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Stability and Convergence Analysis for a New Class of GNOYIP Involving XOR-Operation in Ordered Positive Hilbert Spaces

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Abstract. In the setting of real ordered positive Hilbert spaces, a new class of general nonlinear ordered Yosida inclusion problem involving $\oplus$ operation has been considered and solved by employing a perturbed two step-iterative algorithm. The stability and convergence analysis of solution of new class of Yosida inclusion problem involving $\oplus$ operation has been substantiated by applying a new resolvent operator and Yosida approximation operator method with XOR-operation technique. The iterative algorithm and results demonstrated in this article have witnessed, a significant improvement for many previously known results of this domain. Further, we give a numerical example in support of our main result by using MATLAB programming.

1. Introduction

A useful and important generalization of variational inequalities is a mixed type variational inequalities involving nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence and algorithm of solutions for the mixed type variational inequalities. In 1994, Hassouni and Moudafi [16] used the resolvent operator technique form maximal monotone mapping to study a class of mixed type variational inequalities with single-valued mappings which were called variational inclusions and developed a perturbed algorithm for finding approximate solutions of the mixed variational inequalities. It has been proved that the theory of variational inequalities (inclusions) is quite application oriented and thus generalized in several different directions. This theory is used to solve efficiently many problems related to economics, optimization, transportation, elasticity, basic and applied sciences, etc., see [1–5, 7, 9, 11–13, 15, 24–31] and references therein.

Focussing on the work done related to ordered set-valued mapping, it is worth to mention that work done by Li [20] and Li et al. [19, 23] is quite interesting and applicable in pure and applied sciences. In 2009, Li [17] introduced and studied a new class of general nonlinear ordered variational inequalities

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(ordered equations), and established an existence theorem in real ordered Banach spaces by using the B-restricted-accretive mappings.

By using different kind of mappings such as RME set-valued mapping, weak-ANODD mapping, ordered (α, λ)-NODM set-valued mapping, (γ_G, λ)-weak-GRD set-valued mapping and ordered (α_A, λ)-ANODM set-valued mappings with strong comparison mapping and their respective resolvent operators, Li et al. [18–23] studied different classes of nonlinear inclusion problems and obtained their solutions in real ordered Hilbert spaces. Very recently, Ahmad et al. [6, 8] considered some classes of ordered variational inclusions involving XOR-operator in different settings.

It is well known that the monotone operators on Hilbert spaces can be regularized into single-valued Lipschitzian monotone operators through a process known as Yosida approximation, see [3–5, 9, 12]. An existence of this process in Banach spaces satisfying some assumptions can be found in [3, 5, 10–13, 29–31].

Yosida inclusions are similar to variational inclusions involving Yosida approximation operator. Motivated and inspired by on going research in this direction, we introduce a new class of general nonlinear ordered Yosida inclusion problem involving XOR-operation in real ordered positive Hilbert spaces. Using the concept of XOR-operation, we propose a perturbed two-step iterative algorithm which is more general than the previous iterative algorithms considered by Li et al. [18, 21, 23]. Furthermore, we prove the existence of solution of general nonlinear ordered Yosida inclusion problem involving XOR-operation and analyze the convergence criteria of the iterative sequences of the proposed algorithm. Finally, we discuss stability analysis. We also construct a numerical example and a convergence graph by using MATLAB programming.

2. Preliminaries

Throughout this paper, we suppose that $\mathcal{H}_p$ is a real ordered positive Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$ and $2^{\mathcal{H}_p}$ is the family of all nonempty subsets of $\mathcal{H}_p$.

For the presentation of the results, let us demonstrate some known definitions and results.

Definition 2.1 ([14, 32]). *A nonempty subset C of $\mathcal{H}_p$ is called*

(i) *a normal cone if there exists a constant $N > 0$ such that for $0 \leq a \leq b$, we have $\|a\| \leq N\|b\|$, for any $a, b \in \mathcal{H}_p$;*
(ii) *for any $a, b \in \mathcal{H}_p$, $a \leq b$ if and only if $b - a \in C$;*
(iii) *a and b are said to be comparative to each other if and only if, we have either $a \leq b$ or $b \leq a$ and is denoted by $a \propto b$.*

Definition 2.2 ([32]). *For arbitrary elements $a, b \in \mathcal{H}_p$, $lub\{a, b\}$ and $glb\{a, b\}$ mean least upper bound and greatest upper bound of the set $\{a, b\}$. Suppose $lub\{a, b\}$ and $glb\{a, b\}$ exist, some binary operations are defined as follows:*

(i) $a \vee b = lub\{a, b\}$;
(ii) $a \wedge b = glb\{a, b\}$;
(iii) $a \oplus b = (a - b) \vee (b - a)$;
(iv) $a \odot b = (a - b) \wedge (b - a)$.

The operations $\oplus$, $\odot$ $\vee$ and $\wedge$ are called XOR, XNOR, OR and AND operations, respectively.

Lemma 2.3 ([14]). *For any natural number n, $a \propto b_n$ and $b_n \to b^*$ as $n \to \infty$, then $a \propto b^*$.*

Proposition 2.4 ([18, 20, 22, 23]). *Let $\odot$ be an XNOR-operation and $\oplus$ be an XOR-operation. Then the following relations hold:*

(i) $a \odot a = 0$, $a \odot b = b \odot a = -(a \oplus b) = -(b \oplus a)$;
(ii) *if* $a \propto 0$, *then* $-a \oplus 0 \leq a \leq a \oplus 0$;

(iii) $(\lambda a) \oplus (\lambda b) = |\lambda|(a \oplus b)$;
(iv) $0 \leq a \oplus b$, *if* $a \propto b$;
(v) *if* $a \propto b$, *then* $a \oplus b = 0$ *if and only if* $a = b$;
(vi) $(a + b) \odot (u + v) \geq (a \odot u) + (b \odot v)$;
(vii) *if* a, b *and* w *are comparative to each other, then* $(a \oplus b) \leq a \oplus w + w \oplus b$;
(viii) *if* $a \propto b$, *then* $((a \oplus 0) \oplus (b \oplus 0)) \leq (a \oplus b) \oplus 0 = a \oplus b$;
(ix) $(ma) \oplus (na) = |m - n|a = (m \oplus n)a$, *for all* $a, b, u, v, w \in \mathcal{H}_p$ *and* $n, m, \lambda \in \mathbb{R}$.

Proposition 2.5 ([14]). *Let C be a normal cone in* $\mathcal{H}_p$ *with normal constant N, then for each* $a, b \in \mathcal{H}_p$, *the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$;
(ii) $\|a \vee b\| \leq \|a\| \vee \|b\| \leq \|a\| + \|b\|$;
(iii) $\|a \oplus b\| \leq \|a - b\| \leq N\|a \oplus b\|$;
(iv) *if* $a \propto b$, *then* $\|a \oplus b\| = \|a - b\|$.

Definition 2.6 ([17, 20, 22]). *Let* $A : \mathcal{H}_p \to \mathcal{H}_p$ *be a single-valued mapping. Then*

(i) *A is said to be strongly comparison mapping, if A is a comparison mapping and* $A(a) \propto A(b)$ *if and only if* $a \propto b$, *for all* $a, b \in \mathcal{H}_p$;
(ii) *A is said to be* γ*-ordered non-extended mapping, if there exists* $\gamma > 0$ *such that*

$$\gamma(a \oplus b) \leq A(a) \oplus A(b), \text{ for all } a, b \in \mathcal{H}_p.$$

Definition 2.7 ([17, 20]). *A mapping* $A : \mathcal{H}_p \to \mathcal{H}_p$ *is said to be* β*-ordered compression mapping, if A is a comparison mapping and*

$$A(a) \oplus A(b) \leq \beta(a \oplus b), \text{ for } 0 < \beta < 1.$$

Definition 2.8 ([23]). *A mapping* $F : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ *is said to be* (κ, ν)*-ordered Lipschitz continuous, if* $a \propto b$ *and* $u \propto v$, *then* $N(a, u) \propto N(b, v)$ *and there exist constants* $\kappa, \nu > 0$ *such that*

$$F(a, u) \oplus F(b, v) \leq \kappa(a \oplus b) + \nu(u \oplus v), \text{ for all } a, b, u, v \in \mathcal{H}_p.$$

