Analytical Solutions of Some General Classes of Differential and Integral Equations by Using the Laplace and Fourier Transforms

H. M. Srivastava[a], Mohammad Masjed-Jamei[b], Rabia Aktaş[c]

*[a]Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia V8W 3R4, Canada*

and

*Department of Medical Research, China Medical University Hospital,
China Medical University, Taichung 40402, Taiwan, Republic of China*

and

*Department of Mathematics and Informatics, Azerbaijan University,
71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan*

[b]Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16315-1618, Tehran 16315-1618, Iran

[c]Department of Mathematics, Faculty of Science, Ankara University, TR-06100 Tandogan, Ankara, Turkey

Abstract. This article deals with a general class of differential equations and two general classes of integral equations. By using the Laplace transform and the Fourier transform, analytical solutions are derived for each of these classes of differential and integral equations. Some illustrative examples and particular cases are also considered. The various analytical solutions presented in this article are potentially useful in solving the corresponding simpler differential and integral equations.

1. Introduction, Motivation and Preliminaries

Since Laplace and Fourier transforms were introduced in 19th century, they have been extensively studied and applied in the literature. Each of these integral transforms provides powerful and effective tools for finding the solutions of many problems in mathematics, physics, statistics and engineering.

The Laplace transform is usually used for solving several families of ordinary, partial, difference and integral equations. Along with these applications, some of the well-known applications of the Laplace transform are in heating, ventilation, air conditioning system modeling, modeling of radioactive decay in nuclear physics, electrical circuits and in signal processing. It has also applications in economics and management such as evaluation of payments, reliability and maintenance strategies, the choice of investments, the theory of system and element behavior, assessing econometric models, dynamical economic systems and inventory processes (see, for example, [1] to [8]).

The applications of Fourier transform are also numerous, ranging from solving some ordinary, partial, difference and integral equations to designing filters for noise reduction in audio-signals (such as music or speech) (see, for details, [1] to [8]). Both the Laplace transform and the Fourier transform also play an important role in Analytic Number Theory.

Let $f(x)$ be a continuous real-valued function for all $x \geqq 0$. The Laplace transform of $f(x)$ is defined by

$$L\big(f(x)\big) = \int_0^\infty e^{-sx} f(x)\, dx = F(s) \qquad (\forall\, s > 0) \tag{1}$$

and the inverse Laplace transform is given formally by

$$f(x) = L^{-1}\big(F(s)\big) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{sx} F(s)\, ds \qquad (\forall\, s > 0). \tag{2}$$

The Fourier transform of a function $g(x)$ and its inversion formula are given as follows:

$$\mathbf{F}\big(g(x)\big) = \int_{-\infty}^\infty e^{-ipx} g(x)\, dx = G(p)$$

$$\iff \quad g(x) = \mathbf{F}^{-1}\big(G(p)\big) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ipx} G(p)\, dp. \tag{3}$$

In this paper, we directly obtain the analytical solutions of a class of differential equations of the form:

$$\sum_{k=0}^m (a_k x + b_k)\, y^{(k)}(x) = r(x), \tag{4}$$

where $\{a_k\}_{k=0}^m$ and $\{b_k\}_{k=0}^m$ are finite sequences of known parameters, $y(x)$ is the unknown function and the function $r(x)$ is suitably prescribed. We also directly obtain the analytical solutions of two classes of integral equations of the convolution type whose kernels are given by (see, for details, [9]) $K(x,t) = t\,\kappa(x-t)$, that is,

$$\sum_{k=0}^m (a_k x + b_k)\, y^{(k)}(x) = r(x) + \int_0^x t\,\kappa(x-t)\, y^{(n)}(t)\, dt \tag{5}$$

and

$$\sum_{k=0}^m (a_k x + b_k)\, y^{(k)}(x) = r(x) + \int_{-\infty}^\infty t\,\kappa(x-t)\, y^{(n)}(t)\, dt. \tag{6}$$

