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Difference Gap Functions and Global Error Bounds for Random Mixed Equilibrium Problems

Nguyen Van Hung[a,b], Xiaolong Qin[c], Vo Minh Tam[d], Jen-Chih Yao[e]

[a]*Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[b]*Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[c]*General Education Center, National Yunlin University of Science and Technology, Douliou, Taiwan*
[d]*Department of Mathematics, Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam*
[e]*Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung, Taiwan*

Abstract. The aim of this paper is to study the difference gap (in short, D-gap) function and error bounds for a class of the random mixed equilibrium problems in real Hilbert spaces. Firstly, we consider regularized gap functions of the Fukushima type and Moreau-Yosida type. Then difference gap functions are established by using these terms of regularized gap functions. Finally, the global error bounds for random mixed equilibrium problems are also developed. The results obtained in this paper are new and extend some corresponding known results in literatures. Some examples are given for the illustration of our results.

1. Introduction

In 1976, Auslender [5] first introduced the concept of gap functions for the following variational inequality problem:

$$\pi(x) = \sup_{y\in K}\{\langle f(x), x-y\rangle\},$$

where $\pi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, $K \subset \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$. Based on the gap function of Auslender [5], Fukushima [17] extended it to the concept of regularized gap functions for the following variational inequality problem:

$$\pi(x;\alpha) = \sup_{y\in K}\{\langle f(x), x-y\rangle - \alpha\|x-y\|^2\},$$

where α is a nonnegative parameter and $\pi(\cdot;0) = \pi(\cdot)$. Yamashita and Fukushima [35] developed the regularized function of the Moreau-Yosida type based on the ideal of Fukushima [17] as follows:

$$\delta_\pi(x;\alpha,\lambda) = \inf_{w\in K}\{\pi(z;\alpha) + \lambda\|x-w\|^2\},$$

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The second author was supported by the Feng Tay project of the National Yunlin University of Science and Technology, Taiwan
Email addresses: nguyenvanhung2@tdtu.edu.vn (Nguyen Van Hung), qinxl@yuntech.edu.cn (Xiaolong Qin), vmtam@dthu.edu.vn (Vo Minh Tam), yaojc@mail.cmu.edu.tw (Jen-Chih Yao)

where λ is a positive parameter. By using the strong monotonicity assumptions, they also proposed error bounds for the variational inequality problems via the regularized gap functions. Error bound is known as an upper estimate of the distance of an arbitrary feasible point to the solution set of a certain problem. It plays an vital role in convergence analysis of iterative algorithms for solving variational inequalities. Later, the regularized gap functions and error bounds have been established by many authors in different ways for various kinds of variational inequality problems and equilibrium problems, etc., see [1–3, 6, 13–15, 20–25, 33] and the references therein.

Recently, Peng [31] established the D-gap (Difference gap) functions for unconstrained reformulation of the variational inequality problem based on the difference of two regularized gap functions given by Fukushima [17]. Yamashita et al. [36] extended the results of Peng [31] and developed various properties of the D-gap function. A global error bound result for D-gap functions under the strong monotonicity assumption was given Peng and Fukushima [32]. Since then, the D-gap function and error bounds have been studied for various kinds of variational inequality problems and equilibrium problems, see, e.g., [9, 16, 28]. Besides, many authors studied various kinds of random variational inequalities and random complementarity problems for different topics, such as, iterative algorithms, the existence of solutions, etc., see [8, 12, 26, 30]. However, to the best of our knowledge, up to now, there is no any work on the gap functions in terms of Moreau-Yosida regularized and D-gap functions and their error bounds in random environment.

Motivated by the research works mentioned above, in this paper, we study a class of the random mixed equilibrium problems (for short, (RMEP)) in real Hilbert spaces. From applying similar to ideas in [28], the regularized gap function of Fukushima type is established without the projection operator method. Moreover, we will construct the gap functions in terms of Moreau-Yosida regularized and D-gap functions by using the regularized gap function of Fukushima type for (RMEP). Further, the global error bounds for (RMEP) are also obtained the previous gap functions. The results obtained in this paper are new and extend some corresponding known results in literatures. Some examples are given for the illustration of our results.

2. Preliminaries

Throughout the paper, unless otherwise stated, we suppose that $\mathbb{R}$ denotes the set of real numbers and $(\Omega, \mathcal{A})$ is a measurable space, where Ω is a set and $\mathcal{A}$ is a σ-algebra of subsets of Ω. Let H be a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathrm{CB}(H)$ be the family of all nonempty bounded closed subsets of H and $K \in \mathrm{CB}(H)$. We denote by $\mathcal{B}(H)$ the class of Borel σ-fields on H.

The following definitions and concepts are needed in the sequel.

Definition 2.1. (See [19])

(a) A mapping $x : \Omega \to K$ is said to be *measurable* if, for any $B \in \mathcal{B}(H)$, $\{t \in \Omega : x(t) \in B\} \in \mathcal{A}$.

(b) A mapping $S : \Omega \times K \to H$ is said to be a *random operator* if, for any $x \in K, S(t,x) = x(t)$ is measurable.

(c) We say that $S : \Omega \times K \to H$ is *Lipschitz continuous* (resp., *convex, monotone, linear, bounded*) if, for any $t \in \Omega$, the mapping $S(t,\cdot) : K \to H$ is Lipschitz continuous (resp., convex, monotone, linear, bounded).

Definition 2.2. A random function $F : \Omega \times K \times K \to \mathbb{R}$ is said to be

(a) $\alpha(t)$-*strongly monotone* if there exists a measurable function $\alpha : \Omega \to \mathbb{R}_+$ such that

$$F\Big(t, x(t), y(t)\Big) + F\Big(t, y(t), x(t)\Big) \leq -\alpha(t)\|x(t) - y(t)\|^2, \quad \forall t \in \Omega, \forall x(t), y(t) \in K;$$

(b) $\beta(t)$-*strongly convex uniformly in the second argument* if there exists a measurable function $\beta : \Omega \to \mathbb{R}_+$ such that, for any $t \in \Omega, x(t) \in K$,

$$F\Big(t, x(t), y_1(t)) - F(t, x(t), y_2(t)\Big) \geq \langle \nabla_3 F\Big(t, x(t), y_2(t)\Big), y_1(t) - y_2(t)\rangle + \beta(t)\|y_1(t) - y_2(t)\|^2,$$
$$t \in \Omega, \forall y_1(t), y_2(t) \in K;$$

(c) $(L_1(t), L_2(t))$-*mixed Lipschitz continuous* if there exist measurable functions $L_1, L_2 : \Omega \to \mathbb{R}_+$ such that

$$\|F(t, x(t), z_1(t)) - F(t, y(t), z_2(t))\| \leq L_1(t)\|x(t) - y(t)\| + L_2(t)\|z_1(t) - z_2(t)\|,$$
$$\forall x(t), y(t), z_1(t), z_2(t) \in K.$$

Note that $\nabla_i F$ denotes the partial differential of F with respect to the ith variable of F, $i = 1, 2, 3$.

