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Idempotents Generated by Weighted Generalized Inverses in Rings With Involution

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Abstract. Let R be an associate ring with unity 1 and involution. In this paper, we investigate several properties and characterizations of idempotents generated by weighted Moore-penrose inverses and weighted pseudo core inverses in R. Moreover, several new characterizations about weighted EP elements and existence criteria of weighted pseudo core inverses are also given.

1. Introduction

It is well known that idempotents are a class of important elements and has a close relationship with generalized inverses. Many researchers have considered questions concerning the idempotents in various fields, such as in complex matrices, Banach algebras, rings, etc (see [3], [4], [6], [7], [9], [10]). Therein, characterizations of idempotents generated by Moore-Penrose inverses and weighted Moore-Penrose inverses of elements over various sets attract wide interest from many scholars. For example, Hartwig and Spindelböck [3] investigated the characterization of $A^\dagger A = BB^\dagger$, where A, B are two $n \times n$ complex matrices. Then, Patrício and Araújo [10] further considered the case of $aa^\dagger = bb^\dagger$ for $a, b \in R^\dagger$ by units in rings with involution. Shortly after, Mosić and Djordjević [9] derived the characterization of $aa^\dagger_{e,f} = bb^\dagger_{e,f}$, when $a, b \in R^\dagger_{e,f}$. Recently, Zhu and Peng [14] gave several characterizations of $a^{(1,4)}a = bb^{(1,3)}$ for $a \in R^{(1,4)}$ and $b \in R^{(1,3)}$. In this paper, we mainly consider properties and characterizations of idempotents generated by weighted Moore-Penrose generalized inverses and weighted pseudo core inverses, respectively.

The paper is organized as follows. In Section 2, we consider how to characterize idempotents generated by weighted Moore-Penrose inverses. We present some characterizations of $a^{(1,4f)}a = bb^{(1,3e)}$ in Theorem 2.4. Moreover, some new characterizations of weighted EP element are given in Theorem 2.8.

In Section 3, we consider the idempotent generated by weighted pseudo core inverses. Some characterizations of $aa^{(1,3e)} = bb^{(1,3e)}$ are given by units. The existence criteria of weighted pseudo core inverses and some characterizations of $aa^{e,\mathbb{D}} = bb^{e,\mathbb{D}}$ will be presented.

Throughout this paper, let R be a unital $*$-ring, that is, a ring with unity 1 and an involution $x \mapsto x^*$ satisfying $(a^*)^* = a, (ab)^* = b^*a^*, (a+b)^* = a^* + b^*$ for all $a, b \in R$. An element $a \in R$ is Hermitian if $a^* = a$. In what follows, we assume that all $e, f \in R$ are invertible Hermitian elements.

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Definition 1.1. [5] *An element $a \in R$ is called weighted Moore-Penrose invertible with weights e, f if there exists some $x \in R$ such that*

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3e)\ (eax)^* = eax, \quad (4f)\ (fxa)^* = fxa.$$

Such an element x is called a weighted Moore-penrose inverse of a with weights e and f. It is unique if it exists and denoted by $a^{\dagger}_{e,f}$. The set of all weighted Moore-Penrose invertible elements with weights e, f in R is denoted by $R^{\dagger}_{e,f}$.

Definition 1.2. [13] *Let $a \in R$. An element a is pseudo e-core invertible if there exists some $x \in R$ and some positive integer n such that*

$$(5)\ xa^{n+1} = a^n, \quad (6)\ ax^2 = x, \quad (3e)\ (eax)^* = eax.$$

Such an element x is called a pseudo e-core inverse of a. It is unique if it exists and denoted by $a^{e,\textcircled{D}}$. The smallest positive integer n is called the pseudo e-core index of a and denoted by $\mathrm{EI}(a)$. The set of all pseudo e-core invertible elements in R is denoted by $R^{e,\textcircled{D}}$. In particular, if $\mathrm{EI}(a) = 1$, then a is e-core invertible and the e-core inverse of a is denoted by $a^{e,\textcircled{\#}}$.

If $\delta \subseteq \{1, 2, 3e, 4f, 5, 6\}$ and $x \in R$ satisfies the equation (i) for all $i \in \delta$, then x is called a δ-inverse of a. The set of all δ-inverse elements of a is denoted by $a\{\delta\}$. The sets of all $\{1, 3e\}$-invertible, $\{1, 4f\}$-invertible elements in R are denoted by $R^{(1,3e)}$, $R^{(1,4f)}$, respectively.

An element $a \in R$ is Drazin invertible, if there exists an $x \in R$ and some nonnegative n satisfying the equations $xax = x$, $xa^{n+1} = a^n$ and $xa = ax$. Then such an x is called a Drazin inverse of a. It is unique if it exists and is denoted by a^D. The smallest nonnegative integer n is called the Drazin index of a, and is denoted by $\mathrm{ind}(a)$. The set of all Drazin invertible elements in R is denoted by R^D. In particular, if $\mathrm{ind}(a) = 1$, a is called group invertible. The group inverse of a is unique if it exists and is denoted by $a^{\#}$. The set of all group invertible elements in R is denoted by $R^{\#}$. It is known that $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$.

Following [8], an element $a \in R$ is weighted EP with respect to (e, f) if $a \in R^{\#} \cap R^{\dagger}_{e,f}$ and $a^{\#} = a^{\dagger}_{e,f}$. We use the symbol $[a, b] = ab - ba$ to denote the commutator of a and b, and R^{-1} to denote the set of all invertible elements in R.

