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Automorphisms and Isomorphisms of Enhanced Hypercubes

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Abstract. Let $\mathbb{Z}_2^n$ be the elementary abelian 2-group, which can be viewed as the vector space of dimension n over F_2. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}_2^n$ and $\epsilon_k = e_k + \cdots + e_n$ for some $1 \leq k \leq n-1$. Denote by $\Gamma_{n,k}$ the Cayley graph over $\mathbb{Z}_2^n$ with generating set $S_k = \{e_1, \ldots, e_n, \epsilon_k\}$, that is, $\Gamma_{n,k} = \mathrm{Cay}(\mathbb{Z}_2^n, S_k)$. In this paper, we characterize the automorphism group of $\Gamma_{n,k}$ for $1 \leq k \leq n-1$ and determine all Cayley graphs over $\mathbb{Z}_2^n$ isomorphic to $\Gamma_{n,k}$. Furthermore, we prove that for any Cayley graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^n, T)$, if Γ and $\Gamma_{n,k}$ share the same spectrum, then $\Gamma \cong \Gamma_{n,k}$. Note that $\Gamma_{n,1}$ is known as the so called n-dimensional folded hypercube FQ_n, and $\Gamma_{n,k}$ is known as the n-dimensional enhanced hypercube $Q_{n,k}$.

1. Introduction

It is known that an interconnection network is conveniently represented by an undirected graph. For instance, the vertices of the graph represent the nodes of the network and the edges represent the links of the network. The *n-dimensional hypercube* Q_n is the graph whose vertices are the n-bit binary strings $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1\}$ for $1 \leq i \leq n$, and whose edges are pairs of vertices differing in exactly one position. As a topology for an interconnection network of a multiprocessor system, the hypercube is a widely used and well-known model, since it possesses many attractive properties such as regularity, symmetry, logarithmic diameter, high connectivity, recursive construction, ease of bisection, and relatively low link complexity [10, 13, 16]. There are many invariants of Q_n, for instance, generalized hypercube, folded hypercube, twisted hypercube, argument hypercube and enhanced hypercube. In this paper, we focus on the folded hypercube and the enhance hypercube. As a variant of the hypercube, the *n-dimensional folded hypercube* FQ_n, proposed first by El-Amawy and Latifi [2], is obtained from the hypercube Q_n by making each vertex u adjacent to its complementary vertex, denoted by $\bar{u}$ and obtained from u by subtracting each bit from 1. Some properties of the folded hypercube FQ_n are discussed in [11, 16–18]. Another variant of hypercube, the so called *enhanced hypercube* $Q_{n,k}$, proposed first by Tzeng and Wei [15], is obtained from Q_n by adding the edges $\{x, y\}$ if $x = (x_1, \ldots, x_n)$ and $y = (x_1, \ldots, x_{k-1}, \overline{x_k}, \ldots, \overline{x_n})$ where $\overline{x_j} = x_j + 1$. It is clear that the folded hypercube FQ_n is the special case of the enhanced hypercube $Q_{n,k}$ for $k = 1$, that is, $FQ_n = Q_{n,1}$.

Let $\mathbb{Z}_2^n$ be the elementary abelian 2-group. It can be viewed as the n-dimensional vector space over the filed F_2 and $\{e_1, \ldots, e_n\}$ forms the orthonormal basis of $\mathbb{Z}_2^n$, where e_i is the vector whose i-th entry is 1 and other entries are 0s. From the definition of the enhanced hypercube $Q_{n,k}$, each vertex of $Q_{n,k}$ is a vector in

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$\mathbb{Z}_2^n$. Moreover, two vertices v and v' are adjacent if one of the followings holds:
(1) v and v' are distinct in exactly one position, which means $v' - v = e_i$ for some $1 \leq i \leq n$.
(2) the first $k-1$ entries of v and v' are the same but the rest $n-k+1$ entries of them are all distinct, which means that $v' - v = e_k + \cdots + e_n$.

Recall that the *Cayley graph* over a group G with the generating set $S \subset G$, denoted by $\mathrm{Cay}(G,S)$, is the graph with vertex set G and two vertices x and y are adjacent if $yx^{-1} \in S$. Therefore, the n-dimensional enhanced hypercube $Q_{n,k}$ is just the Cayley graph $\Gamma_{n,k} = \mathrm{Cay}(\mathbb{Z}_2^n, S_k)$, where $S_k = \{e_1, \ldots, e_n, \epsilon_k\}$ and $\epsilon_k = e_k + \cdots + e_n$. Particularly, the n-dimensional folded hypercube FQ_n is the Cayley graph $\Gamma_{n,1} = \mathrm{Cay}(\mathbb{Z}_2^n, S_1)$, where $S_1 = \{e_1, \ldots, e_n, \epsilon_1\}$ and $\epsilon_1 = e_1 + \cdots + e_n$.

An isomorphism α from a graph Γ to Γ' is a bijection from $V(\Gamma)$ to $V(\Gamma')$ such that $u \sim v$ in Γ if and only if $\alpha(u) \sim \alpha(v)$ in Γ'. If Γ and Γ' are the same, then α is called an *automorphism* of Γ. The set of all automorphisms of Γ forms the *automorphism group* of Γ, denoted by $Aut(\Gamma)$ [4]. There exists strong connection between the automorphism of a graph and the structure of the graph. For example, a graph with high symmetry always has a large automorphism group and a graph with little symmetry always has a small automorphism group. Therefore, we would like to investigate the automorphism group of the hypercube $Q_{n,k}$. For a Cayley graph $\Gamma = \mathrm{Cay}(G,S)$, let $Aut_e(\Gamma) = \{\alpha \in Aut(\Gamma) \mid \alpha(e) = e\}$ and $Aut_S(G) = \{\beta \in Aut(G) \mid \beta(S) = S\}$, where the subscript e denotes the identity element of G. If G is abelian, we write $Aut_0(\Gamma)$ for $Aut_e(\Gamma)$ since the identity element of abelian groups is always denoted by 0. It is clear that $Aut_S(G) \leq Aut_e(\Gamma)$. If $Aut_S(G) = Aut_e(\Gamma)$ then Γ is called *normal*. In this paper, we characterize the automorphism group of $\Gamma_{n,k}$. Moreover, we determine all Cayley graphs over $\mathbb{Z}_2^n$ isomorphic to $\Gamma_{n,k}$. Furthermore, we give the spectrum of $\Gamma_{n,k}$ and show that $\Gamma_{n,k}$ is determined by its spectrum among the Cayley graphs over $\mathbb{Z}_2^n$, that is, for any Cayley graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^n, T)$, if Γ and $\Gamma_{n,k}$ share the same spectrum, then $\Gamma \cong \Gamma_{n,k}$.

