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Boundary Schwarz Lemma and Rigidity Property for Holomorphic Mappings of the Unit Polydisc in $\mathbb{C}^n$

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Abstract. In this paper, we generalize the classical Schwarz lemma at the boundary from the unit disk D in the complex plane to the unit polydisc D^n in higher-dimensional complex space. Two boundary Schwarz lemmas for holomorphic mappings of D^n and corresponding rigidity properties are established without the restriction of the interior fixed point.

1. Introduction

For notations, let D be the unit disk in the complex plane $\mathbb{C}$. Denote by $\mathbb{C}^n$ the n-dimensional complex Hilbert space with the inner product and the norm given by

$$\langle z, \omega \rangle = \sum_{i=1}^{n} z_i \bar{\omega}_i,$$

and

$$\|z\| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\frac{1}{2}} = \langle z, z \rangle^{\frac{1}{2}},$$

where $z = (z_1, z_2, \cdots, z_n)^T, \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \in \mathbb{C}^n$. Let $B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the open unit ball with the unit sphere $\partial B^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. Moreover, denote $\|z\|_\infty = \max_{1 \le i \le n} |z_i|$, then the unit polydisc can be represented as $D^n = D \times \cdots \times D = \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}$ directly. Let T be the unit circle of D, then $T^n = T \times \cdots \times T = \{z \in \mathbb{C}^n : |z_i| = 1, 1 \le i \le n\}$ stands for the distinguished boundary of D^n.

Denote the set of all holomorphic self-mappings of the unit polydisc D^n as $H(D^n)$. For any $f = (f_1, f_2, \cdots, f_n)^T \in H(D^n)$, the Jacobian matrix of f at $z \in D^n$ is defined by

$$Df(z) = (Df_i(z))_{n \times 1} = \left(\frac{\partial f_i}{\partial z_j}(z) \right)_{n \times n}$$

where f_i is a holomorphic function from D^n to $\mathbb{C}$, and $i, j = 1, 2, \cdots, n$. It is clear that $Df(z)$ is a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$.

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The Schwarz lemma is regarded as one of the most important results in complex analysis. Various versions of Schwarz lemma have been rapidly developed in recent years, such as the Schwarz-Pick lemma for various holomorphic mappings [1-2] and the Ahlfors-Schwarz lemma for quasiconformal Euclidean harmonic functions [3]. With the development of Schwarz lemma, some extended results associated with the boundary Schwarz lemma have also been presented based on different mappings, such as real-valued functions in $C^2(D)$[4], harmonic self-mappings of the unit disk [5], holomorphic self-mappings of the unit ball [6], pluriharmonic self-mappings of the unit ball [6] and pluriharmonic mappings between the unit polydiscs with any dimensions [7]. Nevertheless, these papers provide no results about the rigidity theorem.

The well-known rigidity theorem is form the work of Cartan. He stated that if Ω is a bounded domain in $\mathbb{C}^n$ and $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z) = z + o(\|z - z_0\|)$ as $z \to z_0$ for some $z_0 \in \Omega$, then $f(z) \equiv z$. It is a natural task to consider a boundary version of the rigidity theorem, which is closely related to the Schwaez lemma at the boundary.

In the case of one complex variable, the classical boundary Schwarz lemma is described as follow:

Theorem 1.1. [8] *Let $f : D \to D$ be a holomorphic mapping. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then*
(1) $f^{'}(1) \geq 1$;
(2) $f^{'}(1) = 1$, *if and only if* $f(z) \equiv z$.

Theorem 1.1 has the following extension.

Theorem 1.2. [9] *Let $f : D \to D$ be a holomorphic mapping. If f is holomorphic at $z = \alpha \in T$ with $f(0) = 0$ and $f(\alpha) = \beta \in T$, then*
(1) $\overline{\beta} f^{'}(\alpha)\alpha \geq 1$;
(2) $\overline{\beta} f^{'}(\alpha)\alpha = 1$, *if and only if* $f(z) \equiv e^{i\theta} z$, *where* $e^{i\theta} = \beta\alpha^{-1}$ *and* $\theta \in \mathbb{R}$.

When $\alpha = \beta = 1$, Theorem 1.2 reduces to Theorem 1.1. In fact, the sharp conditions (2) in two theorem are just the rigidity theorem for holomorphic mappings of D.

On the other hand, in the case of several complex variables, Burns and Krantz [10] studied a new boundary Schwarz lemma, which gives a new rigidity theorem for holomorphic mappings of a bounded strongly pseudo-convex domain.

Theorem 1.3. [10] *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudo-convex domain, and $p \in \partial\Omega$. If $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z) = z + O(\|z - p\|^4)$ as $z \to p$, then $f(z) \equiv z$.*

Huang [11] further studied a sharp boundary version of the Schwarz lemma for holomorphic mappings of strongly convex domain with an interior fixed point.