2. Idempotents generated by weighted Moore-Penrose inverses

In [13], one can know that if $a \in R^{(1,3e)}$, then its $\{1, 3e\}$-inverses are not unique. However the product of a and its difference $\{1, 3e\}$-inverses is equal. A similar fact is true for $\{1, 4f\}$-inverses. In this section, we give several equivalent characterizations of $a^{(1,4f)}a = bb^{(1,3e)}$. Before we state results, some auxiliary lemmas will be given, which play an important role in the sequel.

Lemma 2.1. *Let $a, b \in R$. Then the following conditions hold*:

(i) *If $a \in R^{(1,4f)}$, then $f^{-1}a^*R = a^{(1,4f)}aR$ and $a^*R = (a^{(1,4f)}a)^*R$.*

(ii) *If $b \in R^{(1,3e)}$, then $ebR = (bb^{(1,3e)})^*R$ and $bR = bb^{(1,3e)}R$.*

Proof. (i) From $a \in R^{(1,4f)}$, it follows that $f^{-1}a^*R = a^{(1,4f)}aR$. Indeed, $f^{-1}a^*R = a^{(1,4f)}af^{-1}a^*R \subseteq a^{(1,4f)}aR = f^{-1}a^*(a^{(1,4f)})^*fR \subseteq f^{-1}a^*R$. $a^*R = (a^{(1,4f)}a)^*R$ is clear.

(ii) If $b \in R^{(1,3e)}$, then $bR = bb^{(1,3e)}R$ by $b = bb^{(1,3e)}b$. Also as $ebR = (bb^{(1,3e)})^*ebR \subseteq (bb^{(1,3e)})^*R = ebb^{(1,3e)}e^{-1}R \subseteq ebR$, then we obtain $ebR = (bb^{(1,3e)})^*R$. □

Lemma 2.2. *Let $a \in R^{(1,4f)}$ and $b \in R^{(1,3e)}$. Then the following conditions hold*:

(i) *If $f^{-1}a^*R = bR$, then $a^{(1,4f)}a = bb^{(1,3e)}a^{(1,4f)}a$ and $bb^{(1,3e)} = a^{(1,4f)}abb^{(1,3e)}$.*

(ii) *If $ebR = a^*R$, then $bb^{(1,3e)} = bb^{(1,3e)}a^{(1,4f)}a$ and $a^{(1,4f)}a = a^{(1,4f)}abb^{(1,3e)}$.*

Proof. (i) As $f^{-1}a^*R = bR$, then $a^{(1,4f)}aR = bb^{(1,3e)}R$ by Lemma 2.1. This implies that $bb^{(1,3e)} = a^{(1,4f)}ax$ and $a^{(1,4f)}a = bb^{(1,3e)}y$ for some x, $y \in R$. Left-multiplying $bb^{(1,3e)} = a^{(1,4f)}ax$ by $a^{(1,4f)}a$, and left-multiplying $a^{(1,4f)}a = bb^{(1,3e)}y$ by $bb^{(1,3e)}$ give $a^{(1,4f)}abb^{(1,3e)} = bb^{(1,3e)}$ and $a^{(1,4f)}a = bb^{(1,3e)}a^{(1,4f)}a$, respectively.

(ii) From $ebR = a^*R = (bb^{(1,3e)})^*R = (a^{(1,4f)}a)^*R$, we get $Rbb^{(1,3e)} = Ra^{(1,4f)}a$. The rest can be proved in a similar way of (i). □

Lemma 2.3. *Let $a \in R^{(1,4f)}$ and $b \in R^{(1,3e)}$. Then the following conditions hold*:
(i) *If $a^{(1,4f)}a = bb^{(1,3e)}a^{(1,4f)}a$ and $Raf^{-1}eb = Rb$, then $f^{-1}a^*R = bR$.*
(ii) *If $bb^{(1,3e)} = a^{(1,4f)}abb^{(1,3e)}$ and $abR = aR$, then $f^{-1}a^*R = bR$.*
(iii) *If $bb^{(1,3e)} = bb^{(1,3e)}a^{(1,4f)}a$ and $af^{-1}ebR = aR$, then $ebR = a^*R$.*
(iv) *If $a^{(1,4f)}a = a^{(1,4f)}abb^{(1,3e)}$ and $Rab = Rb$, then $ebR = a^*R$.*

Proof. (i) As $a \in R^{(1,4f)}$, then by Lemma 2.1, $f^{-1}a^*R = a^{(1,4f)}aR$. From $f^{-1}a^*R = a^{(1,4f)}aR$, $Raf^{-1}eb = Rb$ and $a^{(1,4f)}a = bb^{(1,3e)}a^{(1,4f)}a$, we have $b^*ef^{-1}a^*R = b^*R$ and

$$\begin{aligned} f^{-1}a^*R &= a^{(1,4f)}aR \\ &= bb^{(1,3e)}a^{(1,4f)}aR \\ &= bb^{(1,3e)}f^{-1}a^*R \\ &= e^{-1}(b^{(1,3e)})^*b^*ef^{-1}a^*R \\ &= e^{-1}(b^{(1,3e)})^*b^*R \\ &= bb^{(1,3e)}R \\ &= bR. \end{aligned}$$

(ii) If $bb^{(1,3e)} = a^{(1,4f)}abb^{(1,3e)}$ and $abR = bR$, then $bR = bb^{(1,3e)}R = a^{(1,4f)}abb^{(1,3e)}R = a^{(1,4f)}abR = a^{(1,4f)}aR = f^{-1}a^*R$ by Lemma 2.1.
(iii) and (iv) can be proved by a similar way of (i) and (ii). □

Next, we give some characterizations of idempotents generated by $\{1,3e\}$-inverses and $\{1,4f\}$-inverses.

