w} \leq w(y)\|f\|_{\Phi,w}$. Then $L^\Phi_w(K)$ is a Banach algebra with respect to the usual convolution. Further, the algebra $L^\Phi_w(K)$ is commutative if K is commutative.*

Proof. Let $L^\Phi_w(K) \subseteq L^1_w(K)$ and $f, g \in L^\Phi_w(K)$. Then for each function v with $\int_K \Psi(|v|)\, dm \leq 1$, we have

$$\begin{aligned}
\|(g * f)wv\|_1 &= \int_K |(g * f)(x)v(x)|w(x)\, dm(x) \\
&\leq \int_K \int_K |f(y^- * x)||g(y)||v(x)|w(x) dm(y) dm(x) \\
&= \int_K \left(\int_K |f(y^- * x)|\, |v(x)|w(x)\, dm(x) \right) |g(y)| dm(y) \\
&= \int_K \left(\int_K |f(y^- * x)|\, |v(x)|w(x)\, dm(x) \right) w(y)^{-1}|g(y)|w(y) dm(y).
\end{aligned}$$

So

$$\begin{aligned}
\|g * f\|_{\Phi,w} &= \|(g * f)w\|_\Phi \\
&= \sup \left\{ \|(g * f)wv\|_1 : \int_K \Psi(|v|)\, dm \leq 1 \right\} \\
&\leq \sup \Big\{ \int_K \left(\int_K |f(y^- * x)|\, |v(x)|w(x)\, dm(x) \right) w(y)^{-1}|g(y)|w(y) dm(y) : \\
&\qquad\qquad \int_K \Psi(|v|)\, dm \leq 1 \Big\} \\
&\leq \int_K \|L_y f\|_{\Phi,w} w(y)^{-1}|g(y)|w(y) dm(y) \\
&\leq \|g\|_{1,w}\|f\|_{\Phi,w}.
\end{aligned}$$

For each $f \in L^\Phi_w(K)$ we define $\|f\| := \|f\|_{1,w} + \|f\|_{\Phi,w}$. Then $(L^\Phi_w(K), \|\cdot\|)$ is a Banach space and the identity mapping from $(L^\Phi_w(K), \|\cdot\|)$ to $(L^\Phi_w(K), \|\cdot\|_{\Phi,w})$ is continuous and one-to-one. So by the open mapping theorem, there is a constant $c > 0$ such for each $f \in L^\Phi_w(K)$,

$$\|f\|_{1,w} \leq \|f\| \leq c\|f\|_{\Phi,w}.$$

By renorming $L^\Phi_w(K)$ we can assume that $c = 1$, and so for each $f \in L^\Phi_w(K)$ we have the inequality

$$\|f * g\|_{\Phi,w} \leq \|f\|_{\Phi,w}\|g\|_{\Phi,w}.$$

The commutativity of $L^\Phi_w(K)$ immediately follows from that of K. □

4. Approximate Identities

In this section we transfer the results of [13] concerning the existence of an approximate identity for the weighted L^Φ algebra of a locally compact group to the hypergroup case. We assume that Φ is a Young function with $\Phi \in \Delta_2$.

Theorem 4.1. *Let $L^\Phi_w(K)$ be a weighted Orlicz algebra and V be a compact neighborhood of e in K. Let $\mathcal{U}$ be a neighborhood basis of e in K such that each $U \in \mathcal{U}$ satisfies $U \subseteq V$. Then there exists a net $(e_U)_{U\in\mathcal{U}}$ consisting of compactly supported functions in $L^\Phi_w(K)$ such that for each $f \in C_c(K)$, $e_U * f \to f$ in $L^\Phi_w(K)$.*

Proof. Under the hypothese, for each $U \in \mathcal{U}$ we put $e_U := m(U)^{-1}\chi_U$. There is a $B > 0$ such that $w \leq B$ on V. Then for each $U \in \mathcal{U}$,

$$\int_K e_U w|v|dm = m(U)^{-1}B \int_K \chi_U|v|dm \leq m(U)^{-1}B\|\chi_U\|_\Phi < \infty,$$

where $v \in L^\Psi(K)$ with $\int_K \Psi(|v|)dm$. So $\|e_U\|_{\Phi,w} < \infty$ i.e. $e_U \in L^\Phi_w(K)$.