2. Analytical Solutions of the Differential Equation (4)

In order to obtain the analytical solutions of the equation (4), we first consider some properties of the Laplace transform, which can be derived directly from the definition (1). Since it is known that

$$L\big(x\, f(x)\big) = -\frac{d}{ds} L\big(f(x)\big)$$

and

$$L\big(f^{(n)}(x)\big) = F(s)\, s^n - \sum_{k=0}^{n-1} f^{(k)}(0)\, s^{n-1-k}$$

$$\iff \quad \lim_{x \to \infty} f^{(k)}(x)\, e^{-sx} = 0, \tag{7}$$

we have

$$L\left(x\, y'(x)\right) = -\frac{d}{ds}\, L\left(y'(x)\right) = -s\frac{d}{ds}\, L\left(y(x)\right) - L\left(y(x)\right) = -sY'(s) - Y(s)$$

and

$$L\left(x\, y^{(n)}(x)\right) = -\frac{d}{ds}\left(Y(s)\, s^n - \sum_{k=0}^{n-1} y^{(k)}(0)\, s^{n-1-k}\right)$$

$$= -s^n Y'(s) - n\, s^{n-1} Y(s) + \sum_{k=0}^{n-2} (n-1-k)y^{(k)}(0)\, s^{n-2-k}. \tag{8}$$

By noting (7) and (8), and taking the Laplace transform of both sides of the equation (4), we find that

$$\sum_{k=0}^{m} \left[a_k L\left(x\, y^{(k)}(x)\right) + b_k L\left(y^{(k)}(x)\right)\right] = L\left(r(x)\right) = R(s)$$

$$\implies \sum_{k=0}^{m} a_k\left(-s^k Y'(s) - k\, s^{k-1} Y(s) + \sum_{j=0}^{k-2} (k-1-j)y^{(j)}(0)\, s^{k-2-j}\right)$$

$$+ \sum_{k=0}^{m} b_k\left(Y(s)\, s^k - \sum_{j=0}^{k-1} y^{(j)}(0)\, s^{k-1-j}\right) = R(s)$$

$$\implies \left(\sum_{k=0}^{m} a_k s^k\right) Y'(s) + \left(\sum_{k=0}^{m} k a_k s^{k-1} - b_k s^k\right) Y(s)$$

$$= -R(s) + \sum_{k=0}^{m} a_k\left(\sum_{j=0}^{k-2} (k-1-j)y^{(j)}(0)\, s^{k-2-j}\right)$$

$$- \sum_{k=0}^{m} b_k\left(\sum_{j=0}^{k-1} y^{(j)}(0)\, s^{k-1-j}\right), \tag{9}$$

which can be simplified as follows:

$$\left(\sum_{k=0}^{m} a_k s^k\right) Y'(s) + \left(\sum_{k=1}^{m} (k a_k - b_{k-1})s^{k-1} - b_m s^m\right) Y(s)$$

$$= -R(s) + \sum_{k=0}^{m} a_k\left(\sum_{j=0}^{k-2} (k-1-j)y^{(j)}(0)\, s^{k-2-j}\right)$$

$$- b_k\left(\sum_{j=0}^{k-1} y^{(j)}(0)\, s^{k-1-j}\right). \tag{10}$$

The equation (10) is a special case of the following first-order non-homogeneous differential equation:

$$p(x)\, y'(x) + q(x)\, y(x) = r(x), \tag{11}$$

whose exact solution is given by

$$y(x) = \exp\left(-\int \frac{q(x)}{p(x)}\, dx\right)\left[\int \frac{r(x)}{p(x)} \exp\left(\int \frac{q(x)}{p(x)}\, dx\right) dx + A\right] \tag{12}$$

for an arbitrary constant A. Hence, clearly, the exact solution of the equation (10) takes the following form:

$$
Y(s) = L\big(y(x)\big) = \exp\left[-\int \frac{\sum\limits_{k=1}^{m}\big(ka_k - b_{k-1}\big)s^{k-1} - b_m s^m}{\sum\limits_{k=0}^{m} a_k s^k}\,ds \right.
$$

$$
\cdot\left[\int \frac{-R(s) + \sum\limits_{k=0}^{m}\left[a_k\left(\sum\limits_{j=0}^{k-2}(k-1-j)y^{(j)}(0)\,s^{k-2-j}\right) - b_k\left(\sum\limits_{j=0}^{k-1} y^{(j)}(0)\,s^{k-1-j}\right)\right]}{\sum\limits_{k=0}^{m} a_k s^k}\right.
$$

$$
\left.\left. \cdot \exp\left(\int \frac{\left(\sum\limits_{k=1}^{m}\big(ka_k - b_{k-1}\big)s^{k-1}\right) - b_m s^m}{\sum\limits_{k=0}^{m} a_k s^k}\right) ds + A \right]\right], \tag{13}
$$

where A is an arbitrary constant.