Theorem 2.4. *Let $a \in R^{(1,4f)}$ and $b \in R^{(1,3e)}$. Then the following conditions are equivalent*:
(i) *$a^{(1,4f)}a = bb^{(1,3e)}$.*
(ii) *$a^*R = ebR$, $f^{-1}a^*R = bR$.*
(iii) *$[a^{(1,4f)}a, bb^{(1,3e)}] = 0$, $a^*R = ebR$.*
(iv) *$[a^{(1,4f)}a, bb^{(1,3e)}] = 0$, $f^{-1}a^*R = bR$.*
(v) *$[a^{(1,4f)}a, bb^{(1,3e)}] = 0$, $abR = aR$ and $Raf^{-1}eb = Rb$.*
(vi) *$[a^{(1,4f)}a, bb^{(1,3e)}] = 0$, $Rab = Rb$ and $af^{-1}ebR = aR$.*

Proof. (i)⇒ (ii), (i)⇒(iii), (i)⇒(iv) are obvious by Lemma 2.1. (ii)⇒ (i), (iii)⇒ (i) and (iv)⇒ (i) are clear by Lemma 2.2. Next, it is sufficient to prove that any one of conditions (i)-(iv) is equivalent to (v) and (vi).
(iv)⇒ (v) From $[a^{(1,4f)}a, bb^{(1,3e)}] = 0$ and $f^{-1}a^*R = bR$, we obtain $a^{(1,4f)}a = bb^{(1,3e)}$ by Lemma 2.2. Notice that $a^{(1,4f)}a = bb^{(1,3e)}$ implies $a = abb^{(1,3e)}$. Then $aR = abb^{(1,3e)}R = abR$. Hence, we have $Rb = Rbb^{(1,3e)}b = Re^{-1}(bb^{(1,3e)})^*eb = R(a^{(1,4f)}a)^*eb = Rfa^{(1,4f)}af^{-1}eb \subseteq Raf^{-1}eb$ by Lemma 2.1. Thus, $Raf^{-1}eb = Rb$.
(v)⇒ (vi) From the condition (v), we have $f^{-1}a^*R = bR$ by (i) and (ii) of Lemma 2.3. It implies $a^{(1,4f)}a = bb^{(1,3e)}$ by Lemma 2.2. Then it is easy to get $Rab = Rb$. Also, $aR = aa^{(1,4f)}aR \subseteq af^{-1}(a^{(1,4f)}a)^*R = af^{-1}(bb^{(1,3e)})^*R \subseteq af^{-1}ebR$ by Lemma 2.1. Thus, $af^{-1}ebR = aR$.
(vi)⇒ (iii) It can be proved by (iii) and (iv) of Lemma 2.3. □

Applying Theorem 2.4, we get the following results.

Corollary 2.5. *Let $a, b \in R^{\dagger}_{e,f}$. Then the following conditions are equivalent*:
(i) *$a^{\dagger}_{e,f}a = bb^{\dagger}_{e,f}$.*
(ii) *$a^*R = ebR$, $f^{-1}a^*R = bR$.*
(iii) *$[a^{\dagger}_{e,f}a, bb^{\dagger}_{e,f}] = 0$, $a^*R = ebR$.*
(iv) *$[a^{\dagger}_{e,f}a, bb^{\dagger}_{e,f}] = 0$, $f^{-1}a^*R = bR$.*
(v) *$[a^{\dagger}_{e,f}a, bb^{\dagger}_{e,f}] = 0$, $abR = aR$ and $Raf^{-1}eb = Rb$.*
(vi) *$[a^{\dagger}_{e,f}a, bb^{\dagger}_{e,f}] = 0$, $Rab = Rb$ and $af^{-1}ebR = aR$.*
In particular, if one of the above conditions holds, then we have
(vii) *$(ab) \in R^{\dagger}_{e,f}$ and $(ab)^{\dagger}_{e,f} = b^{\dagger}_{e,f}a^{\dagger}_{e,f}$.*

Proof. As $a^{\dagger}_{e,f} \in a\{1,4f\}$ and $b^{\dagger}_{e,f} \in b\{1,3e\}$, then equivalences of (i)− (vi) can be deduced by Theorem 2.4. Next, it suffices to show (i)⇒ (vii). One can check that $b^{\dagger}_{e,f}a^{\dagger}_{e,f}$ satisfies the four equations of the weighted Moore-Penrose inverse of ab. Indeed, from $a^{\dagger}_{e,f}a = bb^{\dagger}_{e,f}$, we have

(1) $abb^{\dagger}_{e,f}a^{\dagger}_{e,f}ab = aa^{\dagger}_{e,f}abb^{\dagger}_{e,f}b = ab$,

(2) $b^{\dagger}_{e,f}a^{\dagger}_{e,f}abb^{\dagger}_{e,f}a^{\dagger}_{e,f} = b^{\dagger}_{e,f}bb^{\dagger}_{e,f}a^{\dagger}_{e,f}aa^{\dagger}_{e,f} = b^{\dagger}_{e,f}a^{\dagger}_{e,f}$,

(3) $eabb^{\dagger}_{e,f}a^{\dagger}_{e,f} = eaa^{\dagger}_{e,f}aa^{\dagger}_{e,f} = eaa^{\dagger}_{e,f}$,

(4) $fb^{\dagger}_{e,f}a^{\dagger}_{e,f}ab = fb^{\dagger}_{e,f}bb^{\dagger}_{e,f}b = fb^{\dagger}_{e,f}b$.