Let $f \in L^\Phi_w(K)$. For each $U \in \mathcal{U}$ and $v \in L^\Psi(K)$ with $\int_K \Psi(|v|)dm \leq 1$, we have

$$\begin{aligned}(e_U * f - f)(x) &= \int_K \frac{\chi_U(y)}{m(U)} f(y^- * x)dm(y) - f(x)\\ &= m(U)^{-1}\int_U \Big(L_y f(x) - f(x)\Big)dm(y)\end{aligned}$$

and so

$$\begin{aligned}\int_K |(e_U * f - f)wv|dm &\leq m(U)^{-1}\int_K\int_U \Big|L_y f(x) - f(x)\Big| w(x)|v(x)|dm(y)dm(x)\\ &= m(U)^{-1}\int_U \left(\int_K \Big|(L_y f - f)(x)\Big| w(x)|v(x)|dm(x)\right)dm(y)\\ &\leq m(U)^{-1}\int_U \|L_y f - f\|_{\Phi,w}dm(y).\end{aligned}$$

Hence,

$$\|e_U * f - f\|_{\Phi,w} \leq m(U)^{-1}\int_U \|L_y f - f\|_{\Phi,w}dm(y).$$

By Lemma 3.3, for each $\varepsilon > 0$ there exists a $W \in \mathcal{U}$ such that for all $y \in W$, $\|L_y f - f\|_{\Phi,w} < \varepsilon$. Thus for each $U \in \mathcal{U}$ with $U \subseteq W$, we have $\|e_U * f - f\|_{\Phi,w} < \varepsilon$.
□

Proposition 4.2. *If K is a discrete hypergroup, then $L^\Phi_w(K) \subseteq L^\infty_w(K)$.*

Proof. Let $f \in L^\Phi_w(K)$. By the definition of Orlicz space, for some $\alpha > 0$ we have $A := \sum_{t\in K}\Phi(\alpha|f(t)|w(t))\frac{1}{(\delta_{t^-}*\delta_t)(\{e\})} < \infty$. Also, for each $x \in K$, $0 < (\delta_x * \delta_{x^-})(\{e\}) \leq 1$. So,

$$\Phi(\alpha|f(x)|w(x)) \leq \Phi(\alpha|f(x)|w(x))\frac{1}{(\delta_{x^-} * \delta_x)(\{e\})} \leq A$$

for all $x \in K$. Since $\lim_{x\to\infty}\Phi(x) = \infty$, the function $\alpha f w$ is bounded on K i.e., $f \in L^\infty_w(K)$. □

Theorem 4.3. *Let K be first countable. If the weighted Orlicz algebra $L^\Phi_w(K)$ has an identity in $C_c(K)$, then K is discrete.*

Proof. Fix a compact neighborhood V of e in K, let $h \in C_c(K)$ be the identity element of $L^\Phi_w(K)$, and let $\varepsilon > 0$. By Theorem 4.1, there is an $f \in L^\Phi_w(K)$ such that $\|f * h - h\|_{\Phi,w} < \varepsilon$ and outside a neighborhood $U \subseteq V$ of e, $f = 0$. So, since h is the identity, we have

$$\begin{aligned}\varepsilon > \|f - h\|_{\Phi,w} &\geq \int_{K\setminus U} |f - h|\,|v|\,w\,dm\\ &= \int_{K\setminus U} |h|\,|v|\,w\,dm\\ &\geq \int_{K\setminus V} |h|\,|v|\,w\,dm,\end{aligned}$$

where $v \in L^{\Psi}(K)$ with $\int_K \Psi(|v|)dm \leq 1$. We consider $u \in L^{\Psi}(K)$ such that $u \neq 0$ on $K\backslash V$, and $\int_K \Psi(|u|)dm \leq 1$. Since $\varepsilon > 0$ is arbitrary, we have $\int_{K\backslash V} |h|\,|u|\,w\,dm = 0$, and so $|h|\,|u|\,w\chi_{K\backslash V} = 0$ m–a.e. Then, since w and u are non-zero on $K\backslash V$, we have $h = 0$ m–a.e on $K\backslash V$. Let $(V_n)_{n=1}^{\infty}$ be a countable neighborhood basis at e. For each $n \in \mathbb{N}$, we put $A_n := \{x \in K\backslash V_n : h(x) \neq 0\}$. Then, $m(A_n) = 0$, and so setting $A := \cup_{n\in\mathbb{N}} A_n$, we have $m(A) = 0$, and if we redefine $h = 0$ on A, then $\mathrm{supp} h \subseteq \{e\}$. Therefore $m(\{e\}) > 0$, and K is discrete. □