Definition 2.9 ([18, 19, 22, 23]). *Let* $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ *be a set-valued mapping. Then*

(i) *M is said to be a comparison mapping, if for any* $v_a \in M(a)$, $a \propto v_a$, *and if* $a \propto b$, *then for any* $v_a \in M(a)$ *and* $v_b \in M(b)$, $v_a \propto v_b$, $\forall\, a, b \in \mathcal{H}_p$;
(ii) *a comparison mapping M is said to be* α*-non-ordinary difference mapping, if for each* $a, b \in \mathcal{H}_p$, $v_a \in M(a)$ *and* $v_b \in M(b)$ *such that*

$$(v_a \oplus v_b) \oplus \alpha(a \oplus b) = 0;$$

(iii) *a comparison mapping M is said to be* λ*-XOR-weak-ordered strongly compression mapping, if* $a \propto b$, *then there exists a constant* $\lambda > 0$ *such that*

$$\lambda(v_a \oplus v_b) \geq a \oplus b, \forall a, b \in \mathcal{H}_p, v_a \in M(a), v_b \in M(b),$$

holds.

Now, we introduce a new resolvent operator associated with XOR-weak-NODSM mapping as well as Yosida approximation operator based on the new resolvent operator.

Definition 2.10. *A comparison set-valued mapping* $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ *is said to be* (α, λ)*-XOR-weak-NODSM set-valued mapping, if M is a* α*-non-ordinary difference mapping and* λ*-XOR-weak-ordered strongly monotone mapping and* $[I \oplus \lambda M](\mathcal{H}_p) = \mathcal{H}_p$ *for* $\lambda > 0$, *where I is identity mapping on* $\mathcal{H}_p$.

Definition 2.11. *Let M be a (α, λ)-XOR-weak-NODSM set-valued mapping. The resolvent operator $R_\lambda^M : \mathcal{H}_p \to \mathcal{H}_p$ associated with M is defined by*

$$R_\lambda^M(a) = [I \oplus \lambda M]^{-1}(a), \quad \forall a \in \mathcal{H}_p, \tag{1}$$

where $\lambda > 0$ is a constant.

Now, we define the Yosida approximation operator based on the resolvent operator defined by (1).

Definition 2.12. *The Yosida approximation operator of M is defined by*

$$J_\lambda^M(a) = \frac{1}{\lambda}\left[I \oplus R_\lambda^M\right](a), \quad \forall a \in \mathcal{H}_p, \tag{2}$$

where $\lambda > 0$ is a constant.

Definition 2.13 ([30]). *Let $T : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping, $a_0 \in \mathcal{H}_p$ and let*

$$a_{n+1} = S(T, a_n)$$

defines an iterative sequence which yields a sequence of points $\{a_n\}$ in $\mathcal{H}_p$. Suppose that $F(T) = \{a \in \mathcal{H}_p : Ta = a\} \neq \emptyset$ and $\{a_n\}$ converges to a fixed point a^* of T. Let $\{u_n\} \subset \mathcal{H}_p$ and

$$\vartheta_n = \|u_{n+1} - S(T, a_n)\|.$$

If $\lim_{n\to\infty} \vartheta_n = 0$, which implies that $u_n \to a^*$, then the iterative sequence $\{a_n\}$ is said to be T-stable or stable with respect to T.

3. Some Basic Properties

In this section, we prove some basic properties of resolvent operator and Yosida approximation operator defined by (1) and (2), respectively.

Lemma 3.1. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a α-non-ordinary difference comparison mapping with $\alpha > \frac{1}{\lambda}$. Then, the resolvent operator $R_\lambda^M : \mathcal{H}_p \to \mathcal{H}_p$ is a single-valued, for all $\lambda > 0$.*

Proof. For any given $u \in \mathcal{H}_p$ and a constant $\lambda > 0$, let $a, b \in [I \oplus \lambda M]^{-1}(u)$. Then

$$\begin{aligned} v_a &= \frac{1}{\lambda}(a \oplus u) \in M(a) \\ v_b &= \frac{1}{\lambda}(b \oplus u) \in M(b). \end{aligned}$$

Since M is α-non-ordinary difference comparison mapping, we have

$$\begin{aligned} (v_a \oplus v_b) \oplus \alpha(a \oplus a) &= 0 \\ \left[\left(\frac{1}{\lambda}(a \oplus u)\right) \oplus \left(\frac{1}{\lambda}(b \oplus u)\right)\right] \oplus \alpha(a \oplus b) &= 0 \\ \left[\frac{1}{\lambda}(a \oplus b)\right] \oplus [\alpha(a \oplus b)] &= 0 \\ \left(\frac{1}{\lambda} \oplus \alpha\right)(a \oplus b) &= 0 \\ (a \oplus b) &= 0. \end{aligned}$$

Therefore $x = y$, i.e., the resolvent operator $R_\lambda^M = [I \oplus \lambda M]^{-1}$ is single-valued, for $\alpha > \frac{1}{\lambda}$. $\square$

Lemma 3.2. *The Yosida approximation operator* $J_\lambda^M(a) = \frac{1}{\lambda}\left[I \oplus R_\lambda^M\right](a)$ *is a single-valued, for all* $\lambda > 0$, *where* $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ *is a* α*-non-ordinary difference comparison mapping and* R_λ^M *is the resolvent operator is defined by (1).*

Proof. For any given $u \in \mathcal{H}_p$ and $\lambda > 0$, let $a, b \in J_\lambda^M(u)$. Then

$$a \in J_\lambda^M(u) = \frac{1}{\lambda}\left[I \oplus R_\lambda^M\right](u).$$

Then

$$\begin{aligned} \lambda a &\in \left(u \oplus R_\lambda^M(u)\right) \\ u \oplus \lambda a &\in R_\lambda^M(u) = [I \oplus \lambda M]^{-1}(u). \end{aligned}$$

Thus,

$$\begin{aligned} u &\in [I \oplus \lambda M](u \oplus \lambda a) \\ u &\in (u \oplus \lambda a) \oplus \lambda M(u \oplus \lambda a) \\ a &\in M(u \oplus \lambda a). \end{aligned} \tag{3}$$

Suppose that $u \oplus \lambda a = z_1 \implies a = \frac{1}{\lambda}(z_1 \oplus u) = v_{z_1}$. Thus,

$$v_{z_1} = \frac{1}{\lambda}(u \oplus z_1) \in M(z_1). \tag{4}$$

Similarly, let $b \in J_\lambda^M(u)$ and consequently $b \in M(u \oplus \lambda b)$. Continuing the above arguments and taking $u \oplus \lambda b = v_{z_2}$, we have

$$v_{z_2} = \frac{1}{\lambda}(u \oplus z_2) \in M(z_2). \tag{5}$$

Combining (4) and (5), we have

$$\begin{aligned} v_{z_1} \oplus v_{z_2} &= \left[\frac{1}{\lambda}(u \oplus z_1)\right] \oplus \left[\frac{1}{\lambda}(u \oplus z_2)\right] \\ &= \frac{1}{\lambda}(z_1 \oplus z_2). \end{aligned}$$

Since M is α-non-ordinary difference mapping, we have

$$\begin{aligned} (v_{z_1} \oplus v_{z_2}) \oplus \alpha(z_1 \oplus z_2) &= 0 \\ \left[\frac{1}{\lambda}(z_1 \oplus z_2)\right] \oplus \alpha(z_1 \oplus z_2) &= 0 \\ \left(\frac{1}{\lambda} \oplus \alpha\right)(z_1 \oplus z_2) &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} z_1 \oplus z_2 = 0 &\implies z_1 = z_2 \\ &\implies \lambda a = \lambda b \\ &\implies (u \oplus \lambda x) = (u \oplus \lambda y) \\ &\implies u \oplus u \oplus \lambda a = u \oplus u \oplus \lambda b \\ &\implies 0 \oplus \lambda a = 0 \oplus \lambda b \\ &\implies (0 \oplus \lambda)a = (0 \oplus \lambda)b \\ &\implies a = b. \end{aligned}$$