Thus, $(ab)^{\dagger}_{e,f} = b^{\dagger}_{e,f}a^{\dagger}_{e,f}$. □

We know that a is weighted-EP with respect to (e, f) if and only if $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$. In Corollary 2.5, taking $b = a$, some characterizations of a to be weighted EP with respect to (e, f) can be obtained.

About idempotent generated by Moore-Penrose inverses, weighted Moore-Penroses and core inverses, several properties and characterizations were given by units in [9], [10] and [12]. Inspired by this, we extend the results to the idempotent generated by $\{1, 4e\}$-inverses and $\{1, 3e\}$-inverses.

Proposition 2.6. *Let $a \in R^{(1,4e)}$ and $b \in R^{(1,3e)}$. Then the following conditions are equivalent:*

(i) $a^{(1,4e)}a = bb^{(1,3e)}$.

(ii) $a^*R = ebR$.

(iii) $[a^{(1,4e)}a, bb^{(1,3e)}] = 0$, $abR = aR$ *and* $Rab = Rb$.

(iv) $a^{(1,4e)}a = a^{(1,4e)}abb^{(1,3e)}$, $u = a^{(1,4e)}a + 1 - bb^{(1,3e)} \in R^{-1}$.

(v) $[a^{(1,4e)}a, bb^{(1,3e)}] = 0$, $u = a^{(1,4e)}a + 1 - bb^{(1,3e)} \in R^{-1}$ *and* $v = bb^{(1,3e)} + 1 - a^{(1,4e)}a \in R^{-1}$.

Proof. The proof of (i)-(iii) can be derived by Theorem 2.4. (i)⇒(iv) and (i)⇒(v) are clear. Next, it suffices to prove that (iv)⇒(i) and (v)⇒(i).

(iv)⇒(i) It follows from $a^{(1,4e)}a = a^{(1,4e)}abb^{(1,3e)}$ that $a^{(1,4e)}a = bb^{(1,3e)}a^{(1,4e)}a$. Thus, from $ubb^{(1,3e)} = a^{(1,4e)}a = ua^{(1,4e)}a$, we obtain $a^{(1,4e)}a = bb^{(1,3e)}$ by $u \in R^{-1}$.

(v)⇒(i) It follows from $[a^{(1,4e)}a, bb^{(1,3e)}] = 0$ that $ubb^{(1,3e)} = a^{(1,4e)}abb^{(1,3e)} = ubb^{(1,3e)}a^{(1,4e)}a$ and $va^{(1,4e)}a = bb^{(1,3e)}a^{(1,4e)}a = va^{(1,4e)}abb^{(1,3e)}$. Hence, by invertibility of u and v, we get $a^{(1,4e)}a = bb^{(1,3e)}$. □

It is natural to ask whether we can extend the condition $a^{(1,4e)}a = bb^{(1,3e)}$ in Proposition 2.6 to the case of $a^{(1,4f)}a = bb^{(1,3e)}$. The following example shows the impossibility.

Example 2.7. Let $R = M_{2\times2}(\mathbb{Z}_3)$ be the ring of all 2 by 2 matrices over $\mathbb{Z}_3$ and let the involution be transpose. Take $E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \in R$. Then E and F are invertible and Hermitian matrices. Take $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in R$. Then we get $A^{(1,4f)} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $B^{(1,3e)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. It is easy to get $A^{(1,4f)}A = A^{(1,4f)}ABB^{(1,3e)}$ and $A^{(1,4f)}A + I - BB^{(1,3e)} \in R^{-1}$. But $A^{(1,4f)}A \neq BB^{(1,3e)}$.

In 2004, Patrício and Puystens [11, Corollary 3] gave some characterizations about EP elements and proved that $a \in R$ is EP if and only if $a \in R^{\#}, aR = a^*R$ if and only if $a \in R^{\dagger}, aR = a^*R$. In 2008, Mosić et al. [8, Theorem 2.4] extended the results to the case of weighted EP elements. Next, we give more characterizations of weighted EP elements in rings.

Theorem 2.8. *Let $a \in R$. Then the following conditions are equivalent:*

(i) *a is weighted EP with respect to (e, e).*

(ii) $a \in R^{(1,3e)}, eaR = a^*R$.

(iii) $a \in R^{(1,4e)}, eaR = a^*R$.

(iv) $Ra^2 = Ra = Ra^*e$.

(v) $a^2R = aR = e^{-1}a^*R$.

(vi) $a \in R^{\sharp}, a^k = a^{k+1}a^{(1,3e)}$ *for any positive integer k.*

(vii) $a \in R^{\sharp}, a^k = a^{(1,4e)}a^{k+1}$ *for any positive integer k.*

Proof. It suffices to prove that (i)⇒ (ii) ⇔ (iv) ⇒ (vi) ⇒ (i). The proof of (i)⇒ (iii) ⇔ (v) ⇒ (vii) ⇒ (i) can be proved by a similar way.

(i)⇒(ii) As a is weighted EP with respect to (e, e), we obtain $a \in R^{(1,3e)}$ and $a^{\dagger}_{e,e}a = aa^{\dagger}_{e,e}$. This implies $e^{-1}a^*R = a^{\dagger}_{e,e}aR = aa^{\dagger}_{e,e}R = aR$ by Lemma 2.1. Hence, $eaR = a^*R$.