Theorem 4.4. *If K is a discrete hypergroup, then $\chi_{\{e\}}$ is the identity of $L^{\Phi}_w(K)$.*

Proof. Let K be a discrete hypergroup. Then $\chi_{\{e\}} \in L^{\Phi}_w(K)$, and by [8, 7.1A], the measure m with $m(\{x\}) = \frac{1}{(\delta_{x^-} * \delta_x)(\{e\})}$ $(x \in K)$ is a left Haar measure for K. So for all $f \in L^{\Phi}_w(K)$ and $x \in K$,

$$(\chi_{\{e\}} * f)(x) = \sum_{y\in K} \chi_{\{e\}}(y) f(y^- * x) \frac{1}{(\delta_{y^-} * \delta_y)(\{e\})} = f(x).$$

□

5. Maximal Ideal Spaces

In this section, we describe the maximal ideal space of the Banach algebra $L^{\Phi}_w(K)$ in the commutative case.

Definition 5.1. *Let $L^{\Phi}_w(K)$ be an algebra. A bounded linear functional $\psi : L^{\Phi}_w(K) \to \mathbb{C}$ is called multiplicative if for all $f, g \in L^{\Phi}_w(K)$, $\psi(f * g) = \psi(f)\psi(g)$.*

We denote

$$X_b(K) := \{\xi \in C_b(K) : \text{for all } x, y \in K, \xi(x * y) = \xi(x)\xi(y)\},$$

and

$$X^w_b(K) := \{\xi \in X_b(K) : \xi/w \in L^{\Psi}(K)\}.$$

Proposition 5.2. *Let K be a commutative hypergroup, and $L^{\Phi}_w(K)$ be an algebra. For each $\xi \in X^w_b(G)$, we define*

$$\psi_\xi(f) := \int_K f(x)\xi(x)\,dm(x) \quad (f \in L^{\Phi}_w(K)).$$

Then ψ_ξ is a multiplicative functional on $L^{\Phi}_w(K)$.

Proof. Clearly, ψ_ξ is linear. Let $f, g \in L^{\Phi}_w(K)$. Then

$$\begin{aligned}
\psi_\xi(f * g) &= \int_K (f * g)(x)\xi(x)\,dm(x) \\
&= \int_K \int_K f(y)g(y^- * x)\xi(x)\,dm(y)\,dm(x) \\
&= \int_K f(y) \int_K g(y^- * x)\xi(x)\,dm(x)\,dm(y) \\
&= \int_K f(y) \int_K g(x)\xi(y * x)\,dm(x)\,dm(y) \\
&= \int_K f(y) \int_K g(x)\xi(y)\xi(x)\,dm(x)\,dm(y) \\
&= \int_K g(x)\xi(x)\,dm(x) \int_K f(y)\xi(y)\,dm(y) \\
&= \psi_\xi(f)\psi_\xi(g).
\end{aligned}$$

Lastly, since $\xi/w \in L^{\Psi}(K)$, ψ_ξ is a bounded linear functional on $L^{\Phi}_w(K)$. □

Theorem 5.3. *Under the hypothesis of Proposition 5.2, if ψ is a non-zero multiplicative functional on $L_w^\Phi(K)$, then there is a unique $\xi \in X_b^w(K)$ such that $\psi = \psi_\xi$.*