Theorem 1.4. [11] *Let $\Omega \subset \mathbb{C}^n$ be a strongly convex domain, and $p \in \partial\Omega$. If $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z_0) = z_0$ for some $z_0 \in \Omega$ and $f(z) = z + o(\|z - p\|^2)$ as $z \to p$, then $f(z) \equiv z$.*

Huang [12] also obtained a semi-rigidity property for holomorphic mappings between unit balls with different dimensions. In [13], Krantz explored versions of the Schwarz lemma at the boundary, and gave some boundary rigidity theorems. Recently, Liu et al. [9] presented a boundary version of the rigidity theorem for holomorphic mappings of the unit ball B^n in $\mathbb{C}^n$ with the fixed point condition $f(0) = 0$. Furthermore, in [14] they generalized the result with the fixed point origin to the case of the holomorphic with the interior fixed point in the unit ball B^n.

Theorem 1.5. [9] *Let $f : B^n \to B^n$ be a holomorphic mapping with $f(0) = 0$. Then the following two conclusions hold:*
(1) *If f is holomorphic at $z = \alpha \in \partial B^n$ with $f(\alpha) = \beta \in \partial B^n$, then*

$$\beta^H Df(\alpha)\alpha \geq 1;$$

(2) *If there exist linearly independent* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \partial B^n$ *such that* f *is holomorphic at* $z = \alpha_k \in \partial B^n$ *with* $f(\alpha_k) = \beta_k \in \partial B^n (k = 1, 2, \cdots, n)$*, then the following n equalities*

$$\beta_k^H Df(\alpha_k)\alpha_k = 1$$

hold if and only if

$$f(z) \equiv Uz$$

and $U = (\beta_1, \beta_2, \cdots, \beta_n)(\alpha_1, \alpha_2, \cdots, \alpha_n)^{-1}$ *is a unitary matrix of order n.*

Theorem 1.6. [14] *Let* $f : B^n \to B^n$ *be a holomorphic mapping and let* $f(a) = a$ *for some* $a \in B^n$*. Then the following two conclusions hold:*

(1) *If* f *is holomorphic at* $z = \alpha \in \partial B^n$ *with* $f(\alpha) = \beta \in \partial B^n$*, then*

$$\beta^H Df(\alpha)\alpha \geq \frac{|1 - a^H\beta|^2}{|1 - a^H\alpha|^2};$$

(2) *If there exist linearly independent* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \partial B^n$ *such that* $\alpha_1 - a, \alpha_2 - a, \cdots, \alpha_n - a$ *are linearly independent and* f *is holomorphic at* $z = \alpha_k \in \partial B^n$ *with* $f(\alpha_k) = \beta_k \in \partial B^n (k = 1, 2, \cdots, n)$*, then the following n equalities*

$$\beta_k^H Df(\alpha_k)\alpha_k = \frac{|1 - a^H\beta_k|^2}{|1 - a^H\alpha_k|^2}$$

hold if and only if

$$f(z) \equiv \varphi_a(U\varphi_a(z))$$

where $\varphi_a(z)$ *is a biholomorphic automorphism of* B^n *and* $U = (\varphi_a(\beta_1), \cdots, \varphi_a(\beta_n))(\varphi_a(\alpha_1), \cdots, \varphi_a(\alpha_n))^{-1}$ *is a unitary matrix of order n.*

These results are based on the conditions about some fixed points, such as $f(0) = 0$ and $f(a) = a$. In order to draw a general conclusion, we consider removing the restriction and providing a universal result. In the case of one complex variable, Theorem1.1 can be generalized to the following estimate by removing the condition $f(0) = 0$:

Theorem 1.7. [15] *Let* $f : D \to D$ *be a holomorphic mapping. If* f *is holomorphic at* $z = 1$ *with* $f(1) = 1$*, then*

(1) $f'(1) \geq \frac{|1-\overline{f(0)}|^2}{1-|f(0)|^2} > 0$;

(2) $f'(1) = \frac{|1-\overline{f(0)}|^2}{1-|f(0)|^2}$*, if and only if* $f(z) \equiv z$.

Furthermore, Mateljevic [16] introduced a finite Blaschke product $\mathcal{B}$ and presented the theorem connected with the rigidity of holomorphic mappings on the unit disk, which offered the sufficient condition that a holomorphic function f is of the form $f(z) = 1 + \mathcal{B} + o(z - a)^2$.