(ii)⇔(iv) Notice that $a \in R^{(1,3e)}$ is equivalent to $a^*R = a^*eaR$ by [13, Lemma 3.15]. Suppose $a \in R^{(1,3e)}$. Then $a^*R = a^*eaR$. Also as $eaR = a^*R$, it is easy to get $a^*R = (a^*)^2R$. Hence, applying involution, we get $Ra^2 = Ra = Ra^*e$. Conversely, it is need to prove that $a \in R^{(1,3e)}$. From $Ra^2 = Ra = Ra^*e$, it is clear to get $a^*R = (a^*)^2R = a^*eaR$. Thus $a \in R^{(1,3e)}$.

(ii)⇒(vi) As $a \in R^{(1,3e)}, eaR = a^*R$. Then $a^*R = eaR = (aa^{(1,3e)})^*R$ by Lemma 2.1. By involution, $Ra = Raa^{(1,3e)}$ implies $a = zaa^{(1,3e)}$ for some $z \in R$. Then right-multiplying by $aa^{(1,3e)}$ gives $a^2a^{(1,3e)} = a$. Therefore, $a^k = a^kaa^{(1,3e)}$ for any positive integer k and $a \in a^2R \cap Ra^2$, i.e. $a \in R^{\#}$.

(vi) ⇒ (i) As $a \in R^{\#}$ and $a^k = a^kaa^{(1,3e)}$ for any positive integer k. Then multiplying this expression on the left-hand side by $a(a^{\#})^{(k+1)}$, we get $aa^{\#} = aa^{(1,3e)}$. This implies $a \in R^{(1,4e)}$ and $a^{\#}a = a^{(1,4e)}a = aa^{(1,3e)}$. Then a is weighted EP with respect to (e, e). □

As a special case, set $e = 1$, the weighted EP element with respect to (e, e) is EP. Thus, some characterizations for EP elements can be obtained.

Corollary 2.9. *Let $a, b \in R$. Then the following conditions are equivalent:*
(i) *a is EP.*
(ii) $a \in R^{(1,3)}, aR = a^*R$.
(iii) $a \in R^{(1,4)}, aR = a^*R$.
(iv) $a \in R^{\#}, a^k = a^{k+1}a^{(1,3)}$ *for any positive integer k.*
(v) $a \in R^{\#}, a^k = a^{(1,4)}a^{k+1}$ *for any positive integer k.*

Corollary 2.10. *Let $a \in R^{\dagger}_{e,f} \cap R^{\dagger}_{f,e}$. Then the following results hold:*
(i) *If $a^{\dagger}_{f,e}a = aa^{\dagger}_{e,f}$, then a is weighted EP with respect to (e, e).*
(ii) *If $aa^{\dagger}_{f,e} = a^{\dagger}_{e,f}a$, then a is weighted EP with respect to (f, f).*

In [8, Theorem 2.5], Mosić et al. derived the relationship between weighted EP with respect to (e, e) and weighted EP with respect to (e, f). From Corollary 2.10, we give another characterization of weighted EP elements.

Proposition 2.11. *Let $a \in R$. Then the following conditions are equivalent:*
(i) *a is weighted EP with respect to (e, f) ($aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$).*
(ii) *a is weighted EP with respect to (f, e) ($aa^{\dagger}_{f,e} = a^{\dagger}_{f,e}a$).*
(iii) *a is weighted EP with respect to (e, e) and weighted EP with respect to (f, f).*
(iv) *$a^{\dagger}_{f,e}a = aa^{\dagger}_{e,f}$ and $aa^{\dagger}_{f,e} = a^{\dagger}_{e,f}a$.*

Proof. The proof of (i)-(iii) were given in [8, Theorem 2.5]. For completeness, we shall give a brief proof.

(i)⇔ (ii) is clear. In this case, we can get $a^{\sharp} = a^{\dagger}_{e,f} = a^{\dagger}_{f,e}$. Then (i)⇒ (iv) can be proved.

(i)⇔(iii) Suppose $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$. Then $eaa^{\dagger}_{e,f} = ea^{\dagger}_{e,f}a$. As $eaa^{\dagger}_{e,f}$ is Hermitian, then $ea^{\dagger}_{e,f}a$ is Hermitian. Hence, a is weighted EP with respect to (e, e). Similarly, a is weighted EP with respect to (f, f).

Conversely, suppose that a is weighted EP with respect to (e, e) and weighted EP with respect to (f, f). Then we get $a^{\#} = a^{\dagger}_{e,e} = a^{\dagger}_{f,f}$. Set $b = a^{\dagger}_{f,f}aa^{\dagger}_{e,e}$. It can be verified that b is the weighted Moore-Penrose inverse with respect to (e, f) of a. Also as $b = a^{\dagger}_{f,f}aa^{\dagger}_{e,e}$ implies $b = a^{\#}$. Hence, a is weighted EP with respect to (e, f).