2. Basic properties

In this part, we present some properties of Cayley graphs which will be used latter. At first, we introduce the automorphism group of $\mathbb{Z}_2^n$.

Lemma 2.1 ([1]). *The automorphism group of $\mathbb{Z}_2^n$ is isomorphic to $GL(n, F_2)$ where $GL(n, F_2)$ is the set of invertible matrices of order n over F_2. Furthermore, we have*

$$|Aut(\mathbb{Z}_2^n)| = |GL(n, F_2)| = \prod_{k=1}^{n} (2^n - 2^{k-1}).$$

In fact, each element v of $\mathbb{Z}_2^n$ is a $(0,1)$-vector. For any $\sigma \in Aut(\mathbb{Z}_2^n)$, there exists $M_\sigma \in GL(n, F_2)$ corresponding to σ such that $\sigma(v) = M_\sigma v$. By simple observations, one can obtain the following well-known result.

Lemma 2.2. *For any two non-zero vectors $x, y \in \mathbb{Z}_2^n$, there exists $A \in GL(n, F_2)$ such that $Ax = y$ and thus $GL(n, F_2)$ acting on $\mathbb{Z}_2^n$ has two orbits: $\{0\}$ and $(\mathbb{Z}_2^n \setminus \{0\})$.*

Let $\Gamma = \mathrm{Cay}(G,S)$ be a Cayley graph over G with generating set S satisfying $S = S^{-1}$ and $\langle S \rangle = G$. The *right regular representation* of the group G is defined as $R(G) = \{r_g \colon x \mapsto xg (\forall x \in G) \mid g \in G\}$. Clearly, $R(G) \leq Aut(\Gamma)$ acts transitively on Γ. Recall that $Aut_e(\Gamma) = \{\alpha \in Aut(\Gamma) \mid \alpha(e) = e\}$. The following result follows.

Lemma 2.3 ([6]). *The automorphism group $Aut(\Gamma)$ of a normal Cayley graph Γ is given by $Aut(\Gamma) = R(G) \rtimes Aut_e(\Gamma)$.*

Now we present a sufficiency for a Cayley graph over an abelian group to be normal. Many mathematicians have noticed this result, see, for example, [19]. We still give a proof here for the convenience of the reader.

Lemma 2.4. *Let $\Gamma = \mathrm{Cay}(G,S)$ be a connected Cayley graph on the abelian group G. If $N(s) \cap N(t) = \{0, s+t\}$ for any distinct $s, t \in S$ then Γ is normal, that is, $Aut_0(\Gamma) = Aut_S(G)$.*

Proof. It is clear that $Aut_S(G) \leq Aut_0(\Gamma)$ and it remains to show that $Aut_0(\Gamma) \leq Aut_S(G)$. For any $\sigma \in Aut_0(\Gamma)$, we have $\sigma(S) = S$ and thus it only needs to show that $\sigma \in Aut(G)$. Clearly, σ is a bijection from G to itself, and thus it needs to show that $\sigma(x + y) = \sigma(x) + \sigma(y)$ for any $x, y \in G$.

For any distinct $s, t \in S$, since $N(s) \cap N(t) = \{0, s + t\}$, we have $N(\sigma(s)) \cap N(\sigma(t)) = \{0, \sigma(s + t)\}$. Note that $\sigma(s), \sigma(t) \in S$. We also have $N(\sigma(s)) \cap N(\sigma(t)) = \{0, \sigma(s) + \sigma(t)\}$. It leads to that $\sigma(s+t) = \sigma(s)+\sigma(t)$. In addition, we show that $\sigma(2s) = \sigma(s) + \sigma(s)$. Since $s \sim 2s$, we have $\sigma(s) \sim \sigma(2s)$. It means that $\sigma(2s) = \sigma(s) + \sigma(s')$. If $s' \neq s$ then it is proved that $\sigma(s) + \sigma(s') = \sigma(s + s')$ and thus $\sigma(2s) = \sigma(s) + \sigma(s') = \sigma(s + s')$. It leads to $2s = s + s'$ and thus $s = s'$, a contradiction. Hence, we have $\sigma(s + t) = \sigma(s) + \sigma(t)$ for any $s, t \in S$, where s and t may be equal.