Therefore, the Yosida approximation operator J_λ^M of M is single-valued for $\alpha > \frac{1}{\lambda}$. □

Lemma 3.3. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-weak-NODSM set-valued mapping with respect to R^M_λ. Then, the resolvent operator $R^M_\lambda : \mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof. Let M be a (α, λ)-XOR-weak-NODSM set-valued mapping with respect to R^M_λ. That is, M is α-non-ordinary difference and λ-XOR-weak-ordered strongly different comparison mapping with respect to R^M_λ, so that $a \propto R^M_\lambda(a)$. For any $a, b \in \mathcal{H}_p$, let $a \propto b$ and

$$v_a = \frac{1}{\lambda}\left(a \oplus R^M_\lambda(a)\right) \in \left(R^M_\lambda(a)\right) \tag{6}$$

and

$$v_b = \frac{1}{\lambda}\left(b \oplus R^M_\lambda(b)\right) \in \left(R^M_\lambda(b)\right). \tag{7}$$

Since M is λ-XOR-weak-ordered strongly compression mapping, using (6) and (7), we have

$$\begin{aligned}
a \oplus b &\leq \lambda(v_a \oplus v_b) = \left(a \oplus R^M_\lambda(a)\right) \oplus \left(b \oplus R^M_\lambda(b)\right)\\
a \oplus b &\leq (a \oplus b) \oplus \left(R^M_\lambda(a) \oplus R^M_\lambda(b)\right)\\
0 &\leq R^M_\lambda(a) \oplus R^M_\lambda(b) = \left[R^M_\lambda(a) - R^M_\lambda(b)\right] \vee \left[R^M_\lambda(b) - R^M_\lambda(a)\right]\\
0 &\leq R^M_\lambda(a) - R^M_\lambda(b) \quad \text{or} \quad 0 \leq R^M_\lambda(y) - R^M_\lambda(x).
\end{aligned}$$

Thus, we have

$$R^M_\lambda(a) \leq R^M_\lambda(b) \quad \text{or} \quad R^M_\lambda(b) \leq R^M_\lambda(a),$$

which implies that

$$R^M_\lambda(a) \propto R^M_\lambda(b).$$

Therefore, the resolvent operator R^M_λ is a comparison mapping. □

Lemma 3.4. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-weak-NODSM mapping with respect to R^M_λ and the resolvent operator R^M_λ be a comparison mapping. Then, the Yosida approximation operator J^M_λ is also a comparison mapping.*

Proof. For any $a, b \in \mathcal{H}_p$, let $a \propto b$, then obviously $I(a) \propto I(b)$. As R^M_λ is a comparison mapping, we have $R^M_\lambda(a) \propto R^M_\lambda(b)$. Thus, we have

$$\begin{aligned}
\left(I(a) \oplus R^M_\lambda(a)\right) &\propto \left(I(b) \oplus R^M_\lambda(b)\right)\\
\frac{1}{\lambda}\left(I(a) \oplus R^M_\lambda(a)\right) &\propto \frac{1}{\lambda}\left(I(b) \oplus R^M_\lambda(b)\right),
\end{aligned}$$

wich implies that

$$J^M_\lambda(a) \propto J^M_\lambda(b),$$

i.e. $a \propto b$ implies that $J^M_\lambda(a) \propto J^M_\lambda(b)$. Therefore, the Yosida approximation operator J^M_λ is a comparison mapping. □

Lemma 3.5. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-weak-NODSM mapping with respect to R^M_λ, for $\alpha\lambda > \mu$ and $\mu \geq 1$. Then the resolvent operator R^M_λ satisfying the following condition:*

$$R^M_\lambda(a) \oplus R^M_\lambda(b) \leq \frac{\mu}{(\alpha\lambda \oplus \mu)}(a \oplus b),\ \forall a, b \in \mathcal{H}_p, \tag{8}$$

i.e., the resolvent operator R^M_λ is $\frac{\lambda}{(\alpha\lambda \oplus \mu)}$-ordered Lipschitz type continuous mapping.

Proof. Let $a, b \in \mathcal{H}_p$, $u_a = R_\lambda^M(a)$, $u_b = R_\lambda^M(b)$, and let

$$v_a = \frac{1}{\lambda}(a \oplus u_a) \in M(u_a) \ \textit{and} \ v_b = \frac{1}{\lambda}(b \oplus u_b) \in M(u_b).$$

As M be an (α, λ)-XOR-weak-NODSM set-valued mapping with respect to R_λ^M. It follows that M is also an α-non-ordinary difference mapping with respect to R_λ^M, we have

$$(v_a \oplus v_b) \oplus \alpha(u_a \oplus u_b) = 0, \tag{9}$$

and

$$\begin{aligned} v_a \oplus v_b &= \frac{1}{\lambda}[(a \oplus u_a) \oplus (b \oplus u_b)] \\ &= \frac{1}{\lambda}[(a \oplus b) \oplus (u_a \oplus u_b)] \\ &\leq \frac{\mu}{\lambda}[(a \oplus b) \oplus (u_a \oplus u_b)], \quad \text{for } \mu \geq 1. \end{aligned}$$

From (9), we have

$$\begin{aligned} \alpha(u_a \oplus u_b) &= v_a \oplus v_b \\ &\leq \frac{\mu}{\lambda}[(a \oplus b) \oplus (u_a \oplus u_b)] \\ \left[\frac{\alpha\lambda}{\mu} \oplus 1\right](u_a \oplus u_b) &\leq (a \oplus b). \end{aligned}$$

It follows that $u_a \oplus u_b \leq \left(\frac{\mu}{(\alpha\lambda \oplus \mu)}\right)(a \oplus b)$ and consequently, we have

$$R_\lambda^M(a) \oplus R_\lambda^M(b) \leq \frac{\mu}{(\alpha\lambda \oplus \mu)}(a \oplus b), \quad \forall a, b \in \mathcal{H}_p.$$

Therefore, the resolvent operator R_λ^M is $\frac{\lambda}{(\alpha\lambda \oplus \mu)}$-ordered Lipschitz type continuous mapping. □

Lemma 3.6. *Let M be a (α, λ)-XOR-weak-NODSM set-valued mapping with respect to R_λ^M, for $\alpha\lambda > \mu$ and $\mu \geq 1$, and the resolvent operator R_λ^M satisfied the condition (8). Then, the Yosida approximation operator J_λ^M satisfying the following condition:*

$$J_\lambda^M(a) \oplus J_\lambda^M(b) \leq \frac{\alpha}{(\alpha\lambda \oplus \mu)}(a \oplus b), \quad \forall a, b \in \mathcal{H}_p,$$

i.e. the Yosida approximation operator J_λ^M is $\frac{\alpha}{(\alpha\lambda \oplus \mu)}$-ordered Lipchitz type continuous mapping.

Proof. For any $a, b \in \mathcal{H}_p$ the resolvent operator R_λ^M satisfied the condition (8), we have

$$\begin{aligned} J_\lambda^M(a) \oplus J_\lambda^M(b) &= \left[\frac{1}{\lambda}\left[I \oplus R_\lambda^M\right](a)\right] \oplus \left[\frac{1}{\lambda}\left[I \oplus R_\lambda^M\right](b)\right] \\ &= \frac{1}{\lambda}\left[a \oplus R_\lambda^M(a)\right] \oplus \frac{1}{\lambda}\left[b \oplus R_\lambda^M(b)\right] \\ &= \frac{1}{\lambda}\left[(a \oplus b) \oplus \left(R_\lambda^M(a) \oplus R_\lambda^M(b)\right)\right] \\ &\leq \frac{1}{\lambda}\left[(a \oplus b) \oplus \left(\frac{\mu}{(\alpha\lambda \oplus \mu)}\right)(a \oplus b)\right] \\ &= \frac{1}{\lambda}\left[\left(1 \oplus \frac{\mu}{(\alpha\lambda \oplus \mu)}\right)\right](a \oplus b) \\ &= \frac{\alpha}{(\alpha\lambda \oplus \mu)}(a \oplus b) \end{aligned}$$

i.e.,

$$J_\lambda^M(a) \oplus J_\lambda^M(b) \le \frac{\alpha}{(\alpha\lambda \oplus \mu)}(a \oplus b), \quad \forall a, b \in \mathcal{H}_p.$$

Hence, the Yosida approximation operator J_λ^M is $\frac{\alpha}{\alpha\lambda\oplus\mu}$-ordered Lipschitz type continuous. □

4. Formulation of the problem and existence result

Let $\mathcal{H}_p$ be a real ordered positive Hilbert space and C be a normal cone with normal constant N. Let $F : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ and $g : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued comparison mappings. Let $M : \mathcal{H}_p \to 2^{\mathcal{H}}$ be a (α, λ)-XOR-weak-NODSM set-valued mapping. We consider the problem:

For some $\omega \ge 0$ and any $\xi \in \mathbb{R}$, find $a \in \mathcal{H}_p$ such that

$$\omega \in J_\lambda^M(a) \oplus M(a) - \xi F(a, g(a)). \tag{10}$$

We call this problem as general nonlinear ordered Yosida inclusion problem involving $\oplus$ operation (in short, GNOYIP).