Remark 2.3. If $F(t, x, y) \equiv F(x, y)$, $\forall x, y \in K$, then the $\alpha(t)$-strong monotonicity, the $\beta(t)$-strong uniformly convexity and the $(L_1(t), L_2(t))$-mixed Lipschitz continuity reduce to the α-strong monotonicity in [10], the β-strong uniformly convexity in [34] and the (L_1, L_2)-mixed Lipschitz continuity in [27], respectively, where $\alpha, \beta, L_1, L_2 \in \mathbb{R}_+$.

Picking up the ideas from [15, 29] we propose a concept related to strongly nonexpanding random operator.

Definition 2.4. A random operator $g : \Omega \times K \to H$ is said to be $\kappa(t)$-*strongly nonexpanding*, if for each $t \in \Omega$, there exists a measurable function $\kappa : \Omega \to \mathbb{R}_+$ such that

$$\|g(t, x(t)) - g(t, y(t))\| \geq \kappa(t)\|x(t) - y(t)\|, \quad \forall x(t), y(t) \in K.$$

Definition 2.5. (See [19])

(a) A multi-valued mapping $\Gamma : \Omega \rightrightarrows H$ is said to be *measurable* if for any $B \in \mathcal{B}(H)$, $\Gamma^{-1}(B) = \{t \in \Omega : \Gamma(t) \cap B \neq \emptyset\} \in \mathcal{A}$.

(b) A mapping $u : \Omega \to K$ is called a *measurable selection* of a multi-valued measurable mapping $\Gamma : \Omega \rightrightarrows H$ if u is measurable and for any $t \in \Omega$, $u(t) \in \Gamma(t)$.

Definition 2.6. (See [19]) Let $Q : \Omega \times K \to H$ be a random mapping, $T : \Omega \times K \to \mathrm{CB}(H)$ be a multi-valued measurable mapping. Then

(a) Q is said to be $\zeta(t)$-*strongly* g-*monotone with respect to* T, if there exists a measurable function $\zeta : \Omega \to \mathbb{R}_+$ such that for any $t \in \Omega$,

$$\langle Q(t, w_1(t)) - Q(t, w_2(t)), g(t, x_1(t)) - g(t, x_2(t))\rangle \geq \zeta(t)\|x_1(t) - x_2(t)\|^2,$$
$$\forall x_1(t), x_2(t) \in K, w_1(t) \in T(t, x_1(t)), w_2(t) \in T(t, x_2(t));$$

(b) Q is said to be $\sigma(t)$-*Lipschitz continuous*, if there exists a measurable function $\sigma : \Omega \to \mathbb{R}_+$ such that, for any $t \in \Omega$,

$$\|Q(t, x(t)) - Q(t, y(t))\| \leq \sigma(t)\|x(t) - y(t)\|, \quad \forall x(t), y(t) \in K;$$

(c) T is said to be $\mathcal{H}$-*Lipschitz continuous with measurable function* $\tau(t)$, if there exists a measurable function $\tau : \Omega \to \mathbb{R}_+$ such that for any $t \in \Omega$ and $x(t), y(t) \in K$,

$$\mathcal{H}(T(t, x(t)), T(t, y(t))) \leq \tau(t)\|x(t) - y(t)\|,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $\mathrm{CB}(H)$ defined as follows: for any given $A, B \in \mathrm{CB}(H)$,

$$\mathcal{H}(A, B) = \max\left\{\sup_{x\in A} \inf_{y\in B} \|x - y\|, \sup_{y\in B} \inf_{x\in A} \|x - y\|\right\}.$$

Definition 2.7. (See [19]) A random multi-valued mapping $T : \Omega \times K \rightrightarrows H$ is said to be *measurable*, if for any $x \in K$, $T(\cdot, x)$ is measurable. T is said to be $\mathcal{H}$-*continuous*, if for any $t \in \Omega$, $T(t, \cdot)$ is continuous in the Hausdorff metric.

Note that T is upper semicontinuous if, for each $t \in \Omega$, the correspondence $T(t, \cdot) : K \rightrightarrows H$ is upper semicontinuous.

Lemma 2.8. (See [11]) *Let $T : \Omega \times H \to \mathrm{CB}(H)$ be an $\mathcal{H}$-continuous random multi-valued mapping. Then for any measurable mapping $x : \Omega \to H$, the multi-valued mapping $T(\cdot, x) : \Omega \to \mathrm{CB}(H)$ is measurable.*

Lemma 2.9. (See [11]) *Let $T, Q : \Omega \times H \to \mathrm{CB}(H)$ be two measurable multi-valued mappings, let $\varepsilon > 0$ be a constant and let $u : \Omega \to H$ be a measurable selection of T. Then there exists a measurable selection $v : \Omega \to H$ of Q such that*

$$\|u(t) - v(t)\| \leq (1 + \epsilon)\mathcal{H}(T(t, \cdot), Q(t, \cdot)), \quad \forall t \in \Omega.$$

Now, we will introduce the model of random mixed equilibrium problems. Throughout the paper, let $F : \Omega \times K \times K \to \mathbb{R}$ be a random function such that $F(t, x, x) = 0$ for all $x \in K$. Moreover, F is continuously differentiable and that $F(t, x, \cdot)$ is convex for each $x \in K$. Let $\varphi : K \to \mathbb{R} \cup \{+\infty\}$ be a convex continuous but not necessarily differentiable function and $g : \Omega \times K \to K$ be a random operator with $\mathrm{Im}(g) \cap \mathrm{dom}(\partial\varphi) \neq \emptyset$, where $\partial\varphi(x) = \{z \in K : \varphi(y) \geq \varphi(x) + \langle z, y - x \rangle, \forall y \in K\}$ is the subdifferential at $x \in K$ and a point $z \in \partial\varphi(x)$ is called a subgradient of φ at x. Moreover, $g(t, \cdot)$ is a continuous affine function, for all $t \in \Omega$. Let $Q : \Omega \times K \to H$ be continuous random single-valued mapping, $T : \Omega \times K \to \mathrm{CB}(H)$ be a random multi-valued mapping such that T is upper semicontinuous on $\Omega \times K$ and $T(t, \cdot)$ has compact values on K. By Lemma 2.8, for any given measurable mapping $x : \Omega \to K$, the multi-valued mapping $T(\cdot, x(\cdot)) : \Omega \to \mathrm{CB}(H)$ is measurable. Hence there exists measurable selection $w : \Omega \to K$ of $T(\cdot, x(\cdot))$ by Himmelberg [18]. We now consider the following *random mixed equilibrium problems*:
(RMEP) find a measurable mapping $x^* : \Omega \to K$ such that

$$\begin{aligned} &\exists w^* : \Omega \to K, w^*(t) \in T\Big(t, x^*(t)\Big), \\ &F\Big(t, g(t, x^*(t)), y(t)\Big) + \Big\langle Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0 \end{aligned}$$

for all $t \in \Omega, y(t) \in K$.