In general, for $s_1, \ldots, s_k \in S$, we will show that $\sigma(s_1 + \cdots + s_k) = \sigma(s_1) + \cdots + \sigma(s_k)$ for any $\sigma \in Aut_0(\Gamma)$. Note that $r_u \in Aut(\Gamma)$ for any $u \in G$. Thus $r_{-\sigma(u)}\sigma r_u \in Aut(\Gamma)$. Note that $r_{-\sigma(u)}\sigma r_u(0) = r_{-\sigma(u)}\sigma(u) = 0$. We have $r_{-\sigma(u)}\sigma r_u \in Aut_0(\Gamma)$. Since $r_{-\sigma(s_k)}\sigma r_{s_k} \in Aut_0(\Gamma)$, by inductive assumption, we have

$$\begin{aligned}
& r_{-\sigma(s_k)}\sigma r_{s_k}(s_1 + \cdots + s_{k-1}) \\
= \; & r_{-\sigma(s_k)}\sigma r_{s_k}(s_1) + \cdots + r_{-\sigma(s_k)}\sigma r_{s_k}(s_{k-1}) \\
= \; & r_{-\sigma(s_k)}\sigma(s_1 + s_k) + \cdots + r_{-\sigma(s_k)}\sigma(s_{k-1} + s_k) \\
= \; & r_{-\sigma(s_k)}(\sigma(s_1) + \sigma(s_k)) + \cdots + r_{-\sigma(s_k)}(\sigma(s_{k-1}) + \sigma(s_{k-1})) \\
= \; & \sigma(s_1) + \cdots + \sigma(s_{k-1}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \sigma(s_1 + \cdots + s_k) \\
= \; & (r_{\sigma(s_k)}(r_{-\sigma(s_k)}\sigma r_{s_k})r_{-s_k})(s_1 + \cdots + s_k) \\
= \; & r_{\sigma(s_k)}(r_{-\sigma(s_k)}\sigma r_{s_k})(s_1 + \cdots + s_{k-1}) \\
= \; & r_{\sigma(s_k)}(\sigma(s_1) + \cdots + \sigma(s_{k-1})) \\
= \; & \sigma(s_1) + \cdots + \sigma(s_{k-1}) + \sigma(s_k).
\end{aligned}$$

Since $\langle S\rangle = G$, for any $x, y \in G$, we have $x = s_1 + \cdots + s_a$ and $y = s'_1 + \cdots + s'_b$ where $s_i, s'_j \in S$. Therefore, we have

$$\begin{aligned}
& \sigma(x + y) = \sigma(s_1 + \cdots + s_a + s'_1 + \cdots + s'_b) \\
= \; & \sigma(s_1) + \cdots + \sigma(s_a) + \sigma(s'_1) + \cdots + \sigma(s'_b) \\
= \; & \sigma(s_1 + \cdots + s_a) + \sigma(s'_1 + \cdots + s'_b) = \sigma(x) + \sigma(y).
\end{aligned}$$

It completes the proof. □

Let X_1 and X_2 be two graphs with $V(X_1) = \{u_1, \ldots, u_m\}$ and $V(X_2) = \{v_1, \ldots, v_n\}$. The Cartesian product $X_1 \square X_2$ is the graph with vertex set $V(X_1) \times V(X_2)$ and two vertices (u_i, v_j) and (u'_i, v'_j) are connected if $u_i \sim u'_i$ in X_1 and $v_j = v'_j$, or $u_i = u'_i$ and $v_j \sim v'_j$ in X_2.

Lemma 2.5. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph over the group G. If $S = T_1 \cup T_2$ such that $G = \langle T_1\rangle \cdot \langle T_2\rangle$ the internal direct product of $\langle T_1\rangle$ and $\langle T_2\rangle$, then $\Gamma \cong \Gamma_1 \square \Gamma_2$ where $\Gamma_1 = \mathrm{Cay}(\langle T_1\rangle, T_1)$ and $\Gamma_2 = \mathrm{Cay}(\langle T_2\rangle, T_2)$.*

Proof. Since $G = \langle T_1\rangle \cdot \langle T_2\rangle$ is the internal direct product of $\langle T_1\rangle$ and $\langle T_2\rangle$, each element $v \in G$ can be uniquely written as $v = v_1v_2$ for $v_1 \in \langle T_1\rangle$ and $v_2 \in \langle T_2\rangle$. Moreover, it is well known that $G \cong \langle T_1\rangle \times \langle T_2\rangle$ the external direct product of $\langle T_1\rangle$ and $\langle T_2\rangle$ [7]. In fact, let ϕ: $G \to \langle T_1\rangle \times \langle T_2\rangle$ be the map defined by $\phi(v) = \phi(v_1v_2) = (v_1, v_2)$, then ϕ is the isomorphism from G to $\langle T_1\rangle \times \langle T_2\rangle$. Therefore, the map ϕ gives a bijection from $V(\Gamma)$ to $V(\Gamma_1 \square \Gamma_2)$. Next, we show that ϕ is also a graph isomorphism from Γ to $\Gamma_1 \square \Gamma_2$.

Since $S = T_1 \cup T_2$, for each $s \in S$, we have $s \in T_1$ or $s \in T_2$. If $s \in T_1$ then $\phi(s) = (s, e_2)$, where e_2 is the identity element of $\langle T_2\rangle$. If $s \in T_2$ then $\phi(s) = (e_1, s)$, where e_1 is the identity element of $\langle T_1\rangle$. Let $v = v_1v_2$ and $v' = v'_1v'_2$ be two vertices of $V(\Gamma)$. It is seen that $v \sim v'$ in Γ if and only if $v'v^{-1} = s \in S$ if and only if $\phi(v'v^{-1}) = (v'_1v_1^{-1}, v'_2v_2'^{-1}) = \phi(s) = (s, e_2)$ when $s \in T_1$, or $\phi(v'v^{-1}) = \phi(s) = (v'_1v_1^{-1}, v'_2v_2'^{-1}) = (e_1, s)$ when $s \in T_2$ if and only if $v'_1v_1^{-1} = s \in T_1$ and $v'_2v_2'^{-1} = e_2$, or $v'_1v_1^{-1} = e_1$ and $v'_2v_2'^{-1} = s \in T_2$ if and only if $v_1 \sim v'_1$ in Γ_1 and $v_2 = v'_2$, or $v_1 = v'_1$ and $v_2 \sim v'_2$ in Γ_2 if and only if $\phi(v) = (v_1, v_2) \sim (v'_1, v'_2) = \phi(v')$ in $\Gamma_1 \square \Gamma_2$. It leads to that $\Gamma \cong \Gamma_1 \square \Gamma_2$.