(iv)⇒(iii) It can be obtained by Corollary 2.10. □

3. Idempotents generated by weighted pseudo core inverses

Recently, Xu et al. [12] considered $aa^{\#} = bb^{\#}$ when $a, b \in R^{\#}$. Inspired by this, we consider the characterization of $aa^{e,\mathbb{D}} = bb^{e,\mathbb{D}}$ for $a, b \in R^{e,\mathbb{D}}$. Moreover, some existence criteria of pseudo e-core inverse of an element in R will be given. Firstly, we give some facts about pseudo e-core inverses. It is known from [13, Theorem 3.9] that a is pseudo e-core invertible with $\mathrm{EI}(a) = m$ if and only if $a \in R^D$ with $\mathrm{ind}(a) = m$ and a^m is {1,3e}-invertible. In this case, $a^{e,\mathbb{D}} = a^D a^m (a^m)^{(1,3e)}$. Then $aa^{e,\mathbb{D}} = aa^D a^m (a^m)^{(1,3e)} = a^m (a^m)^{(1,3e)}$. To consider the idempotents generated by pseudo e-core inverses, it is sufficient to consider idempotents generated by {1,3e}-inverses.

Proposition 3.1. *Let $a, b \in R^{(1,3e)}$. Then the following conditions are equivalent*:

(i) $aa^{(1,3e)} = bb^{(1,3e)}$.

(ii) $aR = bR$.

(iii) $aa^{(1,3e)} = aa^{(1,3e)}bb^{(1,3e)}$ *and* $u = aa^{(1,3e)} + 1 - bb^{(1,3e)} \in R^{-1}$.

(iv) $[aa^{(1,3e)}, bb^{(1,3e)}] = 0$, $u = aa^{(1,3e)} + 1 - bb^{(1,3e)} \in R^{-1}$ *and* $v = bb^{(1,3e)} + 1 - aa^{(1,3e)} \in R^{-1}$.

Proof. (i)⇔ (ii)⇒ (iii) is trivial.

(iii)⇒ (i) From $aa^{(1,3e)} = aa^{(1,3e)}bb^{(1,3e)}$, we get $aa^{(1,3e)} = aa^{(1,3e)}bb^{(1,3e)} = bb^{(1,3e)}aa^{(1,3e)}$ and $bb^{(1,3e)}u = bb^{(1,3e)}aa^{(1,3e)} = bb^{(1,3e)}aa^{(1,3e)}u$. As $u \in R^{-1}$, we get $bb^{(1,3e)} = bb^{(1,3e)}aa^{(1,3e)} = aa^{(1,3e)}$.

(i)⇒ (iv) is clear.

(iv)⇒ (i) As $[aa^{(1,3e)}, bb^{(1,3e)}] = 0$, then $bb^{(1,3e)}u = bb^{(1,3e)}aa^{(1,3e)} = bb^{(1,3e)}aa^{(1,3e)}u$ and $aa^{(1,3e)}v = aa^{(1,3e)}bb^{(1,3e)} = aa^{(1,3e)}bb^{(1,3e)}v$. Hence, $aa^{(1,3e)} = bb^{(1,3e)}$, since $u, v \in R^{-1}$. □

From the above results, we give the following corollary.

Corollary 3.2. *Let $a, b \in R^{e,\mathbb{D}}$. Then the following conditions are equivalent*:

(i) $aa^{e,\mathbb{D}} = bb^{e,\mathbb{D}}$.

(ii) $a^m R = b^m R$ *for* $m = max\{\mathrm{EI}(a), \mathrm{EI}(b)\}$.

(iii) $aa^{e,\mathbb{D}} = aa^{e,\mathbb{D}}bb^{e,\mathbb{D}}$ *and* $u = aa^{e,\mathbb{D}} + 1 - bb^{e,\mathbb{D}} \in R^{-1}$.

(iv) $[aa^{e,\mathbb{D}}, bb^{e,\mathbb{D}}] = 0$, $u = aa^{e,\mathbb{D}} + 1 - bb^{e,\mathbb{D}} \in R^{-1}$ *and* $v = bb^{e,\mathbb{D}} + 1 - aa^{e,\mathbb{D}} \in R^{-1}$.

Next, the existence criterion of pseudo e-core inverse of an element in R will be given. Herein, we present some auxiliary lemmas.

Lemma 3.3. [13, Theorem 3.17] *Let $a, e \in R$. Then $a \in R^{e,\mathbb{D}}$ if and only if $a^n \in R((a^n)^*)^2 ea^n \cap Ra^{2n}$ for some positive integer n. In this case, $a^{e,\mathbb{D}} = a^{2n-1}x^*e$, where $x \in R$ satisfies $a^n = x((a^n)^*)^2 ea^n$.*

Lemma 3.4. [Jacobson's Lemma] *Let $a, b \in R$. Then $1 + ab$ is invertible if and only if $1 + ba$ is invertible. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.*

Lemma 3.5. [2, Lemma 2.1] *Let $a \in R$. If there exists $x \in R$ such that $xa^{m+1} = a^m$ for some positive integer m and $ax^2 = x$. Then we have the following facts:*

(i) $a^k x^k = ax$ *for any positive integer k.*

(ii) $xax = x$.

(iii) $a^k x^k a^k = a^k$ *for any positive integer $k \geq m$.*

Theorem 3.6. *Let $a, e \in R$. Then the following conditions are equivalent*:

(i) $a \in R^{e,\mathbb{D}}$.

(ii) *There exist a unique idempotent $p \in R$ and a positive integer m such that $(ep)^* = ep$, $pa^m = 0$ and $u = p + a^m \in R^{-1}$.*

(iii) *There exist a unique idempotent $p \in R$ and a positive integer m such that $(ep)^* = ep$, $pa^m = 0$ and $w = p + a^m(1 - p) \in R^{-1}$.*

(iv) *There exist a unique idempotent $p \in R$ and a positive integer m such that $(ep)^* = ep$, $pa^m = 0$ and $v = p + a^n \in R^{-1}$ for any positive integer $n \geq m$.*

(v) *There exist a unique $p \in R$ and a positive integer m such that $(ep)^* = ep$, $pa^m = 0$ and $v = p + a^n \in R^{-1}$ for any positive integer $n \geq 2m$.*

In this case, $a^{e,\mathbb{D}} = a^{m-1}(a^m + p)^{-1}(1 - p)$ and $(a^m + p)^{-1} = x^m + 1 - x^m a^m$, where $x = a^{e,\mathbb{D}}$.