The solution set of (RMEP) is defined by $\Sigma(t)$, i.e.

$$\Sigma(t) = \left\{ \begin{array}{l} x^*(t) \in K \mid \exists w^* : \Omega \to H, w^*(t) \in T\Big(t, x^*(t)\Big) \text{ and} \\ F\Big(t, g(t, x^*(t)), y(t)\Big) + \Big\langle Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle \\ \qquad +\varphi\Big(y(t)\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0, \forall t \in \Omega, y(t) \in K. \end{array} \right\}$$

Throughout this paper, we always assume that $\Sigma(t)$ is not empty for all $t \in \Omega$.

To illustrate motivations for these settings, we provide some special cases of the problem.

(a) If $H = K$, $F \equiv 0$, $Q(t, w(t)) = T(x)$ is a single-valued mapping and $g(t, x(t)) = g(x)$ for all $(t, x) \in \Omega \times H$, then the (RMEP) reduces to the generalized mixed variational inequality problem in deterministic case (for short, (GMVIP)) studied in Solodov [33], which consists in finding $\bar{x} \in H$ such that

$$\langle T(\bar{x}), y - g(\bar{x}) \rangle + \varphi(y) - \varphi(g(\bar{x}) \geq 0, \forall y \in H.$$

(b) If $H = \mathbb{R}^n$, $F \equiv 0$, $Q(t, w(t)) = T(x)$ is a single-valued mapping, $g(t, x(t)) = x$ for all $(t, x) \in \Omega \times H$ and $\varphi \equiv 0$, then problem (RMEP) reduces to the variational inequality problem in deterministic case (for short, (VIP)) studied in [32, 35, 36], which consists in finding $\bar{x} \in K$ such that

$$\langle T(\bar{x}), y - \bar{x} \rangle \geq 0, \forall y \in K.$$

3. Difference gap functions

In this section, by using the regularized forms of the Fukushima type in [17] and the Moreau-Yosida type in [35], we give two kinds of regularized gap functions for (RMEP). Finally, we develop the D-gap functions for (RMEP) based on these functions.

Definition 3.1. Let K be the domain of (RMEP). A random function $m : \Omega \times K \to \mathbb{R}$ is said to be a *gap function* for (RMEP), if it satisfies the following properties:

(G_a) $m(t, x(t)) \geq 0, \forall (t, x(t)) \in \Omega \times K$;
(G_b) for any $(t, x^*(t)) \in \Omega \times K$, $m(t, x^*(t)) = 0$, if and only if $x^*(t) \in K$ is a solution of (RMEP).

We now present characterizations of a smooth convex function in real Hilbert spaces. The following Lemma 3.2 is reformatted from Proposition 17.10 in [7] for the random functions.

Lemma 3.2. *For each $(t, x) \in \Omega \times K$, let $F(t, x, \cdot) : K \to \mathbb{R}$ be a differentiable function on K. Then the following are equivalent:*

(i) *$F(t, x, \cdot)$ is convex;*

(ii) *For any $y_1, y_2 \in K$, $\langle \nabla_3 F(t, x, y_1), y_2 - y_1 \rangle \leq F(t, x, y_2) - F(t, x, y_1)$;*

(iii) *For any $y_1, y_2 \in K$, $\langle \nabla_3 F(t, x, y_2) - \nabla_3 F(t, x, y_1), y_2 - y_1 \rangle \geq 0$. In other word, $\nabla_3 F(t, x, \cdot)$ is monotone.*

Picking up the ideas from establishing the regularized gap function of Fukushima type in [28] without the projection operator method, we develop the notion of regularized gap function of the Fukushima type for (RMEP).

Lemma 3.3. *A measurable mapping $x^* : \Omega \to K$ solves the* (RMEP) *if and only if there exists $w^*(t) \in T(t, x^*(t))$ such that for all $t \in \Omega$, $y(t) \in K$,*

$$\Big\langle \nabla_3 F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big) + Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0. \tag{1}$$

Proof. Suppose that x^* solves (RMEP). Then there exists $w^*(t) \in T(t, x^*(t))$ for $x^*(t) \in K$ such that for all $t \in \Omega$, $y(t) \in K$,

$$F\Big(t, g(t, x^*(t)), y(t)\Big) + \Big\langle Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0.$$

That is $g(t, x^*(t))$ solves the following convex minimization problem

$$\min \Big\{F\Big(t, g(t, x^*(t)), y(t)\Big) + \Big\langle Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\Big(y(t)\Big)\Big\}$$
such that $y(t) \in K$.

Using the optimality condition for this problem, we have

$$0 \in \nabla_3 F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big) + Q\Big(t, w^*(t)\Big) + \partial\varphi\Big(g(t, x^*(t))\Big)$$

which implies

$$-\nabla_3 F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big) - Q\Big(t, w^*(t)\Big) \in \partial\varphi\Big(g(t, x^*(t))\Big). \tag{2}$$

By the definition of the subgradient, it follows from (2) that for all $t \in \Omega$, $y(t) \in K$,

$$\varphi\Big(y(t)\Big) \geq \varphi\Big(g(t, x^*(t))\Big) - \Big\langle \nabla_3 F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big) + Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle,$$

which means that for all $t \in \Omega$, $y(t) \in K$,

$$\Big\langle \nabla_3 F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big) + Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0.$$

Conversely, suppose that $x^*(t) \in K$ and $w^*(t) \in T(t, x^*(t))$ solve (1) for all $t \in \Omega$. By the convexity of $F\big(t, x^*(t), \cdot\big)$, for any $y(t) \in K$, it follows form Lemma 3.2(ii) that

$$\Big\langle \nabla_3 F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big), y(t) - g(t, x^*(t))\Big\rangle \leq F\Big(t, g(t, x^*(t)), y(t)\Big) - F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big). \tag{3}$$