An alternative way to solve the equation (4) is to use the Fourier transform and its inversion formula given by (3). Thus, from the definition (3), we first recall the following known formulas:

$$
\mathbf{F}\big(x\,g(x)\big) = i\frac{d}{dp}\,\mathbf{F}\big(g(x)\big) = i\,G'(p) \tag{14}
$$

and

$$
\mathbf{F}\big(g^{(n)}(x)\big) = (ip)^n G(p). \tag{15}
$$

Therefore, we have the following known consequences:

$$
\mathbf{F}\big(x\,y'(x)\big) = i\frac{d}{dp}\,\mathbf{F}\big(y'(x)\big) = -\frac{d}{dp}\big(p\mathbf{F}\big(y(x)\big)\big) = -pY'(p) - Y(p) \tag{16}
$$

and

$$
\mathbf{F}\big(x\,y^{(n)}(x)\big) = i^{n+1}\frac{d}{dp}\big(Y(p)\,p^n\big) = i^{n+1}p^n Y'(p) + i^{n+1}n\,p^{n-1}Y(p). \tag{17}
$$

By noting (15) and (17), and taking the Fourier transform of both sides of the equation (4), we find that

$$
\sum_{k=0}^{m} a_k \mathbf{F}\big(x\,y^{(k)}(x)\big) + b_k\mathbf{F}\big(y^{(k)}(x)\big) = \mathbf{F}\big(r(x)\big) = R(p)
$$

$$
\implies \sum_{k=0}^{m} a_k\,i^{k+1}\big(p^k Y'(p) + k\,p^{k-1}Y(p)\big) + b_k\,i^k\big(p^k Y(p)\big) = R(p)
$$

$$
\implies \left(\sum_{k=0}^{m} a_k i^{k+1}p^k\right) Y'(p) + \left(\sum_{k=0}^{m} ka_k i^{k+1}p^{k-1} + b_k i^k p^k\right) Y(p) = R(p). \tag{18}
$$

Hence, clearly, the exact solution of the equation (18) is given by

$$Y(p) = \mathbf{F}\big(y(x)\big) = \exp\left(-\int \frac{\sum\limits_{k=0}^{m}\big(ka_k i^{k+1} p^{k-1} + b_k i^k p^k\big)}{\sum\limits_{k=0}^{m} a_k i^{k+1} p^k}\, dp\right)$$
$$\cdot\left[\int \frac{R(p)}{\sum\limits_{k=0}^{m} a_k i^{k+1} p^k} \exp\left(\int \frac{\sum\limits_{k=0}^{m}\big(ka_k i^{k+1} p^{k-1} + b_k i^k p^k\big)}{\sum\limits_{k=0}^{m} a_k i^{k+1} p^k}\, dp\right) dp + B\right], \tag{19}$$

where B is an arbitrary constant.

3. Analytical Solutions of the Integral Equations (5) and (6)

By using the above-mentioned technique, we can similarly solve the two integral equations in (5) and (6). For example, with a view to analytically solving the equation (5), it is sufficient to take the Laplace transform of both sides of (5) and use the convolution theorem to get

$$\sum_{k=0}^{m} a_k L\big(x\, y^{(k)}(x)\big) + b_k L\big(y^{(k)}(x)\big) = L\big(r(x)\big) + L\left(\int_0^x t\, \kappa(x-t)\, y^{(n)}(t)\, dt\right)$$
$$= R(s) + K(s)\, L\big(x y^{(n)}(x)\big),$$

which yields

$$\sum_{k=0}^{m} a_k\left[\big(-s^k Y'(s) - k\, s^{k-1} Y(s)\big) + \sum_{j=0}^{k-2}(k-1-j)y^{(j)}(0)\, s^{k-2-j}\right]$$
$$+ \sum_{k=0}^{m} b_k\left[Y(s)\, s^k - \sum_{j=0}^{k-1} y^{(j)}(0)\, s^{k-1-j}\right]$$
$$= R(s) + K(s)\left[-s^n Y'(s) - n\, s^{n-1} Y(s) + \sum_{j=0}^{n-2}(n-1-j)y^{(j)}(0)\, s^{n-2-j}\right] \tag{20}$$