Proof. Let ψ be a non-zero multiplicative functional on $L_w^\Phi(K)$. Then $\psi \in (L_w^\Phi(K))^* \cong L_{w^{-1}}^\Psi(K)$, and so there is a $\xi \in L_{w^{-1}}^\Psi(K)$ such that for all $f \in L_w^\Phi(K)$,

$$\psi(f) = \int_K f(x)\xi(x)\, dm(x). \tag{2}$$

Since ψ is non-zero, for some $g \in C_c(K)$, we have $\psi(g) \neq 0$. For all $f \in L_w^\Phi(K)$, we have

$$\begin{aligned}
\int_K f(x)\xi(x)\psi(g)\, dm(x) &= \psi(f)\psi(g)\\
&= \psi(f * g)\\
&= \int_K (f * g)(x)\xi(x)\, dm(x)\\
&= \int_K \int_K f(y)g(y^- * x)\xi(x)\, dm(y)\, dm(x)\\
&= \int_K f(y) \int_K L_y g(x)\xi(x)\, dm(x)\, dm(y)\\
&= \int_K f(y)\psi(L_y g)\, dm(y),
\end{aligned}$$

and so $\xi(x) = \frac{\psi(L_y g)}{\psi(g)}$ m-a.e. For each $x \in K$, we redefine $\xi(x) = \frac{\psi(L_x g)}{\psi(g)}$. Then ξ is bounded and continuous, and the relations (2) holds. In addition, for all $x, y \in K$, we have

$$\begin{aligned}
\xi(x * y)\psi(g) &= \psi(g) \int_K \xi(t)\, d(\delta_x * \delta_y)(t)\\
&= \int_K \psi(L_t g)\, d(\delta_x * \delta_y)(t)\\
&= \int_K \int_K \xi(s)L_t g(s)\, dm(s)\, d(\delta_x * \delta_y)(t)\\
&= \int_K \xi(s) \int_K L_t g(s)\, d(\delta_x * \delta_y)(t)\, dm(s)\\
&= \int_K \xi(s) \int_K L_{s^-} g(t^-)\, d(\delta_x * \delta_y)(t)\, dm(s)\\
&= \int_K \xi(s)L_{s^-} g(y^- * x^-)\, dm(s)\\
&= \int_K \xi(s)L_y g(x^- * s)\, dm(s)\\
&= \int_K \xi(s)L_x(L_y g)(s)\, dm(s)\\
&= \psi(L_x(L_y g))\\
&= \xi(x)\psi(L_y g)\\
&= \xi(x)\xi(y)\psi(g).
\end{aligned}$$

Finally, since ψ is a bounded linear functional on $L_w^\Phi(K)$, we have $\xi \in X_b^w(K)$. □

Theorem 5.4. *Let $L^\Phi_w(K)$ be a weighted Orlicz algebra. If I is a left invariant closed linear subspace of $L^\Phi_w(K)$, then I is a left ideal of $L^\Phi_w(K)$.*

Proof. Let $f \in I$ and $g \in L^\Phi_w(K)$. If $g*f \notin I$, then by the Hahn-Banach theorem, there is a functional $F \in (L^\Phi_w(K))^*$ such that $F = 0$ on I and $F(g * f) \neq 0$. Since $\Phi \in \Delta_2$, by duality, there is a unique element $\phi \in L^\Psi_{w^{-1}}(K)$ such that for each $h \in L^\Phi_w(K)$,

$$F(h) = \int_K h\,\phi\,dm.$$

So by Fubini's theorem, we have

$$\begin{aligned}
F(g * f) &= \int_K \left(\int_K g(y) f(y^- * x)\,dm(y) \right) \phi(x)\,dm(x) \\
&= \int_K g(y) \left(\int_K L_y f(x) \phi(x)\,dm(x) \right) dm(y) \\
&= \int_K g(y) F(L_y f)\,dm(y) = 0
\end{aligned}$$

since $L_y f \in I$ and $F = 0$ on I, a contradiction. Therefore, for each $f \in I$ and $g \in L^\Phi_w(K)$, we have $g * f \in I$. □

Theorem 5.5. *Let I be a closed left ideal of a weighted Orlicz algebra $L^\Phi_w(K)$. Then for each $x \in Z(K)$ and $f \in I \cap C_c(K)$, we have $L_x f \in I$.*