This completes the proof. □

3. Automorphism group of the enhanced hypercube $Q_{n,k}$

In this part, we first give the automorphism group of the folded hypercube $Q_{n,1}$ and determine the Caylay graphs over $\mathbb{Z}_2^n$ isomorphic to $Q_{n,1}$. Next, we extend such results to the enhanced hypercube $Q_{n,k}$ for any $2 \le k \le n-1$. Keep in mind that that $Q_{n,k}$ is just the Cayley graph $\Gamma_{n,k} = \mathrm{Cay}(\mathbb{Z}_2^n, S_k)$ with $S_k = \{e_1, \ldots, e_n, \epsilon_k\}$ and $\epsilon_k = e_k + \cdots + e_n$.

Lemma 3.1. *Let $\Gamma_{n,1} = \mathrm{Cay}(\mathbb{Z}_2^n, S_1)$ be the n-dimensional folded hypercube with $n \ge 4$. Then $\Gamma_{n,1}$ is normal and $Aut_0(\Gamma_{n,1}) \cong \mathcal{S}_{n+1}$, where $\mathcal{S}_{n+1}$ is the symmetric group of degree $n+1$.*

Proof. Denote by $v_i = e_i$ for $1 \le i \le n$ and $v_{n+1} = \epsilon_1$. Clearly, $\{0, v_i + v_j\} \subseteq N(v_i) \cap N(v_j)$. Now assume that x is a vertex of $N(v_i) \cap N(v_j)$. We have $x = v_i + v_s = v_j + v_t$ for some $v_s, v_t \in S$ and thus $v_i + v_s - v_j - v_t = 0$, that is v_i, v_j, v_s and v_t are linear dependent. If v_i, v_j, v_s and v_t are distinct then $\{v_i, v_j, v_s, v_t\}$ must be linear independent since any n elements of S_1 form a basis of $\mathbb{Z}_2^n$ and $n \ge 4$. Therefore, we have $v_i = v_s$ and $v_j = v_t$, or $v_i = v_t$ and $v_j = v_s$. If the former occurs then $x = 0$; if the latter occurs then $x = v_i + v_j$. Thus, we have $N(v_i) \cap N(v_j) = \{0, v_i + v_j\}$. Since $\mathbb{Z}_2^n$ is abelian, Lemma 2.4 implies that $Aut_0(\Gamma_{n,1}) = Aut_{S_1}(\mathbb{Z}_2^n)$.

In what follows, we show that $Aut_{S_1}(\mathbb{Z}_2^n) \cong \mathcal{S}_{n+1}$. From Lemma 2.1, we have $Aut(\mathbb{Z}_2^n) = GL(n, F_2)$ and thus $Aut_{S_1}(\mathbb{Z}_2^n) = \{A \in GL(n, F_2) \mid AS_1 = S_1\}$. Since $AS_1 = S_1$ for any $A \in Aut_{S_1}(\mathbb{Z}_2^n)$, we have $A(v_1, v_2, \ldots, v_n, v_{n+1}) = (v_{i_1}, v_{i_2}, \ldots, v_{i_n}, v_{i_{n+1}})$, which implies that $A = [v_{i_1}, \ldots, v_{i_n}]$. Define the map σ: $Aut_{S_1}(\mathbb{Z}_2^n) \to \mathcal{S}_{n+1}$ by setting $\sigma_A = (1, i_1)(2, i_2) \ldots (n+1, i_{n+1})$. In fact, σ is well defined since $i_1, i_2, ..., i_{n+1}$ is a reset of $1, 2, ..., n, n+1$. It is clear that $\sigma_A = \sigma_B$ if and only if $(v_{\sigma_A(1)}, \ldots, v_{\sigma_A(n)}) = (v_{\sigma_B(1)}, \ldots, v_{\sigma_B(n)})$ if and only if $A = B$. Besides, for each $\theta \in \mathcal{S}_{n+1}$, we construct $A = [v_{\theta(1)}, \ldots, v_{\theta(n)}]$ and thus $A(v_1, \ldots, v_{n+1}) = (v_{\theta(1)}, \ldots, v_{\theta(n+1)})$ which gives $\sigma_A = \theta$. Therefore, σ is a bijection. Furthermore, since $v_{\sigma_{AB}(i)} = (AB)v_i = A(Bv_i) = Av_{\sigma_B(i)} = v_{\sigma_A\sigma_B(i)}$ for any $1 \le i \le n+1$, we have $\sigma_{AB} = \sigma_A\sigma_B$. It leads to that σ is an isomorphism and thus $Aut_{S_1}(\mathbb{Z}_2^n) \cong \mathcal{S}_{n+1}$.
□

Combining Lemmas 2.3 and 3.1, we get the automorphism group of the folded hypercube $\Gamma_{n,1} = \mathrm{Cay}(\mathbb{Z}_2^n, S_1)$, where $S_1 = \{e_1, \ldots, e_n, \epsilon_1\}$ and $\epsilon_1 = e_1 + \cdots + e_n$.

Theorem 3.2. *The automorphism group of the folded hypercube $\Gamma_{n,1}$ is given by*

$$Aut(\Gamma_{n,1}) = \begin{cases} \mathcal{S}_4 & \text{if } n = 2 \\ (\mathcal{S}_4 \times \mathcal{S}_4) \rtimes \mathcal{S}_2 & \text{if } n = 3 \\ \mathbb{Z}_2^n \rtimes \mathcal{S}_{n+1} & \text{if } n \ge 4 \end{cases}$$

Proof. It is easy to see that $\Gamma_{2,1} \cong K_4$ and $\Gamma_{3,1} \cong K_{4,4}$. Therefore, we have $Aut(\Gamma_{2,1}) = Aut(K_4) = \mathcal{S}_4$ and $Aut(\Gamma_{3,1}) = Aut(K_{4,4}) = (\mathcal{S}_4 \times \mathcal{S}_4) \rtimes \mathcal{S}_2$. For $n \ge 4$, by Lemmas 2.3 and 3.1, we have $Aut(\Gamma_{n,1}) = \mathbb{Z}_2^n \rtimes \mathcal{S}_{n+1}$. □

A Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is called *perfect* if $T = \{t_1, t_2, ..., t_n, t\}$ such that $\{t_1, ..., t_n\}$ is a basis of $\mathbb{Z}_2^n$ and $t = t_1 + \cdots + t_n$. Clearly, $\Gamma_{n,1}$ is perfect.