From (1) and (3) noting $F\Big(t, g(t, x^*(t)), g(t, x^*(t))\Big) = 0$, we obtain

$$F\Big(t, g(t, x^*(t)), y(t)\Big) + \Big\langle Q\Big(t, w^*(t)\Big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0,$$

for all $t \in \Omega$, $y(t) \in K$, i.e., x^* solves (RMEP). Thus, the proof is complete. □

Let the function $\Phi_{\theta(t)} : \Omega \times K \times K \times H \to \mathbb{R}$ be defined by

$$\Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big) = -F\Big(t, g(t, x(t)), y(t)\Big) + \Big\langle Q\Big(t, w(t)\Big), g(t, x(t)) - y(t)\Big\rangle - \varphi\Big(y(t)\Big) + \varphi\Big(g(t, x(t))\Big) - \frac{1}{2\theta(t)}\Big\|g(t, x(t)) - y(t)\Big\|^2$$

where $\theta : \Omega \to (0, +\infty)$ is a measurable function. Then, we consider the function $\Pi_{\theta(t)} : \Omega \times K \to \mathbb{R}$ defined by

$$\Pi_{\theta(t)}\Big(t, x(t)\Big) = \min_{w(t) \in T(t, x(t))} \max_{y(t) \in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big). \tag{4}$$

Since $T\big(t, x(t)\big)$ is compact and $\max_{y(t) \in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big)$ is continuous in the fourth argument (Note that Q is continuous.), there exists $w_\theta(t) \in T\big(t, x(t)\big)$ such that

$$\Pi_{\theta(t)}\Big(t, x(t)\Big) = \max_{y(t) \in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w_\theta(t)\Big) \tag{5}$$

Since $\Phi_{\theta(t)}\big(t, x(t), \cdot, w_\theta(t)\big)$ is strongly concave and continuous, the following inner maximization problem

$$\begin{aligned} &\max \Phi_{\theta(t)}\Big(t, x(t), y(t), w_\theta(t)\Big) \\ &\text{such that } y(t) \in K, \end{aligned} \tag{6}$$

always has a unique solution $y_{\theta(t)}\big(x(t)\big)$. The optimality condition for the problem (6) can be formulated as follows.

Lemma 3.4. *For each $t \in \Omega, x(t) \in K$ and $w(t) \in T(t, x(t))$,*

$$\Big\langle \nabla_3 F\Big(t, g(t, x(t)), y_{\theta(t)}\big(x(t)\big)\Big) + Q\Big(t, w(t)\Big) + \frac{1}{\theta(t)}\Big(y_{\theta(t)}\big(x(t)\big) - g(t, x(t))\Big), y(t) - y_{\theta(t)}\big(x(t)\big)\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(y_{\theta(t)}\big(x(t)\big)\Big) \geq 0, \quad \forall y(t) \in K \tag{7}$$

holds.

Proof. For each $y(t) \in K$, noting that $y_{\theta(t)}\big(x(t)\big)$ is a unique solution of the problem minimize $-\Phi_{\theta(t)}\big(t, x(t), \cdot, w(t)\big)$ on K, and $-\Phi_{\theta(t)}\big(t, x(t), \cdot, w(t)\big)$ is convex, we have

$$0 \in \nabla_3 F\Big(t, g(t, x(t)), y_{\theta(t)}\big(x(t)\big)\Big) + Q\Big(t, w(t)\Big) + \partial\varphi\Big(y_{\theta(t)}\big(x(t)\big)\Big) + \frac{1}{\theta(t)}\Big(y_{\theta(t)}\big(x(t)\big) - g(t, x(t))\Big)$$

which implies

$$-\nabla_3 F\left(t, g(t, x(t)), y_{\theta(t)}\big(x(t)\big)\right) - Q\big(t, w(t)\big) - \frac{1}{\theta(t)}\left(y_{\theta(t)}\big(x(t)\big) - g(t, x(t))\right) \in \partial\varphi\left(y_{\theta(t)}\big(x(t)\big)\right). \tag{8}$$

By the definition of the subgradient, it follows from (8) that

$$\begin{aligned}\varphi\big(y(t)\big) \geq \varphi\left(y_{\theta(t)}\big(x(t)\big)\right) - \Big\langle \nabla_3 F\left(t, g(t, x(t)), y_{\theta(t)}\big(x(t)\big)\right) + Q\big(t, w(t)\big) \\ + \frac{1}{\theta(t)}\left(y_{\theta(t)}\big(x(t)\big) - g(t, x(t))\right), y(t) - y_{\theta(t)}\big(x(t)\big)\Big\rangle\end{aligned}$$

which implies (7). □

Theorem 3.5. *Let $x : \Omega \to K$ be a measurable mapping and $\theta : \Omega \to (0, +\infty)$ be a measurable function. Then for each $t \in \Omega$, the function $\Pi_{\theta(t)}\big(t, x(t)\big)$ defined by (4) is a gap function for* (RMEP).

Proof. (G_a) For each $t \in \Omega$ and $x(t) \in K$, it is easy to see that

$$\max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x(t), y(t), w(t)\big) \geq 0, \forall x(t) \in K, w(t) \in T\big(t, x(t)\big).$$

Suppose on the contrary that there exists $x^*(t) \in K$ and $w_0(t) \in T\big(t, x^*(t)\big)$ such that

$$\max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x^*(t), y(t), w_0(t)\big) < 0.$$

Then

$$0 > \max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x^*(t), y(t), w_0(t)\big) \geq \Phi_{\theta(t)}\big(t, x^*(t), y(t), w_0(t)\big), \forall y(t) \in K.$$

When $y(t) = x^*(t)$, we have a contradiction. Hence

$$\max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x(t), y(t), w(t)\big) \geq 0, \forall x(t) \in K, w(t) \in T\big(t, x(t)\big).$$

Since $w(t) \in T\big(t, x(t)\big)$ is arbitrary, we have

$$\Pi_{\theta(t)}\big(t, x(t)\big) = \min_{w(t)\in T(t,x(t))} \max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x(t), y(t), w(t)\big) \geq 0, \forall x(t) \in K.$$

(G_b) Suppose that $x^*(t)$ is a solution of (RMEP). Then, it follows from (5) that

$$\Pi_{\theta(t)}\big(t, x^*(t)\big) = \max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x^*(t), y(t), w_\theta(t)\big) = \Phi_{\theta(t)}\Big(t, x^*(t), y_{\theta(t)}\big(x^*(t)\big), w_\theta(t)\Big). \tag{9}$$