Proof. Let $\varepsilon > 0$, $x \in Z(K)$ and $f \in I \cap C_c(K)$. By Theorem 4.1, there is a compactly supported function $e_V \in L^\Phi_w(K)$ such that

$$\|e_V * f - f\|_{\Phi,w} \leq \frac{\varepsilon}{w(x)}.$$

So by Lemma 3.3 we have

$$\|L_x(e_V * f) - L_x f\|_{\Phi,w} \leq \varepsilon.$$

On the other hand, since $x \in Z(K)$ and I is a left ideal, we have

$$L_x(e_V * f) = (L_x e_V) * f \in I.$$

Hence $L_x f \in \bar{I} = I$. □

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References

[1] I. Akbarbaglu, S. Maghsoudi, Banach-Orlicz algebras on a locally compact group, Mediterranean Journal of Mathematics, 10(4) (2013) 1937–1947.
[2] W. R. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, De Gruyter, Berlin, 1995.
[3] W. R. Bloom, P. Ressel, Exponentially bounded positive-definite functions on a commutative hypergroup, J. Austral. Math. Soc. (Series A) 61 (1996) 238–248.
[4] C. F. Dunkl, The measure algebra of a locally compact hypergroup, Trans. Amer. Math. Soc. 179 (1973) 331–348.
[5] C. F. Dunkl, D. E. Ramirez, A family of countably compact P_*-hypergroups, Trans. Amer. Math. Soc. 202 (1975) 339–356.
[6] F. Ghahramani, A. R. Medghalchi, Compact multipliers on weighted hypergroup algebras, Math. Proc. Camb. Phil. Soc. 98 (1985) 493–500.
[7] F. Ghahramani, A. R. Medghalchi, Compact multipliers on weighted hypergroup algebras. II, Math. Proc. Camb. Phil. Soc. 100 (1986) 145–149.
[8] R. I. Jewett, Spaces with an abstract convolution of measures, Adv. Math. 18 (1975) 1–101.
[9] V. Kumar, R. Sarma, S. Kumar, Orlicz spaces on hypergroups, Publ. Math. Debrecen 94 (2019) 31–47.
[10] Yu. N. Kuznetsova, Weighted L^p algebras on groups, English transl., Funct. Anal. Appl. 40(3) (2006) 234–236.

[11] M. Lashkarizade Bami, The semisimplicity of $L^1(K,w)$ of a weighted commutative hypergroup K, Acta Mathematica Sinica, English Series Apr. 24 (2008) 607–610.
[12] W. A. Majewski, L.E. Labuschagne, On applications of Orlicz spaces to statical physics, Ann. Henri Poincaré 15 (2014) 1197–1221.
[13] A. Osançlıol, S. Öztop, Weighted Orlicz algebras on locally compact groups, J. Aust. Math. Soc. 99 (2015) 399–414.
[14] S. Öztop, E. Samei, Twisted Orlicz algebras I, Studia Math. 236(3) (2017) 271–296.
[15] S. Öztop, E. Samei, Twisted Orlicz algebras II, Mathematische Nachrichten 292(5) (2019) 1122–1136.
[16] Kh. Pourbarat, Amenable weighted hypergroups, J. Sci. I.R. Iran 7(4) (1996) 273–276.
[17] M. M. Rao, Z. D. Ren, Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.
[18] M. M. Rao, Z. D. Ren, Applications of Orlicz Spaces, Marcel Dekker, New York, 2002.
[19] M. Rösler, M. Voit, Deformations of convolution semigroups on commutative hypergroups, Infinite dimensional harmonic analysis III, 249–264, World Sci. Publ., Hackensack, NJ, 2005.
[20] K. A. Ross, Centers of hypergroups, Trans. Amer. Math. Soc. 243 (1978) 251–269.
[21] S. Saeki, The L^p-conjecture and Young's inequality, Illinois J. Math. 34 (1990) 614–627.
[22] R. Spector, Apercu de la theorie des hypergroups, in Analyse Harmonique sur les Groups de Lie, 643–673, Lec. Notes Math. Ser. vol.497, Springer, 1975.
[23] R. Spector, Measures invariantes sur les hypergroups, Trans. Amer. Math. Soc. 239 (1978) 147–165.
[24] S. M. Tabatabaie, F. Haghighifar, The weigted KPC-hypergroups, Gen. Math. Notes 34(2) (2016) 29–38.
[25] M. Voit, Positive characters on commutative hypergroups and some applications, Math. Z. 198 (1988) 405–421.
[26] J. Wermer, On a class of normed rings, Ark. Mat. 2 (1954) 537–551.