Theorem 3.3. *Let $\Gamma_{n,1} = \mathrm{Cay}(\mathbb{Z}_2^n, S_1)$ be the Cayley graph with generating set $S_1 = \{e_1, \ldots, e_n, \epsilon_1\}$ where $\epsilon_1 = e_1 + \cdots + e_n$. Then the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is isomorphic to $\Gamma_{n,1}$, if and only if $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is perfect.*

Proof. If $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is perfect, then $T = \{t_1, \ldots, t_n, t\}$ where $t_1, ..., t_n$ is a basis of $\mathbb{Z}_2^n$ and $t = t_1 + \cdots + t_n$. Now we define $\varphi(x) = Ax$ for any $x \in V(\Gamma_{n,1})$ where $A = [t_1, \ldots, t_n]$. It is clear that $AS_1 = T$. Since $\{t_1, \ldots, t_n\}$ is a basis of $\mathbb{Z}_2^n$, we have $A \in GL(n, F_2)$. Therefore, φ is a bijection between $V(\Gamma_{n,1})$ and $V(\mathrm{Cay}(\mathbb{Z}_2^n, T))$. Moreover, $\varphi(x) \sim \varphi(y)$ in $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ if and only if $\varphi(y) - \varphi(x) = A(y - x) \in T$ if and only if $(y - x) \in A^{-1}T = S_1$ if and only if $x \sim y$ in Γ_1. It leads to that $\mathrm{Cay}(\mathbb{Z}_2^n, T) \cong \Gamma_{n,1}$.

Conversely, assume that $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is a Cayley graph isomorphic to $\Gamma_{n,1}$. Let φ be the isomorphism from $\Gamma_{n,1}$ to $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ with $\varphi(0) = 0$. We have $T = \varphi(S_1) = \{\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_n), \varphi(\epsilon_1)\}$. Since $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is also connected, we have $\mathbb{Z}_2^n = \langle T \rangle$. Therefore, there is a basis of $\mathbb{Z}_2^n$ contained in T. Without loss of generality, we may assume that $\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_n)$ is a basis. Thus, we have

$$\varphi(\epsilon_1) = \varphi(e_{i_1}) + \varphi(e_{i_2}) + \cdots + \varphi(e_{i_s}),$$

where $i_1, i_2, \ldots, i_s$ are distinct. Now, it only needs to show that $\{i_1, i_2, \ldots, i_s\} = \{1, 2, \ldots, n\}$, i.e., $s = n$. Note that $\{0, \varphi(e_{i_1}), \varphi(e_{i_1}) + \varphi(e_{i_2}), \ldots, \varphi(e_{i_1}) + \cdots + \varphi(e_{i_{s-1}}), \varphi(\epsilon_1)\}$ forms a cycle of length $s+1$ in $\mathrm{Cay}(\mathbb{Z}_2^n, T)$. It follows that $\{0, e_{i_1}, \varphi^{-1}(\varphi(e_{i_1}) + \varphi(e_{i_2})), \ldots, \varphi^{-1}(\varphi(e_1) + \cdots + \varphi(e_{i_{s-1}})), \epsilon_1\}$ is a cycle in $\Gamma_{n,1}$. Therefore, for $2 \le k \le s-1$, we have

$$\varphi^{-1}(\varphi(e_{i_1}) + \cdots + \varphi(e_{i_k})) = \varphi^{-1}(\varphi(e_{i_1}) + \cdots + \varphi(e_{i_{k-1}})) + v_k,$$

and $\epsilon_1 = \varphi^{-1}(\varphi(e_{i_1}) + \cdots + \varphi(e_{i_{s-1}})) + v_s$, where $v_2, \ldots, v_s \in \{e_1, e_2, \ldots, e_n, \epsilon_1\}$. Thus, we have

$$\epsilon_1 = v_1 + v_2 + \cdots + v_s,$$

where $v_1 = e_{i_1}$ and $v_2, \ldots, v_s \in \{e_1, \ldots, e_n, \epsilon_1\}$. Note that $v_1, v_2, \ldots, v_s$ may be not distinct, and $2v_j = 0$ for any $1 \le j \le s$. Each term appears even times vanishes and we have

$$\epsilon_1 = v_1' + v_2' + \cdots + v_l',$$

where $\{v_1', v_2', \ldots, v_t'\} \subseteq \{e_1, \ldots, e_n, \epsilon_1\}$ and $l \le s$. If there is one of $v_1', \ldots, v_l'$ equal to ϵ_1, say $v_l' = \epsilon_1$, then $v_1' + v_2' + \cdots + v_{l-1}' = 0$. It is a contradiction because $v_1', \ldots, v_{l-1}' \in \{e_1, \ldots, e_n\}$, which is linear independent. Therefore, we have $\{v_1', v_2', \ldots, v_l'\} \subseteq \{e_1, e_2, \ldots, e_n\}$. It follows that there are l positions of ϵ_1 is 1, and thus $l = s = n$.

The proof is completed. □

Recall that, if $\Gamma = \Gamma_1 \Box \Gamma_2$, then Γ_1 and Γ_2 are called *factors* of Γ. Two graphs Γ_1 and Γ_2 are called *relatively prime* if there is no non-trivial graph that is a factor of both of them.