Moreover, since $x^*(t)$ is a solution of (RMEP), from the proof of Lemma 3.3, there exists $w^*(t) \in T(t, x^*(t))$ such that

$$\begin{aligned}\Big\langle \nabla_3 F\big(t, g(t, x^*(t)), g(t, x^*(t))\big) + Q\big(t, w^*(t)\big), y_{\theta(t)}\big(x^*(t)\big) - g(t, x^*(t))\Big\rangle \\ + \varphi\left(y_{\theta(t)}\big(x^*(t)\big)\right) - \varphi\big(g(t, x^*(t))\big) \geq 0.\end{aligned} \tag{10}$$

From the result of Lemma 3.4,

$$\begin{aligned}\Big\langle \nabla_3 F\left(t, g(t, x^*(t)), y_{\theta(t)}\big(x^*(t)\big)\right) + Q\big(t, w^*(t)\big) + \frac{1}{\theta(t)}\left(y_{\theta(t)}\big(x^*(t)\big) - g(t, x^*(t))\right), g(t, x^*(t)) \\ - y_{\theta(t)}\big(x^*(t)\big)\Big\rangle + \varphi\big(g(t, x^*(t))\big) - \varphi\left(y_{\theta(t)}\big(x^*(t)\big)\right) \geq 0.\end{aligned} \tag{11}$$

From (10) and (11), we get

$$
\begin{aligned}
&-\frac{1}{\theta(t)}\left\|y_{\theta(t)}\big(x^*(t)\big)-g(t,x^*(t))\right\|^2\\
&\geq \left\langle \nabla_3 F\left(t,g(t,x^*(t)),y_{\theta(t)}\big(x^*(t)\big)\right)-\nabla_3 F\left(t,g(t,x^*(t)),g(t,x^*(t))\right),y_{\theta(t)}\big(x^*(t)\big)-g(t,x^*(t))\right\rangle \geq 0,
\end{aligned}
$$

since $\nabla_3 F(t,g(t,x^*(t)),\cdot)$ is monotone (by Lemma 3.2(iii)). Thus, we get $g(t,x^*(t)) = y_{\theta(t)}\big(x^*(t)\big)$. It follows from (9) that $\Pi_{\theta(t)}\big(t,x^*(t)\big)=0$.

Conversely, for any $t\in\Omega$ such that $x^*(t)\in K$, $\Pi_{\theta(t)}\big(t,x^*(t)\big)=0$. Then there exists $w^*(t)\in T\big(t,x^*(t)\big)$ such that $\max_{y(t)\in K}\Phi_{\theta(t)}\big(t,x^*(t),y(t),w^*(t)\big)=0$. This implies $\Phi_{\theta(t)}\big(t,x^*(t),y(t),w^*(t)\big)\leq 0$, $\forall y(t)\in K$ or

$$
\begin{aligned}
F\big(t,g(t,x^*(t)),y(t)\big)+\left\langle Q\big(t,w^*(t)\big),y(t)-g(t,x^*(t))\right\rangle+\varphi\big(y(t)\big)-\varphi\big(g(t,x^*(t))\big)&\\
+\frac{1}{2\theta(t)}\left\|g(t,x^*(t))-y(t)\right\|^2\geq 0.&
\end{aligned}
$$

Then $x^*(t)$ solves the following convex minimization problem

$$
\min\left\{\begin{array}{r} F\big(t,g(t,x^*(t)),y(t)\big)+\left\langle Q\big(t,w^*(t)\big),y(t)-g(t,x^*(t))\right\rangle+\varphi\big(y(t)\big)\\ +\dfrac{1}{2\theta(t)}\left\|g(t,x^*(t))-y(t)\right\|^2\end{array}\right\}
$$

such that $y(t)\in K$.

Using the optimality condition for this problem, we get

$$
\begin{aligned}
\left\langle \nabla_3 F\big(t,g(t,x^*(t)),g(t,x^*(t))\big)+Q\big(t,w^*(t)\big),y(t)-g(t,x^*(t))\right\rangle+\varphi\big(y(t)\big)&\\
-\varphi\big(g(t,x^*(t))\big)\geq 0,\forall t\in\Omega, y(t)\in K.&
\end{aligned}
$$

By Lemma 3.3, this implies that $x^*(t)$ is a solution of the (RMEP). Thus, $\Pi_{\theta(t)}\big(t,x(t)\big)$ is a gap function for the (RMEP). □

Remark 3.6. Theorem 3.5 extends Theorem 3.1 in [17], Lemma 2.1 in [35] and Theorem 5.1 in [6] in the following aspects:

(i) The problem (RMEP) is a generalization from the variational inequality to the mixed equilibrium problem.

(ii) The problem (RMEP) is established in random environment.

(iii) Our Theorem 3.5 is established without the projection operator method.

From the result Theorem 1.4.16 in [4], we get the following result:

Lemma 3.7. Let a random multi-valued mapping $T:\Omega\times K\to \mathrm{CB}(H)$ and a random function $f: Graph(T)\to\mathbb{R}$ be given. If f and T are upper semicontinuous and $T(t,\cdot)$ has compact values on K for all $t\in\Omega$, then the random function $g:\Omega\times K\to\mathbb{R}\cup\{+\infty\}$ defined by

$$
g(t,x)=\max_{w\in T(t,x)} f(t,x,w)
$$

is upper semicontinuous.

Now we consider the gap function $\Xi_{\Pi_{\theta(t)},\rho(t)} : \Omega \times K \to \mathbb{R}$ based on Moreau-Yosida regularization of the gap function $\Pi_{\theta(t)}$ for (RMEP). Then $\Xi_{\Pi_{\theta(t)},\rho(t)}$ is defined by

$$\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t)) = \min_{z(t)\in K} \left\{ \Pi_{\theta(t)}(t, z(t)) + \rho(t) \left\| g(t, x(t)) - g(t, z(t)) \right\|^2 \right\}, \tag{12}$$

where $x(t) \in K$ and $\rho : \Omega \to (0, +\infty)$ is a measurable function.

Theorem 3.8. *Assume that $g(t, \cdot)$ is $\kappa(t)$-strongly nonexpanding, for all $t \in \Omega$. Then $\Xi_{\Pi_{\theta(t)},\rho(t)}$ defined by* (12) *is a gap function for* (RMEP).

Proof. (G_a) For any measurable functions $\theta, \rho : \Omega \to (0, +\infty)$ and $x(t) \in K$, since $\Pi_{\theta(t)}(t, x(t)) \geq 0$ for all $x(t) \in K$, we conclude that $\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t)) \geq 0$ for all $x(t) \in K$.