Lemma 4.1. *Let $g : \mathcal{H}_p \to \mathcal{H}_p$ and $F : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that g is comparison and δ-ordered compression mapping, F is comparison, and (κ, ν)-ordered Lipschitz continuous mapping with respect to g, respectively. Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α, λ)-XOR-weak-NODSM set-valued mapping. Then the followings are equivalent:*

- (i) *$a \in \mathcal{H}_p$ is a solution of GNOYIP (10);*
- (ii) *$a \in \mathcal{H}_p$ is a fixed point of a mapping $Q : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ defined by*

$$Q(a) = J_\lambda^M(a) \oplus M(a) - \xi F(a, g(a)) - \omega + a;$$

- (iii) *$a \in \mathcal{H}_p$ is a solution of the following equation*

$$a = R_\lambda^M\left[(\lambda\omega + \lambda\xi F(a, g(a))) \oplus R_\lambda^M(a)\right]. \tag{11}$$

Proof. $(i) \implies (ii)$ Adding a to both sides of (10), we have

$$\begin{aligned} 0 &\in J_\lambda^M(a) \oplus M(a) - \xi F(a, g(a)) - \omega \\ &\implies a \in J_\lambda^M(a) \oplus M(a) - \xi F(a, g(a)) - \omega + a = Q(a). \end{aligned}$$

Hence, a is a fixed point of Q.

$(ii) \implies (iii)$ Let a be a fixed point of Q, then

$$\begin{aligned} a &\in Q(a) = J_\lambda^M(a) \oplus M(a) - \xi F(a, g(a)) - \omega + a \\ &\implies \omega \in J_\lambda^M(a) \oplus M(a) - \xi F(a, g(a)) \\ &\implies \omega + \xi F(a, g(a)) \in J_\lambda^M(a) \oplus M(a) \\ &\implies \lambda\omega + \lambda\xi F(a, g(a)) \in a \oplus R_\lambda^M(a) \oplus \lambda M(a) \\ &\implies (\lambda\omega + \lambda\xi F(a, g(a))) \oplus R_\lambda^M(a) \in a \oplus \lambda M(a) \\ &\implies (\lambda\omega + \lambda\xi F(a, g(a))) \oplus R_\lambda^M(a) \in [I \oplus \lambda M](a), \end{aligned}$$

which implies that $a = R_\lambda^M[(\lambda\omega + \lambda\xi F(a, g(a))) \oplus R_\lambda^M(a)]$. Consequently, x is a solution of the GNOYIP (10).

$(iii) \implies (i)$, from (11) we have

$$\begin{aligned} a &= R_\lambda^M[(\lambda\omega + \lambda\xi F(a, g(a))) \oplus R_\lambda^M(a)] \\ a &= [I \oplus \lambda M]^{-1}[(\lambda\omega + \lambda\xi F(a, g(a))) \oplus R_\lambda^M(a)], \end{aligned}$$

so

$$\begin{array}{rcl} [(\lambda\omega + \lambda\xi F(a, g(a))) \oplus R^M_\lambda(a)] & \in & (I \oplus \lambda M)(a), \\ (\lambda\omega + \lambda\xi F(a, g(a))) \oplus R^M_\lambda(a) & \in & I(a) \oplus \lambda M(a), \\ \omega + \xi F(a, g(a)) & \in & \dfrac{1}{\lambda}[I(a) \oplus R^M_\lambda(a)] \oplus M(a), \end{array}$$

which implies

$$\omega \in J^M_\lambda(a) \oplus M(a) - \xi F(a, g(a)).$$

Therefore, $a \in \mathcal{H}_p$ is a solution of problem GNOYIP (10). □

Now, we prove the existence of solution for GNOYIP (10).

Theorem 4.2. *Let $g : \mathcal{H}_p \to \mathcal{H}_p$ and $F : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that g is comparison and δ-ordered compression mapping, F is comparison, and (κ, ν)-ordered Lipschitz continuous mapping with respect to g, respectively.*

Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α, λ)-XOR-weak-NODSM set-valued mapping. In addition, if F, g, M and R^M_λ are compared to each other, the following conditions are satisfied:

$$\begin{cases} N(\lambda|\xi|\kappa + \nu\delta) < \left[\frac{\alpha\lambda\oplus\mu}{N\mu} \oplus \frac{\mu}{\alpha\lambda\oplus\mu}\right], \\ (\alpha\lambda \vee \mu) > 1,\ \mu \geq 1 \text{ and } \alpha\lambda > \mu, \end{cases} \tag{12}$$

then, GNOYIP (10) admits a unique solution $a^* \in \mathcal{H}_p$, which is a fixed point of the resolvent operator $R^M_\lambda\left[(\lambda\omega + \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right]$.

Proof. Using Proposition 2.4, Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned} 0 \leq\ & R^M_\lambda\left[(\lambda\omega + \lambda\xi F(., g(.))) \oplus R^M_\lambda(.)\right](a_1) \oplus R^M_\lambda\Big[(\lambda\omega + \lambda\xi F(., g(.))) \\ & \oplus R^M_\lambda(.)\Big](a_2) \\ \leq\ & \frac{\mu}{\alpha\lambda \oplus \mu}\left[\left(\lambda\omega + \lambda\xi F(a_1, g(a_1)) \oplus R^M_\lambda(a_1)\right) \oplus \left(\lambda\omega + \lambda\xi F(a_2, g(a_2))\right.\right. \\ & \left.\left.\oplus R^M_\lambda(a_2)\right)\right] \\ \leq\ & \frac{\mu}{\alpha\lambda \oplus \mu}\left[\left(\lambda\xi F(a_1, g(a_1)) \oplus \lambda\xi F(a_2, g(a_2))\right) \oplus \left(R^M_\lambda(a_1) \oplus R^M_\lambda(a_2)\right)\right] \\ \leq\ & \frac{\mu}{\alpha\lambda \oplus \mu}\left[\lambda|\xi|\Big(F(a_1, g(a_1)) \oplus F(a_2, g(a_2))\Big) \oplus \left(\frac{\mu}{\alpha\lambda \oplus \mu}\right)(a_1 \oplus a_2)\right] \\ \leq\ & \frac{\mu}{\alpha\lambda \oplus \mu}\left[\Big(\lambda|\xi|\Big(\kappa(a_1 \oplus a_2) + \nu\Big(g(a_1) \oplus g(a_2)\Big)\Big)\Big) \oplus \left(\frac{\mu}{\alpha\lambda \oplus \mu}(a_1 \oplus a_2)\right)\right] \\ \leq\ & \frac{\mu}{\alpha\lambda \oplus \mu}\left[\Big(\lambda|\xi|\kappa + \nu\delta\Big) \oplus \frac{\mu}{\alpha\lambda \oplus \mu}\right](a_1 \oplus a_2), \end{aligned}$$

which implies that

$$\begin{aligned} 0 \leq\ & R^M_\lambda\left[(\lambda\omega + \lambda\xi F(., g(.))) \oplus R^M_\lambda(.)\right](a_1) \oplus R^M_\lambda\Big[(\lambda\omega + \lambda\xi F(., g(.))) \\ & \oplus R^M_\lambda(.)\Big](a_2) \\ \leq\ & \psi(a_1 \oplus a_2), \end{aligned}$$

where

$$\psi = \frac{\mu}{\alpha\lambda \oplus \mu}\left(\left(\lambda|\xi|\kappa + \nu\delta\right) \oplus \frac{\mu}{\alpha\lambda \oplus \mu}\right).$$