Lemma 3.4 ([12, Corollary 6.12]). *If $\Gamma = \Gamma_1 \Box \Gamma_2$ where Γ_1, Γ_2 are two connected relative prime graphs, then $Aut(\Gamma) = Aut(\Gamma_1) \times Aut(\Gamma_2)$.*

Lemma 3.5. *For any $n \ge 2$, the graph $\Gamma_{n,1} = \mathrm{Cay}(\mathbb{Z}_2^n, S_1)$ has no factor K_2.*

Proof. Suppose to the contrary that $\Gamma_{n,1} = K_2 \Box \Gamma$. Therefore, the vertex set of $\Gamma_{n,1}$ can be partitioned as $V(\Gamma_{n,1}) = V \cup V'$, where each of V and V' induces a graph isomorphic to Γ. Moreover, each vertex in V (resp. V') has exactly one neighbor in V' (resp. V). We will use this fact frequently. Without loss of generality, assume that $0 \in V$. Therefore, all but one neighbors of 0 are in V. Without loss of generality, assume that $e_1 \in V'$ and $e_2, \ldots, e_n, \epsilon_1 \in V$.

In what follows, we show that, for any $2 \le i \le n-1$, $e_i + e_{i+1} + \cdots + e_n \in V'$ and $e_1 + e_i + e_{i+1} + \cdots + e_n \in V$. Clearly, $e_1 + e_2 + \cdots + e_n = \epsilon_1 \in V$. Note that $e_1 \in V'$ and 0 is the only neighbor of e_1 in V. We have $e_2 + e_3 + \cdots + e_n \in V'$ because $e_1 \sim e_2 + e_3 + \cdots + e_n$. Therefore, the statement is true for $i = 2$. Now we assume that the statement is true for i. It suffices to show that the statement is also true for the case of $i+1$. Note that $e_i + e_{i+1} + \cdots + e_n \in V'$ and $e_1 + e_i + \cdots + e_n$ is the only neighbor of $e_i + e_{i+1} + \cdots + e_n$ in V. We have $e_{i+1} + \cdots + e_n \in V'$ because $e_i + e_{i+1} + \cdots + e_n \sim e_{i+1} + \cdots + e_n$.

By the arguments above, we have $e_{n-1} + e_n \in V'$. However, if $n = 2$, then $e_{n-1} + e_n = \epsilon_1 \in V$, a contradiction; if $n \ge 3$, then $e_{n-1} + e_n \sim e_{n-1} \in V$ and $e_{n-1} + e_n \sim e_n \in V$, which contradicts the fact that $e_{n-1} + e_n$ has exactly one neighbor in V.

This completes the proof. □

Now we are in the position to present one of our main results.

Theorem 3.6. *The automorphism group of $\Gamma_{n,k}$ is given by*

$$Aut(\Gamma_{n,k}) \cong \begin{cases} \mathcal{S}_4 & \text{if } n = 2 \text{ and } k = 1 \\ \mathcal{S}_2 \times \mathcal{S}_4 & \text{if } n = 3 \text{ and } k = 2 \\ (\mathcal{S}_4 \times \mathcal{S}_3) \rtimes \mathcal{S}_2 & \text{if } n = 3 \text{ and } k = 1 \\ (\mathbb{Z}_2^{k-1} \rtimes \mathcal{S}_{k-1}) \times \mathcal{S}_4 & \text{if } n \ge 4 \text{ and } k = n-1 \\ (\mathbb{Z}_2^{k-1} \rtimes \mathcal{S}_{k-1}) \times ((\mathcal{S}_4 \times \mathcal{S}_4) \rtimes \mathcal{S}_2) & \text{if } n \ge 4 \text{ and } k = n-2 \\ (\mathbb{Z}_2^{k-1} \rtimes \mathcal{S}_{k-1}) \times (\mathbb{Z}_2^{n-k+1} \rtimes \mathcal{S}_{n-k+2}) & \text{if } n \ge 4 \text{ and } k \le n-3 \end{cases}$$

Proof. The cases of $n = 2, k = 1$ and $n = 3, k = 1$ were considered in Theorem 3.2. If $n = 3$ and $k = 2$, then $\Gamma_{3,2} \cong K_2 \Box K_4$. It is clear that K_2 is not a factor of K_4. Lemma 3.4 implies that $Aut(\Gamma_{3,2}) \cong Aut(K_2) \times Aut(K_4) = \mathcal{S}_2 \times \mathcal{S}_4$. Now we consider the case of $n \geq 4$. Let $S_k^{(1)} = \{e_1, e_2, \ldots, e_{k-1}\}$ and $S_k^{(2)} = \{e_k, e_{k+1}, \ldots, e_n, \epsilon_k\}$. It is clear that $S_k = S_k^{(1)} \cup S_k^{(2)}$ and $\mathbb{Z}_2^n = \langle S_k^{(1)} \rangle \cdot \langle S_k^{(2)} \rangle$. Therefore, Lemma 2.5 indicate that $\Gamma_{n,k} = \mathrm{Cay}(\langle S_k^{(1)} \rangle, S_k^{(1)}) \Box \mathrm{Cay}(\langle S_k^{(2)} \rangle, S_k^{(2)})$. Note that $\mathrm{Cay}(\langle S_k^{(1)} \rangle, S_k^{(1)}) \cong Q_{k-1}$ and $\mathrm{Cay}(\langle S_k^{(1)} \rangle, S_k^{(1)}) \cong \Gamma_{n-k+1,1}$. We have $\Gamma_{n,k} \cong Q_{k-1} \Box \Gamma_{n-k+1,1}$. Since Q_{k-1} is the Cartesian product of $k-1$ k_2 and Lemma 3.5 implies that K_2 is not a fact of $\Gamma_{n-k+1,1}$, Lemma 3.4 indicates that $Aut(\Gamma_{n,k}) \cong Aut(Q_{k-1}) \times Aut(\Gamma_{n-k+1,1})$. Note that $Aut(Q_{k-1}) = \mathbb{Z}_2^{k-1} \rtimes \mathcal{S}_{k-1}$ [9] and $Aut(\Gamma_{n-k+1,1})$ is given in Theorem 3.2. The result follows. □

A Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is called *k-perfect* if $T = \{t_1, t_2, ..., t_n, t\}$ such that $t_1, ..., t_n$ is a basis of $\mathbb{Z}_2^n$ and t is a sum of $n-k+1$ elements of $\{t_1, t_2, ..., t_n\}$, i.e., $t = t_{i_1} + \cdots + t_{i_{n-k+1}}$. Particularly, when $k = 1$, the concept of *1-perfect* is coincident with the concept of *perfect*. It is clear that $\Gamma_{n,k} = \mathrm{Cay}(\mathbb{Z}_2^n, S_k)$ is k-perfect, where $S_k = \{e_1, ..., e_n, \epsilon_k\}$ and $\epsilon_k = e_k + e_{k+1} + \cdots + e_n$.

Theorem 3.7. *The Cayley graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^n, T)$ is isomorphic to $\Gamma_{n,k}$ if and only if Γ is k-perfect.*

Proof. From Lemma 2.5, we have $\Gamma_{n,k} = Q_{k-1} \Box \Gamma_{n-k+1,1}$. Note that Q_{k-1} is a $k-1$ Cartesian products of K_2 and Lemma 3.5 indicate that $\Gamma_{n-k-1,1}$ has no factor K_2. It implies that $\Gamma_{n,k} \ncong \Gamma_{n,l}$ if $k \neq l$.

Assume $\mathrm{Cay}(\mathbb{Z}_2^n, T) \cong \Gamma_{n,k}$. By the connectivity of $\mathrm{Cay}(\mathbb{Z}_2^n, T)$, T contains a basis, and the other element in T is a sum of l distinct elements in the basis. Then, $\mathrm{Cay}(\mathbb{Z}_2^n, T) \cong \Gamma_{n,l}$, and hence $\Gamma_{n,l} \cong \Gamma_{n,k}$. It follows that $l = k$, that is, $\mathrm{Cay}(\mathbb{Z}_2^n, T)$ is k-perfect. □

We know that up to isomorphism there is only one n-regular connected Cayley graph over $\mathbb{Z}_2^n$. Theorem 3.7 implies the following result.

Corollary 3.8. *Up to isomorphism, $\Gamma_{n,1}, \Gamma_{n,2}, ..., \Gamma_{n,n-1}$ are the only $(n+1)$-regular connected Cayley graphs over $\mathbb{Z}_2^n$.*

4. The G-DS property of $Q_{n,k}$

For a graph Γ with vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$, its adjacency matrix $A = (a_{ij})_{n\times n}$ is the $n \times n$ matrix with $a_{ij} = 1$ if $v_i \sim v_j$ in Γ and $a_{ij} = 0$ otherwise. The eigenvalues of A are called the eigenvalues of Γ and the set of such eigenvalues together with their multiplicities forms the spectrum of Γ, denoted by $\mathrm{Spec}(\Gamma)$. A graph Γ is called *determined by its spectrum* (DS for short) if, for any graph Γ', $\mathrm{Spec}(\Gamma') = \mathrm{Spec}(\Gamma)$ implies that $\Gamma' \cong \Gamma$. The question 'which graphs are DS ?' goes back for about half a century, and originates from chemistry. In 1956, Günthard and Primas [8] raised the question in a paper that relates the theory of graph spectra to Hückel's theory from chemistry. For more details about this problem, we would like to refer the reader to [5].

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph over G. To investigate whether the Cayley graph Γ is DS, it should be first discussed whether it is determined by its spectrum among the Cayley graphs over G. This thought leads to the concept of 'G-DS'. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called *G-spectrum determined* (G-DS for short) if, for any $\Gamma' = \mathrm{Cay}(G, T)$, $\mathrm{Spec}(\Gamma') = \mathrm{Spec}(\Gamma)$ implies that $\Gamma' \cong \Gamma$. Clearly, if a Cayley graph is DS then it must be G-DS, but the converse is not true. In this part, we discuss the G-DS property of $Q_{n,k}$. Note that the spectra of Cayley graphs over abelian groups are given by Babai.

Lemma 4.1 ([3]). *Let G be an abelian group of order n and S a subset of G such that $1 \notin S$ and $S^{-1} = S$. If $\chi_1, \ldots, \chi_n$ are all irreducible characters of G, then the eigenvalues of the Cayley graph $\mathrm{Cay}(G, S)$ are $\lambda_i = \sum_{s\in S} \chi_i(s)$ for $1 \leq i \leq n$.*

According to Lemma 4.1, to get the eigenvalues of $Q_{n,k} = \Gamma_{n,k}$, we should know the irreducible characters of $\mathbb{Z}_2^n$.

Lemma 4.2 ([14]). *The irreducible characters of $\mathbb{Z}_2^n$ are $\chi_{i_1,\ldots,i_n}$ for $i_j \in \{0, 1\}$ and $1 \leq j \leq n$, where $\chi_{i_1,\ldots,i_n}(v) = (-1)^{a_1 i_1 + \cdots + a_n i_n}$ for $v = (a_1, \ldots, a_n)^T \in \mathbb{Z}_2^n$.*

Now we are ready to give the eigenvalues of $\Gamma_{n,k}$.