(G_b) Suppose that $x^*(t) \in \Sigma(t)$. Theorem 3.5 implies that $\Pi_{\theta(t)}(t, x^*(t)) = 0, \forall t \in \Omega$. Moreover, we have

$$\begin{aligned}
\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x^*(t)) &= \min_{z(t)\in K} \left\{ \Pi_{\theta(t)}(t, z(t)) + \rho(t) \left\| g(t, x^*(t)) - g(t, z(t)) \right\|^2 \right\} \\
&\leq \Pi_{\theta(t)}(t, x^*(t)) + \rho(t) \left\| g(t, x^*(t)) - g(t, x^*(t)) \right\|^2 \\
&= 0.
\end{aligned}$$

Moreover, $\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t)) \geq 0$, we conclude that $\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x^*(t)) = 0, \forall t \in \Omega$.

Conversely, if $\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x^*(t)) = 0$, i.e.,

$$\min_{z(t)\in K} \left\{ \Pi_{\theta(t)}(t, z(t)) + \rho(t) \left\| g(t, x^*(t)) - g(t, z(t)) \right\|^2 \right\} = 0.$$

Thus, for each n, there is $z_n(t) \in K$ such that

$$\Pi_{\theta(t)}(t, z_n(t)) + \rho(t) \left\| g(t, x^*(t)) - g(t, z_n(t)) \right\|^2 < \frac{1}{n}. \tag{13}$$

Since $g(t, \cdot)$ is $\kappa(t)$-strongly nonexpanding, it follows from (13) that

$$0 \leq \Pi_{\theta(t)}(t, z_n(t)) + \rho(t)\kappa^2(t) \left\| x^*(t) - z_n(t) \right\|^2 < \frac{1}{n},$$

and hence, $\Pi_{\theta(t)}(t, z_n(t)) \to 0$ and $\|x^*(t) - z_n(t)\| \to 0, \forall t \in \Omega$. So $z_n(t) \to x^*(t)$. We now prove that $\Pi_{\theta(t)}$ is lower semicontinuous on K. In fact, since F, Q and g are continuous and φ is lower semicontinuous, it is clearly that

$$\begin{aligned}
\Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big) = &- F\Big(t, g(t, x(t)), y(t)\Big) + \Big\langle Q\Big(t, w(t)\Big), g(t, x(t)) - y(t) \Big\rangle \\
&- \varphi\Big(y(t)\Big) + \varphi\Big(g(t, x(t))\Big) - \frac{1}{2\theta(t)} \left\| g(t, x(t)) - y(t) \right\|^2
\end{aligned}$$

is lower semicontinuous in the second and fourth arguments for each $t \in \Omega, y(t) \in K$. So the mapping $\max\limits_{y(t)\in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big)$ is lower semicontinuous. Hence, $-\max\limits_{y(t)\in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big)$ is upper semicontinuous. Moreover, as T is upper semicontinuous with compact values, from Lemma 3.7 we get that

$$\begin{aligned}
\Pi_{\theta(t)}\Big(t, x(t)\Big) &= \min_{w(t)\in T(t,x(t))} \max_{y(t)\in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big) \\
&= - \max_{w(t)\in T(t,x(t))} \left\{ - \max_{y(t)\in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big) \right\}
\end{aligned}$$

is lower semicontinuous. Moreover, $\Pi_{\theta(t)}(t,\cdot)$ is nonnegative. Hence, we get

$$0 \leq \Pi_{\theta(t)}(t, x^*(t)) \leq \liminf_{n\to+\infty} \Pi_{\theta(t)}(t, z_n(t)) = 0,$$

which yields that $\Pi_{\theta(t)}(t, x^*(t)) = 0$. Applying Theorem 3.5, we have $x^*(t) \in \Sigma(t)$. This completes the proof. □

Next, we will establish D-gap functions for (RMEP) by using the regularized gap functions of the Fukushima type and the Moreau-Yosida type given above.

Let the gap functions $\Pi_{\theta(t)}$ and $\Xi_{\Pi_{\theta(t)},\rho(t)}$ be defined by (4) and (12), respectively. Now, we will consider the functions $D_{\theta(t),\vartheta(t)}, D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)} : \Omega \times K \to \mathbb{R}$ defined by

$$D_{\theta(t),\vartheta(t)}\big(t, x(t)\big) = \Pi_{\theta(t)}\big(t, x(t)\big) - \Pi_{\vartheta(t)}\big(t, x(t)\big); \tag{14}$$

$$D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t, x(t)\big) = \Xi_{\Pi_{\theta(t)},\rho(t)}\big(t, x(t)\big) - \Xi_{\Pi_{\vartheta(t)},\varrho(t)}\big(t, x(t)\big). \tag{15}$$

where $\theta, \vartheta, \rho, \varrho : \Omega \to (0, +\infty)$ are measurable functions satisfying $\theta(t) > \vartheta(t)$, $\rho(t) > \varrho(t)$ for all $t \in \Omega$. Then we obtain the following properties of $D_{\theta(t),\vartheta(t)}\big(t, x(t)\big), D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t, x(t)\big)$.

Lemma 3.9. *For all $t \in \Omega$ and $x(t) \in K$, we have*

$$\Big\|g(t, x(t)) - y_{\vartheta(t)}\big(x(t)\big)\Big\|^2 \leq \frac{2\theta(t)\vartheta(t)}{\theta(t) - \vartheta(t)} D_{\theta(t),\vartheta(t)}\big(t, x(t)\big) \leq \Big\|g(t, x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\|^2. \tag{16}$$

Proof. By the definitions of the gap functions $\Pi_{\theta(t)}\big(t, x(t)\big)$, $\Pi_{\vartheta(t)}\big(t, x(t)\big)$ and the function $D_{\theta(t),\vartheta(t)}\big(t, x(t)\big)$ in (14), we get

$$\begin{aligned}
D_{\theta(t),\vartheta(t)}\big(t, x(t)\big) &= \max_{y(t)\in K} \Phi_{\theta(t)}\big(t, x(t), y(t), w_\theta(t)\big) - \max_{y(t)\in K} \Phi_{\vartheta(t)}\big(t, x(t), y(t), w_\vartheta(t)\big)\\
&= \Phi_{\theta(t)}\big(t, x(t), y_{\theta(t)}\big(x(t)\big), w_\theta(t)\big) - \Phi_{\vartheta(t)}\big(t, x(t), y_{\vartheta(t)}\big(x(t)\big), w_\vartheta(t)\big)\\
&\leq \Phi_{\theta(t)}\big(t, x(t), y_{\theta(t)}\big(x(t)\big), w_\vartheta(t)\big) - \Phi_{\vartheta(t)}\big(t, x(t), y_{\vartheta(t)}\big(x(t)\big), w_\vartheta(t)\big)\\
&\leq \Phi_{\theta(t)}\big(t, x(t), y_{\theta(t)}\big(x(t)\big), w_\vartheta(t)\big) - \Phi_{\vartheta(t)}\big(t, x(t), y_{\theta(t)}\big(x(t)\big), w_\vartheta(t)\big)\\
&= \left(\frac{1}{2\vartheta(t)} - \frac{1}{2\theta(t)}\right)\Big\|g(t, x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\|^2.
\end{aligned}$$