Using the definition of normal cone and Proposition 2.5, we conclude that

$$\begin{aligned}\Big\|R^M_\lambda\left[(\lambda\omega + \lambda\xi F(.,g(.))) \oplus R^M_\lambda(.)\right](a_1) \\ \oplus R^M_\lambda\Big[(\lambda\omega + \lambda\xi F(.,g(.))) \oplus R^M_\lambda(.)\Big](a_2)\Big\| \quad &\le \quad |\psi|N\|(a_1 \oplus a_2)\|. \end{aligned} \tag{13}$$

Using the condition (12), we can see that $|\psi| < \frac{1}{N}$. It follows from (13) that the resolvent operator $R^M_\lambda\left[(\lambda\omega + \lambda\xi F(.,g(.))) \oplus R^M_\lambda(.)\right]$ is contraction operator. Hence, there exists a unique $a^* \in \mathcal{H}_p$ such that

$$a^* = R^M_\lambda\left[(\lambda\omega + \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right].$$

From Lemma 4.1, a^* is a unique solution of GNOYIP (10). □

5. Convergence Analysis and Stability

In this section, we suggest the following iterative algorithm based on Lemma 4.1 for finding the approximate solution of GNOYIP (10). We also discuss the convergence and stability analysis of the proposed algorithm.

Algorithm 5.1. *Let $g : \mathcal{H}_p \to \mathcal{H}_p$ and $F : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Given any initial point $a_0 \in \mathcal{H}_p$, assume that $a_1 \propto a_0$. We define the sequence $\{a_n\}$ and let $a_{n+1} \propto a_n$ such that*

$$\left.\begin{aligned} a_{n+1} &= (1-\alpha_n)a_n + \alpha_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(b_n, g(b_n))) \oplus R^M_\lambda(b_n)\right] + \alpha_n e_n \\ b_n &= (1-\beta_n)a_n + \beta_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a_n, g(a_n))) \oplus R^M_\lambda(a_n)\right] + \beta_n d_n. \end{aligned}\right\} \tag{14}$$

Let $\{u_n\}$ be any sequence in $\mathcal{H}_p$ and define the sequence $\{\nu_n\}$ by

$$\left.\begin{aligned} \nu_n &= \Big\|u_{n+1} - \Big[(1-\alpha_n)a_n + \alpha_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(t_n, g(t_n))) \oplus R^M_\lambda(t_n)\right] + \alpha_n e_n\Big]\Big\| \\ t_n &= (1-\beta_n)u_n + \beta_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(u_n, g(u_n))) \oplus R^M_\lambda(u_n)\right] + \beta_n d_n. \end{aligned}\right\} \tag{15}$$

where $0 \le \alpha_n, \beta_n \le 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty, \forall\, n \ge 0$. Here $\{e_n\}$ and $\{d_n\}$ are two sequences in $\mathcal{H}_p$ introduced to take into account the possible inexact computation provided that $e_n \oplus 0 = e_n$ and $d_n \oplus 0 = d_n$, $\forall n \ge 0$.

Remark 5.2. *If $\beta_n = 0, \forall\, n \ge 0$, then Algorithm 5.1 becomes Mann type iterative algorithm. Also, we remark that for suitable choices of operators involved in Algorithm 5.1, we can easily obtain many more algorithms studied by several authors for solving ordered variational inclusion problems, see e.g. [18–20, 22, 23].*

Theorem 5.3. *Let g, F and M be the same as in Theorem 4.2 such that all the conditions of Theorem 4.2 are satisfied. In addition, assume that the following conditions are satisfied:*

$$\begin{cases} \lambda|\xi|\kappa + \nu\delta < \left[\frac{\alpha\lambda\oplus\mu}{\mu} \oplus \frac{\mu}{\alpha\lambda\oplus\mu}\right], \\ \alpha\lambda \vee \mu > 1,\ \mu \ge 1 \text{ and } \alpha\lambda > \mu. \end{cases} \tag{16}$$

If $\lim_{n\to\infty} \|e_n \vee (-e_n)\| = \lim_{n\to\infty} \|d_n \vee (-d_n)\| = 0$, then

(*I*) *The sequence* $\{a_n\}$ *generated by Algorithm 5.1 converges strongly to the unique solution* a^* *of GNOYIP (10).*

(*II*) *Moreover, if* $0 < \kappa < \alpha_n$*, then* $\lim_{n\to\infty} u_n = a^*$ *if and only if* $\lim_{n\to\infty} \nu_n = 0$*, where* ν_n *is defined in (15) i.e., the sequence* $\{a_n\}$ *generated by (14) is stable with respect to* R^M_λ*.*

Proof. (I). Let us suppose that a^* be an unique solution of GNOYIP (10). Then, we have

$$\begin{aligned} a^* &= (1-\alpha_n)a^* + \alpha_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right] \\ &= (1-\beta_n)a^* + \beta_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right]. \end{aligned}$$

Using Algorithm 5.1, Proposition 2.4 and Lemma 3.5, it follows that

$$\begin{aligned} a^* &\le a_{n+1} \oplus a^* \\ &= \Big[(1-\alpha_n)a_n + \alpha_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(b_n, g(b_n))) \oplus R^M_\lambda(b_n)\right] + \alpha_n e_n\Big] \\ &\quad \oplus (1-\alpha_n)a^* + \alpha_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right] \\ &\le (1-\alpha_n)(a_n \oplus a^*) + \alpha_n\Big[R^M_\lambda\Big((\lambda\omega - \lambda\xi F(b_n, g(b_n))) \oplus R^M_\lambda(b_n)\Big) \\ &\quad \oplus R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right]\Big] + \alpha_n(e_n \oplus 0) \\ &\le (1-\alpha_n)(a_n \oplus a^*) + \alpha_n\psi(b_n \oplus a^*) + \alpha_n(e_n \oplus 0), \end{aligned} \tag{17}$$

where

$$\psi = \frac{\mu}{\alpha\lambda \oplus \mu}\left[\Big(\lambda|\xi|\kappa + \nu\delta\Big) \oplus \left(\frac{\mu}{\alpha\lambda \oplus \mu}\right)\right].$$

Using the same argument as for (17), we calculate

$$\begin{aligned} 0 &\le b_n \oplus a^* \\ &= \Big[(1-\beta_n)a_n + \beta_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a_n, g(a_n))) \oplus R^M_\lambda(a_n)\right] + \beta_n d_n\Big] \\ &\quad \oplus \Big[(1-\beta_n)a^* + \beta_n R^M_\lambda\left[(\lambda\omega - \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right]\Big] \\ &\le (1-\beta_n)(a_n \oplus a^*) + \beta_n\psi(a_n \oplus a^*) + \beta_n(d_n \oplus 0) \\ &\le (1-\beta_n(1-\psi))(a_n \oplus a^*) + \beta_n(d_n \oplus 0) \\ &\le (a_n \oplus a^*) + \beta_n(d_n \oplus 0) \quad \text{since } (1-\beta_n(1-\psi)) \le 1. \end{aligned} \tag{18}$$

Using (18), (17) become as

$$\begin{aligned} a^* &\le a_{n+1} \oplus a^* \\ &\le (1-\alpha_n)(a_n \oplus a^*) + \alpha_n\psi\left[(a_n \oplus a^*) + \beta_n(d_n \oplus 0)\right] + \alpha_n(e_n \oplus 0) \\ &\le (1-\alpha_n)(a_n \oplus a^*) + a_n\psi(a_n \oplus a^*) + \alpha_n\psi\beta_n(d_n \oplus 0) + \alpha_n(e_n \oplus 0) \\ &= (1-\alpha_n(1-\psi))(a_n \oplus a^*) + \alpha_n\left[\psi\beta_n(d_n \oplus o) + (e_n \oplus 0)\right]. \end{aligned}$$