Proof. (i)⇒ (ii) As $a \in R^{e,\textcircled{D}}$ with EI$(a) = m$ and let $x \in R$ be a pseudo e-core inverse of a, then we can easily get $axa^m = a^m$, $x^ka^{k+m} = a^m$ and $a^{k+1}x^{k+1} = ax$ for any positive integer k by Lemma 3.5. Let $p = 1-ax$. As $a^mx^m = ax$, $axa^m = a^m$ and $(eax)^* = eax$, one can verify that p satisfies $p^2 = 1 - ax - ax + axax = 1 - ax - ax + axa^mx^m = 1 - ax - ax + a^mx^m = 1 - ax = p$, $pa^m = 0$ and $(ep)^* = ep$. Let $t = x^m + 1 - x^ma^m$. Then t is the inverse of u. Indeed, we get $tu = 1 = ut$. Hence, $u \in R^{-1}$.

Next, to prove the uniqueness of p, we define the set $^\circ a$=$\{x \in R \mid xa = 0\}$. As x is the pseudo e-core inverse of a with EI$(a) = m$ and p satisfies the condition (ii), then we give the following fact, i.e. $^\circ(a^m) = {}^\circ(1 - p)$. Indeed, for any $x \in {}^\circ(a^m)$, we have $x(1 - p)a^m = x(1 - p)(a^m + p) = 0$. As $u = a^m + p \in R^{-1}$, then $x(1 - p) = 0$ and consequently $x \in {}^\circ(1 - p)$. Thus, $^\circ(a^m) \subseteq {}^\circ(1 - p)$. For any $x \in {}^\circ(1 - p)$, as $pa^m = 0$, then we have $xa^m = x(1 - p)a^m = 0$. Hence, $^\circ(1 - p) \subseteq {}^\circ(a^m)$.

Assume that there is an idempotent q satisfying $(eq)^* = eq$, $qa^{m_2} = 0$ and $u = q + a^{m_2} \in R^{-1}$ for some positive integer m_2. It suffices to prove $p = q$. Indeed,
1. We get $^\circ(a^m) = {}^\circ(1 - p)$ and $^\circ(a^{m_2}) = {}^\circ(1 - q)$ by the above fact.
2. $^\circ(a^m) = {}^\circ(a^{m_2})$: From $(p + a^{m_2})a^{m_2} = a^{2m_2}$ and $p + a^{m_2} \in R^{-1}$, we get $Ra^{m_2} = Ra^{m_2+1}$. Also as $a \in R^{e,\textcircled{D}}$ with EI$(a) = m$ gives $a \in R^D$ with ind$(a) = m$, then $m_2 \geq m$ by [1]. Thus, it is easy to get $^\circ(a^m) \subseteq {}^\circ(a^{m_2})$ and $a^mR = a^{m+1}R = ... = a_2^mR$. As $a^mR = a_2^mR$, then there exists $t \in R$ such that $a^m = a^{m_2}t$. Then for any $x \in {}^\circ(a^{m_2})$, $xa^{m_2} = 0$ implies $xa^{m_2}t = 0 = xa^m$, i.e. $^\circ(a^{m_2}) \subseteq {}^\circ(a^m)$. Thus $^\circ(a^m) = {}^\circ(a^{m_2})$.
3. $p = q$: step 1 combines with step 2, $^\circ(1 - p) = {}^\circ(1 - q)$. As the idempotent $p \in {}^\circ(1 - p) = {}^\circ(1 - q)$, then $p = pq$. Similarly, $q = qp$. Also $ep = (ep)^*$, $eq = (eq)^*$, then $p = pq = e^{-1}p^*ee^{-1}q^*e = e^{-1}(qp)^*e = e^{-1}q^*e = q$. Thus, $p = q$.

(ii)⇒ (i) Giving $p \in R$ such that $p^2 = p$, $(ep)^* = ep$ and $u = a^m + p \in R^{-1}$, then we have $1 - p = (p + a^m)^{-1}a^m(1 - p) = a^m(p + a^m)^{-1}$, $pa^{m-1} = pa^{m-1}p(p + a^m)^{-1}$ and $p = p(p + a^m)^{-1} = p(p + a^m)^{-1}p$. Let $x = a^{m-1}(p + a^m)^{-1}(1 - p)$. Then we can check that x satisfies $xa^{2m} = a^{2m-1}$, $ax^2 = x$ and $(eax)^* = eax$. Thus, $a \in R^{e,\textcircled{D}}$.

(ii)⇔ (iii) We show that under the assumption $p^2 = p$, $(ep)^* = ep$ and $pa^m = 0$,

$$u = p + a^m \in R^{-1} \Leftrightarrow w = p + a^m(1 - p) \in R^{-1}.$$

Notice that $w = p + a^m(1 - p) = 1 - 1 + p + a^m(1 - p) = 1 + (a^m - 1)(1 - p) \in R^{-1}$. Then $w \in R^{-1}$ if and only if $u = 1 + (1 - p)(a^m - 1) = p + a^m \in R^{-1}$ by Lemma 3.4.