Hence, the right-hand-side inequality in (16) holds. Similarly, we obtain the left-hand-side inequality in (16). □

Theorem 3.10. *Let $x : \Omega \to K$ be a measurable mapping and $\theta, \vartheta : \Omega \to (0, +\infty)$ be measurable functions such that $\theta(t) > \vartheta(t)$ for all $t \in \Omega$. Then for each $t \in \Omega$ and $x(t) \in K$, the function $D_{\theta(t),\vartheta(t)}\big(t, x(t)\big)$ defined by (14) is a gap function for* (RMEP).

Proof. (G_a) It is clearly follows from (16) that $D_{\theta(t),\vartheta(t)}\big(t, x(t)\big) \geq 0$, for all $x(t) \in K$.

(G_b) Suppose that $x^*(t)$ is a solution of (RMEP). It follows from Theorem 3.5 that $\Pi_{\theta(t)}\big(t, x^*(t)\big) = 0$ and $\Pi_{\vartheta(t)}\big(t, x^*(t)\big) = 0$. Hence $D_{\theta(t),\vartheta(t)}\big(t, x^*(t)\big) = 0$.

Conversely, for any $t \in \Omega$ such that $x^*(t) \in K$, $D_{\theta(t),\vartheta(t)}\big(t, x^*(t)\big) = 0$. From (16), we have $g(t, x^*(t)) = y_{\vartheta(t)}\big(x^*(t)\big)$. Then there exists $w^*(t) \in T(t, x^*(t))$. It follows from (7) that

$$\begin{aligned}
&\Big\langle \nabla_3 F\big(t, g(t, x^*(t)), g(t, x^*(t))\big) + Q\big(t, w^*(t)\big), y(t) - g(t, x^*(t))\Big\rangle + \varphi\big(y(t)\big)\\
&\qquad\qquad - \varphi\big(g(t, x^*(t))\big) \geq 0, \quad \forall y(t) \in K.
\end{aligned}$$

Hence, by Lemma 3.3, we get that $x^*(t)$ is a solution of the (RMEP). Thus, $D_{\theta(t),\vartheta(t)}\big(t,x(t)\big)$ is a gap function for (RMEP). □

Define the function $\Lambda_{\theta(t),\rho(t)}:\Omega\times K\times K\to\mathbb{R}$ as

$$\Lambda_{\theta(t),\rho(t)}(t,x(t),z(t))=\Pi_{\theta(t)}(t,z(t))+\rho(t)\big\|g(t,x(t))-g(t,z(t))\big\|^2.$$

Then, by (12), we have

$$\Xi_{\Pi_{\theta(t)},\rho(t)}(t,x(t))=\min_{z(t)\in K}\Lambda_{\theta(t),\rho(t)}(t,x(t),z(t)).$$

When the function $\Lambda_{\theta(t),\rho(t)}(t,x(t),\cdot)$ is assumed to attain its minima uniquely on K, we have the following result.

Theorem 3.11. *Let $x:\Omega\to K$ be a measurable mapping and $\theta,\vartheta,\rho,\varrho:\Omega\to(0,+\infty)$ be measurable function such that $\theta(t)>\vartheta(t)$ and $\rho(t)>\varrho(t)$ for all $t\in\Omega$. If the function $\Lambda_{\theta(t),\rho(t)}(t,x(t),\cdot)$ (resp. $\Lambda_{\vartheta(t),\varrho(t)}(t,x(t),\cdot)$ attains its unique minimum $z_{\theta(t),\rho(t)}(x(t))$ (resp. $z_{\vartheta(t),\varrho(t)}(x(t))$) on K and $g(t,\cdot)$ is $\kappa(t)$-strongly nonexpanding on K, for all $t\in\Omega$, $x(t)\in K$. Then the function $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}$ defined by (15) is a gap function for* (RMEP).

Proof. (G_a) By the definitions of the gap functions $\Xi_{\Pi_{\theta(t)},\rho(t)}$ and $\Xi_{\Pi_{\vartheta(t)},\varrho(t)}$, we get

$$\begin{aligned}
&D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t,x(t)\big)\\
&\quad=\min_{z(t)\in K}\Lambda_{\theta(t),\rho(t)}(t,x(t),z(t))-\min_{z(t)\in K}\Lambda_{\vartheta(t),\varrho(t)}(t,x(t),z(t))\\
&\quad=\Lambda_{\theta(t),\rho(t)}(t,x(t),z_{\theta(t),\rho(t)})-\Lambda_{\vartheta(t),\varrho(t)}(t,x(t),z_{\vartheta(t),\varrho(t)})\\
&\quad\geq\Lambda_{\theta(t),\rho(t)}(t,x(t),z_{\theta(t),\rho(t)})-\Lambda_{\vartheta(t),\varrho(t)}(t,x(t),z_{\theta(t),\rho(t)})\\
&\quad=\Pi_{\theta(t)}\left(t,z_{\theta(t),\rho(t)}(x(t))\right)+\rho(t)\left\|g(t,x(t))-g\left(t,z_{\theta(t),\rho(t)}(x(t))\right)\right\|^2\\
&\qquad-\Pi_{\vartheta(t)}\left(t,z_{\theta(t),\rho(t)}(x(t))\right)-\varrho(t)\left\|g(t,x(t))-g\left(t,z_{\theta(t),\rho(t)}(x(t))\right)\right\|^2\\
&\quad=D_{\theta(t),\vartheta(t)}\Big(t,z_{\theta(t),\rho(t)}(x(t))\Big)+(\rho(t)-\varrho(t))\left\|g(t,x(t))-g\left(t,z_{\theta(t),\rho(t)}(x(t))\right)\right\|^2\\
&\quad\geq D_{\theta(t),\vartheta(t)}\Big(t,z_{\theta(t),\rho(t)}(x(t))\Big)+(\rho(t)-\varrho(t))\kappa^2(t)\left\|x(t)-z_{\theta(t),\rho(t)}(x(t))\right\|^2.
\end{aligned}\tag{17}$$

By Theorem 3.10, $D_{\theta(t),\vartheta(t)}\Big(t,z_{\theta(t),\rho(t)}(x(t))\Big)\geq 0$. Thus, $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t,x(t)\big)\geq 0$ for all $t\in\Omega$, $x(t)\in K$.