Using definition of normal cone and Proposition 2.5, we have

$$\begin{aligned} \|a_{n+1} - a^*\| &\le (1 - N\alpha_n(1-\psi))\|a_n - a^*\| \\ &\quad + N\alpha_n(1-\psi)\left[\frac{\psi\beta_n\|(d_n) \vee (-d_n)\| + \|(e_n) \vee (-e_n)\|}{(1-\psi)}\right]. \end{aligned} \tag{19}$$

On setting $\eta_n = \frac{\psi\beta_n\|(e_n)\vee(-e_n)\| + \|(d_n)\vee(-d_n)\|}{(1-\psi)}$, $\chi_n = \|a_n - a^*\|$, $G_n = N\alpha_n(1-\psi)$, (19) can written as

$$\chi_{n+1} \le (1 - G_n)\chi_n + G_n\eta_n.$$

From Lemma 3.6 and using the hypothesis $\lim_{n\to\infty} \|e_n \vee (-e_n)\| = \lim_{n\to\infty} \|d_n \vee (-d_n)\| = 0$, we can deduce that $\chi_n \to 0$ as $n \to \infty$, and so $\{a_n\}$ converges strongly to a unique solution a^* of GNOYIP (10).

Proof of (II). Let $S(a^*) = R^M_\lambda \left[(\lambda\omega - \lambda\xi F(a^*, g(a^*))) \oplus R^M_\lambda(a^*)\right]$. Using Algorithm 5.1, Proposition 2.4 and Proposition 3.5, we obtain

$$\begin{aligned}
0 &\leq u_{n+1} \oplus a^* \\
&\leq u_{n+1} \oplus ((1-\alpha_n)a^* + \alpha_n S(a^*)) \\
&\leq u_{n+1} \oplus ((1-\alpha_n)u_n + \alpha_n S(t_n)) + \alpha_n e_n \\
&\quad + ((1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n) \oplus ((1-\alpha_n)a^* + \alpha_n S(a^*)) \\
&\leq u_{n+1} \oplus ((1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n) \\
&\quad + (1-\alpha_n)(u_n \oplus a^*) + \alpha_n (S(t_n) \oplus S(a^*)) + \alpha_n(e_n \oplus 0) \\
&\leq u_{n+1} \oplus ((1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n) \\
&\quad + (1-\alpha_n)(u_n \oplus a^*) + \alpha_n \psi(t_n \oplus a^*) + \alpha_n(e_n \oplus 0),
\end{aligned} \tag{20}$$

where

$$\psi = \frac{\mu}{\alpha\lambda \oplus \mu}\left[\left(\lambda|\xi|\kappa + \nu\delta\right) \oplus \left(\frac{\mu}{\alpha\lambda \oplus \mu}\right)\right].$$

From (20), we have

$$\begin{aligned}
0 &\leq t_n \oplus a^* \\
&= \left[(1-\beta_n)u_n + \beta_n S(u_n) + \beta_n d_n\right] \oplus \left[(1-\beta_n)a^* + \beta_n S(a^*)\right] \\
&\leq (1-\beta_n)(u_n \oplus a^*) + \beta_n(S(u_n) \oplus S(a^*)) + \beta_n(d_n \oplus 0) \\
&\leq (1-\beta_n)(u_n \oplus a^*) + \beta_n \psi(u_n \oplus a^*) + \beta_n(d_n \oplus 0) \\
&\leq (1-\beta_n(1-\psi))(u_n \oplus a^*) + \beta_n(d_n \oplus 0) \\
&\leq (u_n \oplus a^*) + \beta_n(d_n \oplus 0), \quad \text{since } \ 1-\beta_n(1-\psi) \leq 1.
\end{aligned} \tag{21}$$

Using (21), (20) becomes as

$$\begin{aligned}
0 &\leq u_{n+1} \oplus a^* \\
&\leq \left[u_{n+1} \oplus ((1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n)\right] \\
&\quad + \left[(1-\alpha_n(1-\psi))(u_n \oplus a^*) + \alpha_n\left[\psi\beta_n(d_n \oplus 0) + (e_n \oplus 0)\right]\right].
\end{aligned}$$

Using the definition of normal cone and Proposition 2.5, we have

$$\begin{aligned}
\|u_{n+1} - a^*\| &\leq N\|u_{n+1} - [(1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n]\| \\
&\quad + N(1-\alpha_n(1-\psi))\|u_n - a^*\| \\
&\quad + \alpha_n N(1-\psi)\left[\frac{\psi\beta_n\|d_n \vee (-d_n)\| + \|e_n \vee (-e_n)\|}{(1-\psi)}\right] \\
&\leq N\nu_n + N(1-\alpha_n(1-\psi))\|u_n - a^*\| \\
&\quad + \alpha_n N(1-\psi)\left[\frac{\psi\beta_n\|d_n \vee (-d_n)\| + \|e_n \vee (-e_n)\|}{(1-\psi)}\right].
\end{aligned} \tag{22}$$

Since $0 < \kappa \leq \alpha_n$, (22) becomes as

$$\begin{aligned}
\|u_{n+1} - a^*\| &\leq (1 - N\alpha_n(1-\psi))\|u_n - a^*\| \\
&\quad + \alpha_n N(1-\psi)\left[\frac{\nu_n}{\kappa(1-\psi)} + \frac{\psi\beta_n\|d_n \vee (-d_n)\| + \|e_n \vee (-e_n)\|}{(1-\psi)}\right].
\end{aligned}$$

Assume that $\lim_{n\to\infty} \nu_n = 0$, hence

$$\lim_{n\to\infty} u_n = a^*,$$

where
$$\lim_{n\to\infty} \|d_n \vee (-d_n)\| = \lim_{n\to\infty} \|e_n \vee (-e_n)\| = 0.$$
Conversely, suppose that $\lim_{n\to\infty} u_n = a^*$. From (11) and $\lim_{n\to\infty} \|d_n \vee (-d_n)\| = \lim_{n\to\infty} \|e_n \vee (-e_n)\| = 0$, we have

$$\begin{aligned}
0 &\leq u_{n+1} \oplus [(1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n] \\
&\leq (u_{n+1} \oplus a^*) + [((1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n) \oplus a^*] \\
&= (u_{n+1} \oplus a^*) + [((1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n) \\
&\quad \oplus ((1-\alpha_n)a^* + \alpha_n S(a^*))] \\
&\leq u_{n+1} \oplus a^* + (1-\alpha_n)(u_n \oplus a^*) + \alpha_n \left(S(t_n) \oplus S(a^*)\right) + \alpha_n(e_n \oplus 0) \\
&\leq u_{n+1} \oplus a^* + (1-\alpha_n)(u_n \oplus a^*) + a_n \psi(t_n \oplus a^*) \\
&\quad + \alpha_n(e_n \oplus 0) \\
&\leq u_{n+1} \oplus a^* + (1-\alpha_n(1-\psi))(u_n \oplus a^*) + \alpha_n\Big[\psi\beta_n(d_n \oplus 0) \\
&\quad + (e_n \oplus 0)\Big].
\end{aligned} \tag{23}$$

Applying again the definition of normal cone and Proposition 2.5, it follows that

$$\begin{aligned}
\nu_n &= \|u_{n+1} - [(1-\alpha_n)u_n + \alpha_n S(t_n) + \alpha_n e_n]\| \\
&\leq N\|u_{n+1} - a^*\| + N(1-\alpha_n(1-\psi))\|u_n - a^*\| \\
&\quad + \alpha_n N\left[\psi\beta_n\|d_n \vee (-d_n)\| + \|e_n \vee (-e_n)\|\right],
\end{aligned}$$

which implies that
$$\lim_{n\to\infty} \nu_n = 0.$$
Hence, the iterative sequence $\{u_n\}$ generated by (15) is stable with respect to R^M_λ. This completes the proof. □

6. Numerical Example

In this section, we provide a numerical example to illustrate Algorithm 5.1 and justify our main result.