Lemma 4.3. *The eigenvalues of* $\Gamma_{n,k} = \mathrm{Cay}(\mathbb{Z}_2^n, S_k)$ *consist of* $\lambda_{t,o}$ *and* $\lambda_{t,e}$ *for* $0 \le t \le n$ *with multiplicities* $\eta_{t,o}$ *and* $\eta_{t,e}$*, respectively, where*

$$\left\{ \begin{array}{l} \lambda_{t,o} = n - 2t - 1 \\ \lambda_{t,e} = n - 2t + 1 \end{array} \right. \quad and \quad \left\{ \begin{array}{l} \eta_{t,o} = \sum_{s=0}^{t} \binom{k-1}{t-2s-1}\binom{n-k+1}{2s+1} \\ \eta_{t,e} = \sum_{s=0}^{t} \binom{k-1}{t-2s}\binom{n-k+1}{2s} \end{array} \right. .$$

Proof. From Lemmas 4.1 and 4.2, the eigenvalues of $\Gamma_{n,k}$ are given by

$$\lambda_{i_1,\dots,i_n} = \sum_{v \in S_k} \chi_{i_1,\dots,i_n}(v) = \sum_{j=1}^{n} \chi_{i_1,\dots,i_n}(e_j) + \chi_{i_1,\dots,i_n}(\epsilon_k)$$

where $\chi_{i_1,\dots,i_n}$ is the irreducible characters of $\mathbb{Z}_2^n$ given in Lemma 4.2. Denote by $\Lambda_t = \{\chi_{i_1,\dots,i_n} \mid i_1 + \cdots + i_n = t\}$ for $0 \le t \le n$. It is clear that $\eta_t = |\Lambda_t| = \binom{n}{t}$. Furthermore, denote by $\Lambda_{t,o} = \{\chi_{i_1,\dots,i_n} \in \Lambda_t \mid i_1 + \cdots + i_n \equiv 1(\mod 2)\}$ and $\Lambda_{t,e} = \{\chi_{i_1,\dots,i_n} \in \Lambda_t \mid i_1 + \cdots + i_n \equiv 0(\mod 2)\}$. It is clear that $\eta_{t,o} = |\Lambda_{t,o}| = \sum_{s=0}^{t} \binom{k-1}{t-2s-1}\binom{n-k+1}{2s+1}$ and $\eta_{t,e} = |\Lambda_{t,o}| = \sum_{s=0}^{t} \binom{k-1}{t-2s}\binom{n-k+1}{2s}$. Note that $\chi_{i_1,\dots,i_n}(e_j) = (-1)^{i_j}$ and $\chi_{i_1,\dots,i_n}(\epsilon_k) = (-1)^{i_k+\cdots+i_n}$. It is seen that, for any $\chi_{i_1,\dots,i_n} \in \Lambda_t$,

$$\begin{array}{cl} & \sum_{j=1}^{n} \chi_{i_1,\dots,i_n}(e_j) + \chi_{i_1,\dots,i_n}(\epsilon_k) \\ = & \sum_{j=1}^{n} (-1)^{i_j} + (-1)^{i_k+\cdots+i_n} \\ = & \left\{ \begin{array}{ll} n - 2t - 1, & \text{if } \chi_{i_1,\dots,i_n} \in \Lambda_{t,o} \\ n - 2t + 1, & \text{if } \chi_{i_1,\dots,i_n} \in \Lambda_{t,e} \end{array} \right. . \end{array}$$

It means that all characters in $\Lambda_{t,o}$ lead to the same eigenvalue $\lambda_{t,o} = n - 2t - 1$ and all characters in $\Lambda_{t,e}$ lead to the same eigenvalue $\lambda_{t,e} = n - 2t + 1$. This completes the proof. □

From Lemma 4.3, it is easy to see that $\lambda_{1,e} = n - 1$ is an eigenvalue of $\Gamma_{n,k}$ with multiplicity $k - 1$. Therefore, the following result follows immediately.

Corollary 4.4. *The Cayley graphs* $\Gamma_{n,k}$ *and* $\Gamma_{n,k'}$ *cannot share the same spectrum if* $k \ne k'$.

Now we are ready to present the main result of this part.

Theorem 4.5. *The enhanced hypercube* $\Gamma_{n,k}$ *is* $\mathbb{Z}_2^n$*-DS.*

Proof. Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^n, T)$ be the Cayley graph such that $\mathrm{Spec}(\Gamma) = \mathrm{Spec}(\Gamma_{n,k})$. It leads to that Γ is also $n+1$-regular because the two graphs share the same largest eigenvalue which is the valency of them. By Corollary 3.8, it is seen that $\Gamma' \cong \Gamma_{n,k'}$. However, Corollary 4.4 implies that $\mathrm{Spec}(\Gamma_{n,k}) = \mathrm{Spec}(\Gamma_{n,k'})$ if and only if $k = k'$. It follows the result. □

5. Conclusion

The enhanced hypercube $Q_{n,k}$ for $1 \le k \le n - 1$ is an important network topology for parallel processing computer systems. It is proved that a message routed algorithm can always follow a shortest path in any enhanced hypercube. Besides, though the hardware cost to construct enhanced hypercubes is greater than that of the normal hypercubes, the overhead is negligible when the order is large, and thus is more cost-effective when compared to a normal hypercube [15]. Therefore, the structural properties, such as the connectivity, the diameter and so on, of enhanced hypercubes play a very important role in the interconnection network. It is effective to obtain the structural properties of a graph from its algebraic properties. As an important algebraic property, the automorphism group of a graph not only reveals the symmetry of the graph but also reflects the complexity of the construction of the graph. In this paper, we completely determine the automorphism group of the enhanced hypercube $Q_{n,k}$ by regarding $Q_{n,k}$ as a Cayley graph over $\mathbb{Z}_2^n$. Moreover, we prove that all Cayley graphs over $\mathbb{Z}_2^n$ isomorphic to $Q_{n,k}$ must be the so called k-perfect Cayley graphs. Furthermore, we show that no two distinct enhanced hypercube can share the same spectrum, and $Q_{n,k}$ is determined by its spectrum among all Cayley graphs over $\mathbb{Z}_2^n$.

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