(i)⇒ (iv) Let $p = 1 - ax$. By the proof of (i)⇒ (ii), it suffices to prove $v = p + a^n \in R^{-1}$ for any positive integer $n \geq m$. We can directly check that $(x^n + 1 - x^na^n)v = v(x^n + 1 - x^na^n) = 1$. Thus, (i)⇒ (iv), as required.

(iv)⇒ (v) It is a tautology.

(v)⇒ (i) By $(ep)^* = ep$, $pa^m = 0$ and $v = p + a^n \in R^{-1}$, we have $va^m = a^{m+n}$ and $(ev)^* \in R^{-1}$. Hence, $a^m = v^{-1}a^{n-m}a^{2m} \in Ra^{2m}$ by $v \in R^{-1}$. As $(ev)^* = (ea^n)^* + ep$, then $(ev)^*a^m = (ea^n)^*a^m$. Hence, $a^m = ((ev)^*)^{-1}(a^n)^*ea^m = (a^{n-2m}(ev)^{-1})^*(a^{2m})^*ea^m \in R(a^{2m})^*ea^m$ by $(ev)^* \in R^{-1}$. Therefore, we get $a \in R^{e,\textcircled{D}}$ by Lemma 3.3. □

It is known that $a^{-1}(a + b)b^{-1} = a^{-1} + b^{-1}$ for $a, b \in R^{-1}$, this is known as the absorption law. In general, the absorption law for Moore-Penrose inverses, Drazin inverses and pseudo e-core inverses dose not hold. Next, some characterizations of the absorption law for pseudo e-core inverses are presented.

Proposition 3.7. *Let $a, b \in R^{e,\textcircled{D}}$ and m=max*{EI(a),EI(b)}. *Then the following conditions are equivalent*:
(i) $aa^{e,\textcircled{D}} = bb^{e,\textcircled{D}}$.
(ii) $a^{e,\textcircled{D}}(a + b)b^{e,\textcircled{D}} = a^{e,\textcircled{D}} + b^{e,\textcircled{D}}$.
(iii) $(1 - aa^{e,\textcircled{D}})b^m = 0$ *and* $u = 1 - aa^{e,\textcircled{D}} + b^m \in R^{-1}$ *for some positive integer m.*
(iv) $(1 - aa^{e,\textcircled{D}})b^m = 0$ *and* $v = 1 - aa^{e,\textcircled{D}} + b^maa^{e,\textcircled{D}} \in R^{-1}$ *for some positive integer m.*

Proof. (i) ⇔ (ii) Suppose $aa^{e,\textcircled{D}} = bb^{e,\textcircled{D}}$. Then $a^{e,\textcircled{D}}(a + b)b^{e,\textcircled{D}} = a^{e,\textcircled{D}}ab^{e,\textcircled{D}} + a^{e,\textcircled{D}}bb^{e,\textcircled{D}} = a^{e,\textcircled{D}}ab^{e,\textcircled{D}} + a^{e,\textcircled{D}}$. To prove (ii), it suffices to prove $b^{e,\textcircled{D}} = a^{e,\textcircled{D}}ab^{e,\textcircled{D}}$. As $aa^{e,\textcircled{D}} = bb^{e,\textcircled{D}}$, then $b^{e,\textcircled{D}}R = b^mR = a^mR$ by Corollary 3.2. From $b^{e,\textcircled{D}}R = a^mR$, there exists $t \in R$ satisfying $b^{e,\textcircled{D}} = a^mt$. Premultiply this equality by $a^{e,\textcircled{D}}a$, then we get $a^{e,\textcircled{D}}ab^{e,\textcircled{D}} = b^{e,\textcircled{D}}$. Hence, $a^{e,\textcircled{D}}(a + b)b^{e,\textcircled{D}} = a^{e,\textcircled{D}} + b^{e,\textcircled{D}}$.

Conversely, $a^{e,\textcircled{D}}(a + b)b^{e,\textcircled{D}} = a^{e,\textcircled{D}} + b^{e,\textcircled{D}}$ implies $a^{e,\textcircled{D}}ab^{e,\textcircled{D}} + a^{e,\textcircled{D}}bb^{e,\textcircled{D}} = a^{e,\textcircled{D}} + b^{e,\textcircled{D}}$, then post-multiplying the above equality by $bb^{e,\textcircled{D}}$ gives $b^{e,\textcircled{D}} = a^{e,\textcircled{D}}ab^{e,\textcircled{D}}$. Hence, it implies $b^mR = b^{e,\textcircled{D}}R \subseteq a^{e,\textcircled{D}}R = a^mR$. Similarly, left-multiplying $a^{e,\textcircled{D}}ab^{e,\textcircled{D}} + a^{e,\textcircled{D}}bb^{e,\textcircled{D}} = a^{e,\textcircled{D}} + b^{e,\textcircled{D}}$ by $a^{e,\textcircled{D}}a$ gives $a^{e,\textcircled{D}} = a^{e,\textcircled{D}}bb^{e,\textcircled{D}}$, which implies $a^m R \subseteq b^m R$ by $aa^{e,\textcircled{D}} = aa^{e,\textcircled{D}}bb^{e,\textcircled{D}} = bb^{e,\textcircled{D}}aa^{e,\textcircled{D}}$. Thus, $a^m R = b^m R$. Again applying Corollary 3.2, we get $aa^{e,\textcircled{D}} = bb^{e,\textcircled{D}}$.

(i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) can be obtained by Theorem 3.6. $\square$

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