Theorem 1.8. [16] *Let* $\mathcal{B} : D \to D$ *be a finite Blaschke product which equal* ω_0 *on a finite set* $A \subset T$*. Let* f *be a holomoephic function in the unit disk and* $\|f(z) - 1\| < 1$ *for* $\|z\| < 1$*. Suppose that* $f(z) = 1 + \mathcal{B} + o(z - a)^2$ *for all* $a \in A$ *and* $z \in T$ *as* $z \to a$*, then* $u(z) = Re(A_F) - Re(A_B)$ *is continuous on* $D \cup A$ *and satisfies the condition* $\liminf_{z\to\omega} u(z) \geq 0$ *for every* $\omega \in T$*, and it is non-negative on* D*.*

However, in the case of high dimensional variables, there are few results associated with the rigidity of the holomorphic mappings. It is a significant task to obtain the novel rigidity theorem for the holomorphic mappings of the domain in $\mathbb{C}^n$.

In this paper, inspired by Theorem 1.7 and [9], we first present two new boundary Schwarz lemmas for the holomorphic mappings of the unit polydisc in $\mathbb{C}^n$, which extend Theorem 1.7 to higher dimensions. Meanwhile, corresponding rigidity of the holomorphic mappings is established as an important component of the boundary Schwarz lemma. It is worth noting that our main results have precise expressions and need not consider the partial derivatives of order 2 and order 3 at the boundary point, which are essentially different from the work in [7], [8], and [13]. Compared with [6] and [11], this paper provides different results in the different domains without the restrictions of the fixed point.

2. Main results

In this section, we generalize Theorem 1.7 to higher dimension. There are two boundary Schwarz lemmas and corresponding rigidity properties for holomorphic mappings of the unit polydisc D^n in $\mathbb{C}^n$ with different conditions.

Theorem 2.1. *Let $p = (p_1, \cdots, p_n)^T, q = (q_1, \cdots, q_n)^T \in T^n$. If $f : D^n \to D^n$ is a holomorphic mapping with $f(p) = q$, then for any $i = 1, 2, \cdots, n$ we have the following statements:*

(1) $\overline{q_i} Df_i(p)p \geq \frac{|1-\overline{q_i} f_i(0)|^2}{1-|\overline{q_i} f_i(0)|^2}$ where $f_i = f_i(z_1, \cdots, z_n)$ and $Df_i = (\frac{\partial f_i}{\partial z_1}, \cdots, \frac{\partial f_i}{\partial z_n})$;

(2) $\overline{q_i} Df_i(p)p = \frac{|1-\overline{q_i} f_i(0)|^2}{1-|\overline{q_i} f_i(0)|^2}$ if and only if $f(z) \equiv Uz$, and

$$U = diag(q_1, \cdots, q_n) diag(\overline{p_1}, \cdots, \overline{p_n})$$

is a unitary matrix of order n.

Proof. (1) Take

$$g_i(\zeta) = \overline{q_i} f_i(\zeta p), \zeta \in D.$$

It is easy to deduce that $|g_i(\zeta)| \leq |\overline{q_i}||f_i(\zeta p)| < 1$ and $g_i(1) = \overline{q_i} p_i = 1$. Therefore, $g_i : D \to D$ is a holomorphic function and is holomorphic at $\zeta = 1$ with $g(1) = 1$. From theorem 1.7, we have

$$\frac{|1 - \overline{q_i} f_i(0)|^2}{1 - |\overline{q_i} f_i(0)|^2} \leq g_i'(1) = \overline{q_i} Df_i(p)p.$$

The proof of (1) is complete.

(2) Suppose that the equalities $\overline{q_i} Df_i(p)p = \frac{|1-\overline{q_i} f_i(0)|^2}{1-|\overline{q_i} f_i(0)|^2}$ hold for any $i = 1, 2, \cdots, n$. Since

$$\overline{q_i} Df_i(p)p = g_i'(1),$$

and

$$\frac{|1 - \overline{q_i} f_i(0)|^2}{1 - |\overline{q_i} f_i(0)|^2} = \frac{|1 - \overline{g_i(0)}|^2}{1 - |\overline{g_i(0)}|^2},$$

from (1) and theorem 1.7, we have $g_i(\zeta) = \zeta$, which means

$$\overline{q_i} f_i(\zeta p) = \zeta.$$

It follows that for any $i = 1, 2, \cdots, n$

$$f_i(\zeta p) = \zeta q_i.$$

Thus,

$$f(\zeta p) = (f_1(\zeta p), f_2(\zeta p), \cdots, f_n(\zeta p))^T = \zeta (q_1, q_2, \cdots, q_n)^T = \zeta q.$$

Moreover, considering

$$diag(q_1, q_2, \cdots, q_n) = \begin{pmatrix} q_1 & 0 & \ldots & 0 \\ 0 & q_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & q_n \end{pmatrix},$$

$$diag(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n}) = \begin{pmatrix} \overline{p_1} & 0 & \ldots & 0 \\ 0 & \overline{p_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \overline{p_n} \end{pmatrix},$$

we get the matrix

$$U = diag(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n}) diag(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n}) = \begin{pmatrix} q_1\overline{p_1} & 0 & \dots & 0 \\ 0 & q_2\overline{p_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_n\overline{p_n} \end{pmatrix}.$$

Notice that U is a unitary matrix and

$$U(\zeta p) = \zeta(q_1, q_2, \cdots, q_n)^T = \zeta q. \tag{1}$$

Let $z = \zeta p \in D^n$, then equations (3) and (4) yield

$$f(z) \equiv Uz,$$

where U is a unitary matrix of order n.