Example 6.1. *Let $\mathcal{H}_p = \mathbb{R}^+ \cup \{0\}$ with the usual inner product and norm and let $C = \{x \in \mathcal{H}_p : 0 \leq x \leq 1\}$ be a normal cone with normal constant $N = 1$. Let $g : \mathcal{H}_p \to \mathcal{H}_p$ and $F : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the mapping defined by*
$$g(a) = \frac{a}{3} \oplus \frac{1}{6} \ \text{ and } \ F(a, g(a)) = (a + 6g(a)).$$
For each $a, b \in \mathcal{H}_p$, $a \propto b$. Then, it is easy to check that g is $\frac{1}{2}$-ordered compression mapping.
For $a, b, u, v \in \mathcal{H}_p$, $a \propto u$, $b \propto v$, we calculate

$$\begin{aligned}
F(a, g(u)) \oplus F(b, g(v)) &= (a + 6g(u)) \oplus (b + 6g(v)) \\
&\leq a \oplus b + 6(g(u) \oplus g(v)) \\
&= a \oplus b + 6\left(\left(\frac{u}{3} \oplus \frac{1}{6}\right) \oplus \left(\frac{v}{3} \oplus \frac{1}{6}\right)\right) \\
&= a \oplus b + 2(u \oplus v),
\end{aligned}$$

i.e.,
$$F(a, g(u)) \oplus F(b, g(v)) \leq (a \oplus b) + 2(u \oplus v).$$

Hence, F is $(1,2)$-ordered Lipschitz continuous mapping with respect to g. Suppose that $M : \mathcal{H}_p \to CB(\mathcal{H}_p)$ is a set-valued mapping defined by

$$M(p) = \left\{\frac{a}{2} \oplus 1\right\}, \quad \forall a \in \mathcal{H}_p.$$

It can be easily verified that M is a comparison mapping, M is $\frac{1}{2}$-weak non-ordinary difference mapping and 3-XOR-ordered different comparison mapping. Also, it is clear that for $\lambda = \frac{1}{2}$, $[I \oplus \lambda M](\mathcal{H}_p) = \mathcal{H}_p$. Hence, M is an $\left(\frac{1}{2}, 3\right)$-XOR-weak-NODSM set-valued mapping. The resolvent operator and Yosida approximation operator defined by (1) and (2) associated with M are given by

$$R^M_\lambda(a) = 2(a \oplus 3) \ \text{and} \ J^M_\lambda(a) = 2(a \oplus 6), \ \forall a \in \mathcal{H}_p.$$

It is easy to examine that the resolvent operator and Yosida approximation operator defined above are comparison and single-valued mapping. In particular for $\mu = 1$, we obtain

$$\begin{aligned} R^M_\lambda(a) \oplus R^M_\lambda(b) &= [2(a \oplus 3)] \oplus [2(b \oplus 3)] \\ &= (2a \oplus 6) \oplus (2b \oplus 6) \\ &= 2(a \oplus b) \\ &\leq \frac{5}{2}(a \oplus b), \end{aligned}$$

i.e.

$$R^M_\lambda(a) \oplus R^M_\lambda(b) \leq \frac{5}{2}(a \oplus b), \ \forall a, b \in \mathcal{H}_p.$$

That is, the resolvent operator $J^A_{\lambda,M}$ is $\frac{5}{2}$-ordered Lipschitz continuous. Similarly, we can show that the Yosida approximation operator is $\frac{3}{2}$-Lipschitz continuous.
If we take $\omega = 0$ and $\xi = 1$, we calculate

$$\begin{aligned} R^M_\lambda\left[(\lambda\omega + \lambda\xi F(a, g(a))) \oplus R^M_\lambda(a)\right] &= R^M_\lambda\left[3F(a, g(a)) \oplus R^M_\lambda(a)\right] \\ &= R^M_\lambda\left[3((a + (2a \oplus 1))) \oplus (2a \oplus 6)\right] \\ &= 2[((3(a + (2a \oplus 1))) \oplus (2a \oplus 6)) \oplus 3] \\ &= 2[((3a + (6a \oplus 3)) \oplus (2a \oplus 6)) \oplus 3] \end{aligned}$$

clearly, 0 is a fixed point of $R^M_\lambda\left[3F(a, g(a)) \oplus R^M_\lambda(a)\right]$.

Let $\alpha_n = \frac{2n+1}{3n^2+1}$, $\beta_n = \frac{n^2+1}{3n^3+n^2}$, $e_n = \frac{1}{n^2}$ and $d_n = \frac{1}{n^3}$. It is easy to show that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$ and $\{d_n\}$ satisfying the conditions $0 \leq \alpha_n$, $\beta_n \leq 1$,

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \ e_n \oplus 0 = e_n, \ d_n \oplus 0 = d_n.$$

Now, we can estimate the sequences $\{a_n\}$ and $\{b_n\}$ by the following schemes:

$$\begin{aligned} a_{n+1} = \left(\frac{3n^2 - 2n}{3n^2 + 1}\right)a_n &+ \left(\frac{2n + 1}{3n^2 + 1}\right)[((6b_n + (6b_n \oplus 6)) \oplus (4b_n \oplus 12)) \oplus 6] \\ &+ \left(\frac{2n + 1}{3n^4 + n^2}\right), \end{aligned}$$

$$\begin{aligned} b_n = \left(\frac{3n^3 - 1}{3n^3 + n^2}\right)a_n &+ \left(\frac{n^2 + 1}{3n^3 + n^2}\right)[((6a_n + (6a_n \oplus 6)) \oplus (4a_n \oplus 12)) \oplus 6] \\ &+ \left(\frac{n^2 + 1}{3n^6 + n^5}\right). \end{aligned}$$

It is also verified that condition (16) is satisfied. Thus, all the assumptions of Theorem 5.3 are fulfilled. Hence, the sequence $\{a_n\}$ *converges strongly to the unique solution* $a* = 0$ *of the GNOYIP (10).*

All codes are written in MATLAB version 7.13, we have the following different initial values $a_0 = 5, 10$ *and* 15 *which shows that the sequence* $\{a_n\}$ *converge to* $a^* = 0$.

Table 1: The values of a_n with initial values $a_0 = 5$, $a_0 = 10$ and $a_0 = 15$

No. of Iteration	For $a_0 = 5$ a_n	For $a_0 = 10$ a_n	For $a_0 = 15$ a_n
n=1	5	10	15
n=2	14.75000000	27.8750000	41
n=3	15.44436811	29.07417582	42.70398351
n=4	9.90015732	18.60808969	27.31602239
n=5	4.57646612	8.59038575	12.60430528
n=6	1.65662274	3.10401002	4.55139771
n=7	0.49425151	0.92293909	1.35162646
n=8	0.126437212	0.23417848	0.34191996
n=9	0.02906340	0.05258300	0.07610260
n=10	0.0066226	0.01116001	0.01569893
n=13	5.37830012 e-08	5.55419636e-08	5.73007407e-08
n=17	1.14435922e-13	1.99016354e-13	1.14430502e-13
n=20	0	0	0