or, equivalently,

$$\left(\sum_{k=0}^{m} a_k s^k - K(s)\, s^n\right) Y'(s) + \left(\sum_{k=0}^{m}\big(ka_k s^{k-1} - b_k s^k\big) - n\, s^{n-1} K(s)\right) Y(s)$$
$$= -R(s) - \left(\sum_{j=0}^{n-2}(n-1-j)y^{(j)}(0)\, s^{n-2-j}\right) K(s)$$
$$+ \sum_{k=0}^{m} a_k\left(\sum_{j=0}^{k-2}(k-1-j)y^{(j)}(0)\, s^{k-2-j}\right) - b_k\left(\sum_{j=0}^{k-1} y^{(j)}(0)\, s^{k-1-j}\right). \tag{21}$$

Now, if we rearrange this last equation, it follows that

$$\left(\sum_{k=0}^{m} a_k s^k - K(s)\,s^n\right) Y'(s) + \left(\sum_{k=0}^{m}\left(ka_k s^{k-1} - b_k s^k\right) - n\,s^{n-1}K(s)\right) Y(s)$$

$$= -R(s) - \left(\sum_{j=0}^{n-2} (n-1-j)y^{(j)}(0)\,s^{n-2-j}\right) K(s)$$

$$+ \sum_{k=0}^{m}\left[a_k\left(\sum_{j=0}^{k-2}(k-1-j)y^{(j)}(0)\,s^{k-2-j}\right) - b_k\left(\sum_{j=0}^{k-1} y^{(j)}(0)\,s^{k-1-j}\right)\right]. \tag{22}$$

From (12), the exact solution of the equation (22) is given by

$$Y(s) = L\left(y(x)\right) = \exp\left(-\int \frac{\sum\limits_{k=0}^{m} ka_k s^{k-1} - b_k s^k - n\,s^{n-1}K(s)}{\sum\limits_{k=0}^{m} a_k s^k - K(s)\,s^n}\,ds\right)$$

$$\cdot\left[\int \frac{-R(s) - \left(\sum\limits_{j=0}^{n-2}(n-1-j)y^{(j)}(0)\,s^{n-2-j}\right)K(s) + \sum\limits_{k=0}^{m}\left[a_k\left(\sum\limits_{j=0}^{k-2}(k-1-j)y^{(j)}(0)\,s^{k-2-j}\right) - b_k\left(\sum\limits_{j=0}^{k-1} y^{(j)}(0)\,s^{k-1-j}\right)\right]}{\sum\limits_{k=0}^{m} a_k s^k - K(s)\,s^n}\right.$$

$$\left.\cdot\exp\left(\int \frac{\sum\limits_{k=0}^{m}\left(ka_k s^{k-1} - b_k s^k\right) - n\,s^{n-1}K(s)}{\sum\limits_{k=0}^{m} a_k s^k - K(s)\,s^n}\,ds\right) ds + C\right], \tag{23}$$

where C is an arbitrary constant.

In order to find the analytical solutions of the integral equation (6), if we apply the Fourier transform to both sides of (6), we get

$$\sum_{k=0}^{m} a_k \mathbf{F}\left(x\,y^{(k)}(x)\right) + b_k \mathbf{F}\left(y^{(k)}(x)\right) = \mathbf{F}\left(r(x)\right) + \mathbf{F}\left(\int_{-\infty}^{\infty} t\,\kappa(x-t)\,y^{(n)}(t)\,dt\right).$$

So, from (15) and (17), and using the convolution theorem, it follows that

$$\sum_{k=0}^{m} a_k\,i^{k+1}\left(p^k Y'(p) + k\,p^{k-1}Y(p)\right) + b_k\,i^k\left(p^k Y(p)\right) = R(p) + K(p)\mathbf{F}\left(xy^{(n)}\right)$$

or, equivalently,

$$\left(\sum_{k=0}^{m} a_k i^{k+1}p^k\right)Y'(p) + \left(\sum_{k=0}^{m}\left(ka_k i^{k+1}p^{k-1} + b_k i^k p^k\right)\right)Y(p)$$

$$= R(p) + K(p)\left(i^{n+1}p^n Y'(p) + i^{n+1}n\,p^{n-1}Y(p)\right),$$

which can be simplified as follows:

$$\left(\sum_{k=0}^{m} a_k i^{k+1}p^k - i^{n+1}p^n K(p)\right)Y'(p)$$

$$+ \left(\sum_{k=0}^{m}\left(ka_k i^{k+1}p^{k-1} + b_k i^k p^k\right) - i^{n+1}n\,p^{n-1}K(p)\right)Y(p) = R(p). \tag{24}$$