Conversely, suppose that

$$f(z) \equiv Uz,$$

and

$$U = diag(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n}) diag(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n})$$

is a unitary matrix of order n, then

$$\overline{q_i} Df_i(p)p = \overline{q_i}(q_i\overline{p_i})p_i = 1$$

and

$$\frac{|1 - \overline{q_i} f_i(0)|^2}{1 - |\overline{q_i} f_i(0)|^2} = 1.$$

That implies for any $i = 1, 2, \cdots, n$

$$\overline{q_i} Df_i(p)p = \frac{|1 - \overline{q_i} f_i(0)|^2}{1 - |\overline{q_i} f_i(0)|^2}.$$

The proof of Theorem 2.1 has been finished. □

Especially, when $f(p) = p$, the boundary rigidity theorem for holomorphic self-mappings of D^n is deduced from Theorem2.1 that

Theorem 2.2. *Let $p = (p_1, \cdots, p_n)^T \in T^n$. If $f : D^n \to D^n$ is a holomorphic mapping with $f(p) = p$, then for any $i = 1, 2, \cdots, n$ we have the following statements:*

(1) $\overline{p_i} Df_i(p)p \geq \frac{|1-\overline{p_i} f_i(0)|^2}{1-|\overline{p_i} f_i(0)|^2}$ where $f_i = f_i(z_1, \cdots, z_n)$ and $Df_i = (\frac{\partial f_i}{\partial z_1}, \cdots, \frac{\partial f_i}{\partial z_n})$;

(2) $\overline{p_i} Df_i(p)p = \frac{|1-\overline{p_i} f_i(0)|^2}{1-|\overline{p_i} f_i(0)|^2}$ if and only if $f(z) \equiv z$.

Proof. (1) Take

$$g_i(\zeta) = \overline{p_i} f_i(\zeta p), \zeta \in D.$$

It is easy to deduce that $g_i : D \to D$ is a holomorphic function and is holomorphic at $\zeta = 1$ with $g(1) = 1$. From theorem 1.7, we have

$$\frac{|1 - \overline{p_i} f_i(0)|^2}{1 - |\overline{p_i} f_i(0)|^2} \leq g_i'(1) = \overline{p_i} Df_i(p)p.$$

The proof of (1) is complete.

(2) Suppose that the equalities $\overline{p_i} Df_i(p)p = \frac{|1-\overline{p_i} f_i(0)|^2}{1-|\overline{p_i} f_i(0)|^2}$ hold for any $i = 1, 2, \cdots, n$. Since

$$\overline{p_i} Df_i(p)p = g_i'(1),$$

and
$$\frac{|1-\overline{p_i}f_i(0)|^2}{1-|\overline{p_i}f_i(0)|^2}=\frac{|1-\overline{g_i(0)}|^2}{1-|\overline{g_i(0)}|^2},$$
from (1) and theorem 1.7, we have $g_i(\zeta)=\zeta$, which means
$$\overline{p_i}f_i(\zeta p)=\zeta.$$
It follows that for any $i=1,2,\cdots,n$
$$f_i(\zeta p)=\zeta p_i.$$
Thus,
$$f(\zeta p)=(f_1(\zeta p),f_2(\zeta p),\cdots,f_n(\zeta p))^T=\zeta(p_1,p_2,\cdots,p_n)^T=\zeta p.$$
Let $z=\zeta p\in D^n$, then equation (7) deduces
$$f(z)\equiv z.$$

Conversely, suppose that $f(z)\equiv z$, then
$$\overline{p_i}Df_i(p)p=\overline{p_i}p_i=1$$
and
$$\frac{|1-\overline{p_i}f_i(0)|^2}{1-|\overline{p_i}f_i(0)|^2}=1.$$
It implies that for any $i=1,2,\cdots,n$
$$\overline{p_i}Df_i(p)p=\frac{|1-\overline{p_i}f_i(0)|^2}{1-|\overline{p_i}f_i(0)|^2}.$$

The proof of Theorem 2.2 has been finished. □

When $p=q$, Theorem 2.2 is just Theorem 2.1.

Remark 2.3. *When $p=q$, Theorem 2.2 is just Theorem 2.1.*

Remark 2.4. *As far as Theorem 2.1 and Theorem 2.2, we just need to assume that the mapping f is C^1 up to the boundary of the unit polydisc D^n near p.*

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