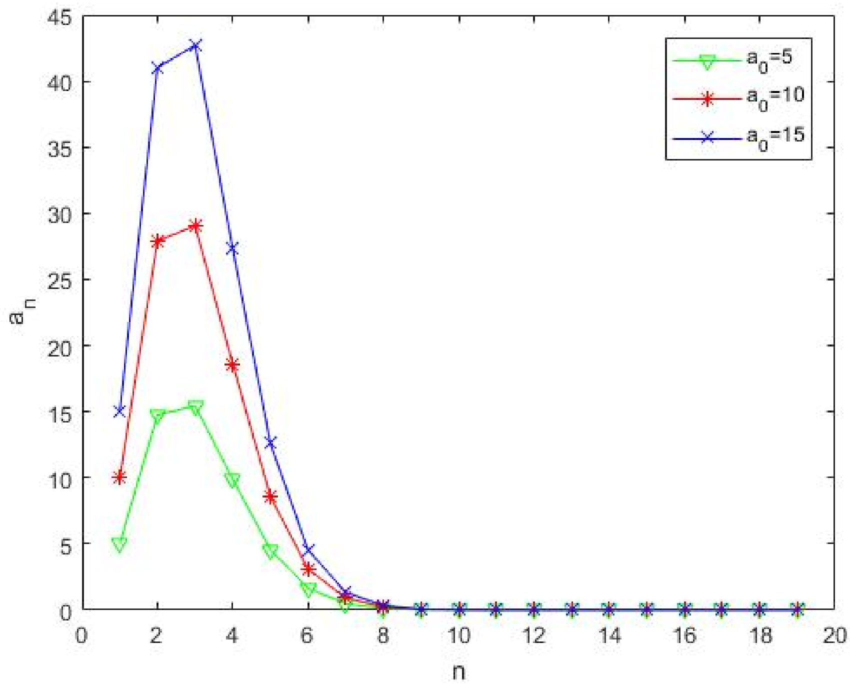

Figure 1: The convergence of a_n with initial values $a_0 = 5$, $a_0 = 10$ and $a_0 = 15$

Remark 6.2. *We choose the same operators as in Example 6.1 and compare our proposed Algorithm 5.1 with Mann-type Algorithm which are mentioned in Remark 5.2. On taking $\beta = 0$, we can calculate the sequence $\{a_n\}$ by the following Mann-type scheme:*

$$a_{n+1} = \left(\frac{3n^2-2n}{3n^2+1}\right)a_n + \left(\frac{2n+1}{3n^2+1}\right)[((6a_n + (6a_n \oplus 6)) \oplus (4a_n \oplus 12)) \oplus 6] + \left(\frac{2n+1}{3n^4+n^2}\right). \tag{24}$$

The iteration methods will be stopped when the stopping criteria $\|a_{n+1} - a_n\| \leq 10^{-11}$ is satisfied. Table 2, Figure 2 and Figure 3 are comparisons of our proposed Algorithm 5.1 with Mann-type Algorithm (24), on taking initial values $a_0 = 10$ and $a_0 = 15$.

Table 2: The values of a_n with initial value $a_0 = 10$

Number of Iteration	Proposed Algo For $a_0 = 10$ a_n	Mann-type Algo For $a_0 = 10$ a_n	Proposed Algo For $a_0 = 15$ a_n	Mann-type Algo For $a_0 = 15$ a_n
n=1	10	10	15	15
n=2	27.875000	41.1950000	41	60.770000
n=3	29.074175	48.177692	42.703983	70.801730
n=4	18.608089	32.583199	27.316022	47.850170
n=5	8.590385	15.635180	12.604305	22.954871
n=6	3.104010	5.939565	4.551397	8.721481
n=7	0.922939	1.996028	1.351626	2.935093
n=8	0.234178	0.733081	0.341919	1.083149
n=9	0.052583	0.397098	0.0761020	0.591064
n=10	0.011160	0.320126	0.015698	0.478654
n=15	1.990163e-13	0.290361	5.7300747e-08	0.450128
n=20	0	0.280616	4.020716e-10	0.305012
n=100	0	0.001664	1.144305e-13	0.001855
n=200	0	1.280347e-13	0	1.302043e-13

The numerical results of Table 2 and graphs of Figure 2 and Figure 3 imply that our proposed Algorithm 5.1 has a good performance and seems to have a competitive advantage. We can conclude that our algorithm is fast, efficient and stable, and it takes average of 13–20 iterations to converge.

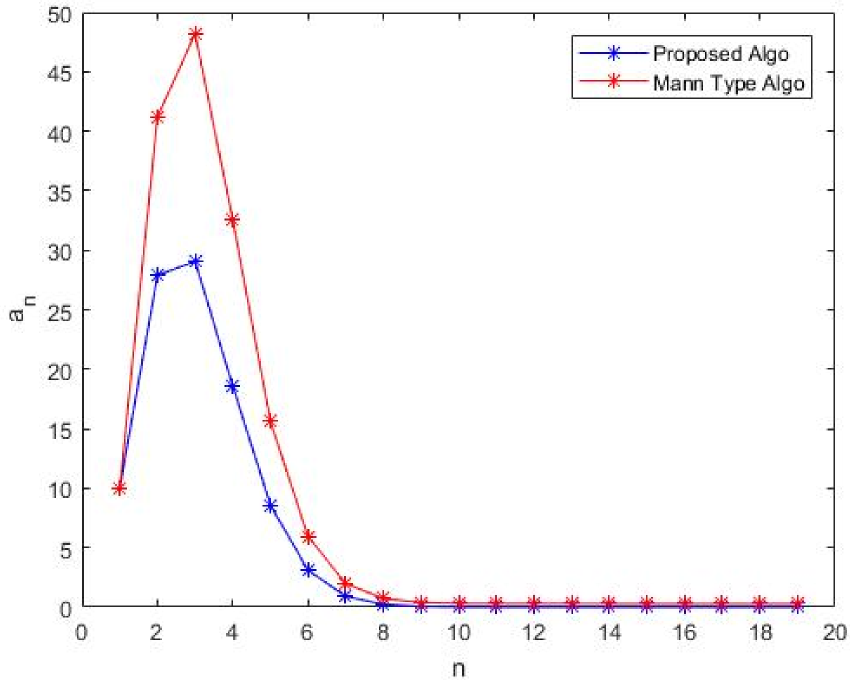

Figure 2: The convergence of a_n with initial value $a_0 = 10$

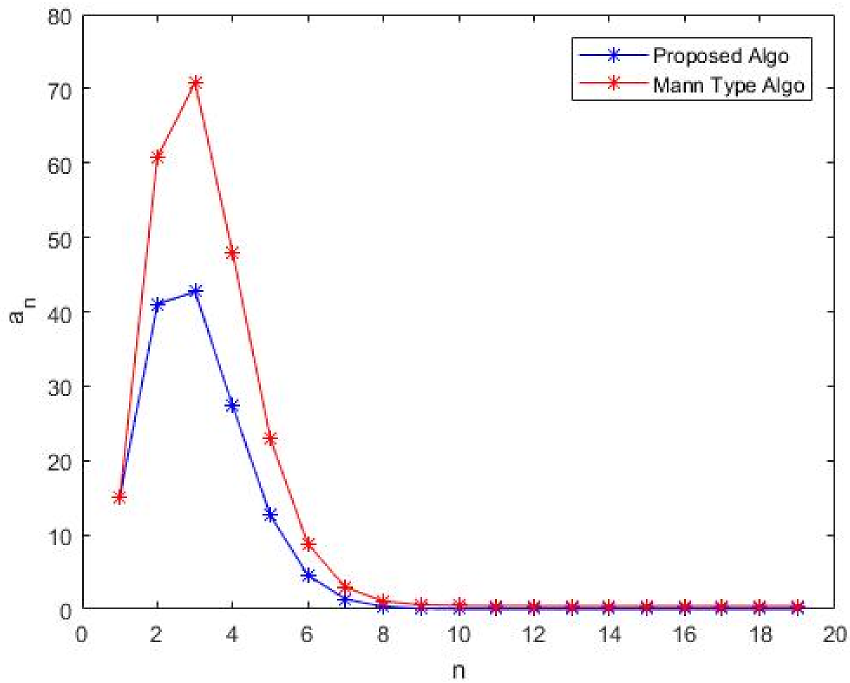

Figure 3: The convergence of a_n with initial value $a_0 = 15$

7. Conclusion

In this article, we study a general nonlinear ordered Yosida inclusion problem involving XOR-operation in a real ordered Hilbert spaces and prove the existence of solution of the problem. We have also constructed a perturbed two-step iterative algorithm for this class of general nonlinear ordered Yosida inclusion problem which is more general than the Mann-type iterative algorithms with errors, and many other iterative schemes studies by several author's, see e.g., [6, 8, 18–23]. We prove the convergence analysis of our proposed algorithm which assumes that the suggested algorithm converges in norm to a unique solution of our considered problem and also show that the convergence is stable. Finally, we give a numerical example in support of our main result. Our obtained results extend and generalize most of the results for different systems existing in the literature.

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