According to (12), the exact solution of the equation (24) is given by

$$Y(p) = \mathbf{F}\big(y(x)\big) = \exp\left(-\int \frac{\sum\limits_{k=0}^{m}\big(ka_k i^{k+1}p^{k-1} + b_k i^k p^k\big) - i^{n+1}n\,p^{n-1}K(p)}{\sum\limits_{k=0}^{m} a_k i^{k+1}p^k - i^{n+1}p^n K(p)}\,dp\right)$$

$$\cdot\left[\int \frac{R(p)}{\sum\limits_{k=0}^{m} a_k i^{k+1}p^k - i^{n+1}p^n K(p)}\,\exp\left(\int \frac{\sum\limits_{k=0}^{m}\big(ka_k i^{k+1}p^{k-1} + b_k i^k p^k\big) - i^{n+1}n\,p^{n-1}K(p)}{\sum\limits_{k=0}^{m} a_k i^{k+1}p^k - i^{n+1}p^n K(p)}\,dp\right)dp + D\right], \quad (25)$$

where D is an arbitrary constant.

4. Examples and Particular Cases

In this section, we consider some particular examples of the equations (4), (5) and (6) and obtain the corresponding analytical solutions.

Example 1. Let $m = 2$, $a_0 = 0$, $b_0 = n$, $a_1 = -1$, $b_1 = 1$, $a_2 = 1$, $b_2 = 0$ and $r(x) = 0$ in the equation (4). Then the familiar Laguerre differential equation emerges as follows:

$$xy'' + (1 - x)\,y' + ny = 0.$$

The equation (10) is, therefore, reduced to the following form:

$$\big(s - s^2\big)Y'(s) + (n + 1 - s)\,Y(s) = 0.$$

From (13), the solution of the above equation is given by

$$Y(s) = \frac{(s-1)^n}{s^{n+1}}.$$

Now, by taking the inverse Laplace transform, we finally get

$$y(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!} =: L_n(x), \tag{26}$$

where $L_n(x)$ is the Laguerre polynomial of degree n in x.

Example 2. Let us consider Example 1 for $r(x) = x^{n-1}$. In this case, the following non-homogenous second-order differential equation emerges:

$$xy'' + (1 - x)\,y' + ny = x^{n-1}. \tag{27}$$

So, the equations (10) and (13) are reduced, respectively, to the following forms:

$$\big(s - s^2\big)Y'(s) + (n + 1 - s)\,Y(s) = \frac{(n-1)!}{s^n}$$

and

$$Y(s) = \frac{n!}{n^2 s^{n+1}} + A\frac{(s-1)^n}{s^{n+1}}.$$

Finally, by applying the inverse Laplace transform, the analytical solution of the equation (27) can be deduced as follows:

$$y(x) = \frac{x^n}{n^2} + AL_n(x), \tag{28}$$

where $L_n(x)$ denotes the Laguerre polynomial given by (26) and A is an arbitrary constant.

Example 3. Let us consider a special case of the equation (5) as follows:

$$xy' - (x+1)y = x^4 + \int_0^x t\,y''(t)\,dt \quad \text{with} \quad y(0) = 0.$$

Hence, the equation (22) is reduced to the form given by

$$Y'(s) = \frac{24}{s^5}. \tag{29}$$

The solution of this last equation (29) is given by

$$y(x) = -x^3. \tag{30}$$

5. Concluding Remarks and Observations

In our investigation here, we have presented the analytical solutions of a general class of differential equations and two general classes of integral equations. Our methodology makes use of the Laplace transform and the Fourier transform, together with their inversion and convolution theorems. We have also considered three illustrative examples and particular cases of our main results. Potential usefulness of the various analytical solutions presented in this article cannot be overemphasized.

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