(G_b) Suppose that $x^*(t)$ is a solution of (RMEP). It follows from Theorem 3.8 that $\Xi_{\Pi_{\theta(t)},\rho(t)}\big(t,x^*(t)\big)=0$ and $\Xi_{\Pi_{\vartheta(t)},\varrho(t)}\big(t,x^*(t)\big)=0$. Hence $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t,x^*(t)\big)=0$.

Conversely, for any $t\in\Omega$ such that $x^*(t)\in K$, $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t,x^*(t)\big)=0$. From (17), we have $\left\|x^*(t)-z_{\theta(t),\rho(t)}(x^*(t))\right\|=0$ and $D_{\theta(t),\vartheta(t)}\Big(t,z_{\theta(t),\rho(t)}(x^*(t))\Big)=0$, i.e., $z_{\theta(t),\rho(t)}(x^*(t))=x^*(t)$ and hence $D_{\theta(t),\vartheta(t)}\big(t,x^*(t)\big)=0$. By Theorem 3.10, we get that $x^*(t)$ is a solution of the (RMEP).

Thus, $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t,x(t)\big)$ is a gap function for (RMEP). □

Remark 3.12. As mentioned in Introduction, up to now there is no any paper devoted to the regularized gap functions of Moreau-Yosida type, D-gap functions for mixed equilibrium problems in random environments. Thus, our results, Theorems 3.8, 3.10 and 3.11 are new. However, if the problem (RMEP) is not random environment, then Theorem 3.8 extends Theorem 2.4 in [35], Theorem 3.10 extends Theorem 3.2 in [36]. Moreover, our results are established without the projection operator method. Note that, Theorem 3.11 is new.

The following example shows that all assumptions imposed in Theorems 3.5, 3.8, 3.10 and 3.11 are satisfied.

Example 3.13. Let $(\Omega, \mathcal{A})$ be a measurable space, where $\Omega = [0,1], \mathcal{A}$ is σ-algebra of subsets of $[0,1]$ and $H = \mathbb{R}$, $K = [0, \frac{1}{2}]$. The random multi-valued mapping $T : [0,1] \times [0, \frac{1}{2}] \to CB(\mathbb{R})$ is defined by $T(t, x(t)) = \{2x(t)\}$. The random single-valued functions $g : [0,1] \times [0, \frac{1}{2}] \to [0, \frac{1}{2}]$, $F : [0,1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \to \mathbb{R}$, $Q : [0,1] \times [0, \frac{1}{2}] \to \mathbb{R}$ and the function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ are defined by

$$\begin{aligned} g(t, x(t)) &= x(t), \\ F(t, x(t), y(t)) &= (t^2+1)(y^2(t) + 3x(t)y(t) - 4x^2(t)), \\ Q(t, w(t)) &= 2tw(t), \\ \varphi(x(t)) &= x^2(t), \forall (x(t), y(t), w(t)) \in \mathbb{R}^3. \end{aligned}$$

We consider (RMEP): find a measurable mapping $x : \Omega \to K$ such that

$$\begin{aligned} &\exists w : \Omega \to H, \quad w(t) \in T(t, x(t)), \\ &F\Big(t, g(t, x(t)), y(t)\Big) + \Big\langle Q\Big(t, w(t)\Big), y(t) - g(t, x(t))\Big\rangle + \varphi\Big(y(t)\Big) - \varphi\Big(g(t, x(t))\Big) \geq 0 \end{aligned}$$

for all $t \in \Omega, y(t) \in K$. That is, find a measurable mapping $x : [0,1] \to [0, \frac{1}{2}]$ such that for all $t \in [0,1]$, $y(t) \in [0, \frac{1}{2}]$,

$$\begin{aligned} &(t^2+1)\Big(y^2(t) + 3x(t)y(t) - 4x^2(t)\Big) + \Big\langle 4tx(t), y(t) - x(t)\Big\rangle + y^2(t) - x^2(t) \\ &\qquad = ((4t^2 + 4t + 5)x(t) + (t^2+2)y(t))(y(t) - x(t)) \geq 0. \end{aligned}$$

It follows from a direct computation that $x(t) = 0, \forall t \in [0,1]$, that is $\Sigma(t) = \{0\}$.

It is easy to verify that the assumptions of Theorems 3.5, 3.8, 3.10 and 3.11 are satisfied with $\kappa(t) = 1$. We now compute the functions $\Pi_{\theta(t)}, \Pi_{\vartheta(t)}, \Xi_{\Pi_{\theta(t)},\rho(t)}, \Xi_{\Pi_{\vartheta(t)},\varrho(t)}, D_{\theta(t),\vartheta}$ and $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}$ for (RMEP).

For any $t \in [0,1]$, $\theta(t) = 2, \vartheta(t) = 1, \rho(t) = 1, \varrho(t) = \dfrac{1}{2}$, we have

$$\begin{aligned} \Pi_{\theta(t)}\Big(t, x(t)\Big) &= \min_{w(t) \in T(t,x(t))} \max_{y(t) \in K} \Phi_{\theta(t)}\Big(t, x(t), y(t), w(t)\Big) \\ &= \max_{y(t) \in [0, \frac{1}{2}]} \Big\{((4t^2+4t+5)x(t) + (t^2+2)y(t))(x(t) - y(t)) - \frac{1}{4}(x(t) - y(t))^2\Big\} \\ &= \frac{1}{4} \max_{y(t) \in [0, \frac{1}{2}]} \Big\{(x(t) - y(t))\Big[\Big(16t^2 + 16t + 19\Big)x(t) + \Big(4t^2 + 9\Big)y(t)\Big]\Big\} \\ &= \frac{1}{4}\Big(16t^2 + 16t + 19\Big)x^2(t); \\ \Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t)) &= \min_{z(t) \in K} \Big\{\Pi_{\theta(t)}(t, z(t)) + \rho(t)\Big\|g(t, x(t)) - g(t, z(t))\Big\|^2\Big\} \\ &= \frac{16t^2 + 16t + 19}{16t^2 + 16t + 23} x^2(t). \end{aligned}$$

Computing $\Pi_{\vartheta(t)}$ and $\Xi_{\Pi_{\vartheta(t)},\varrho(t)}$ similar to $\Pi_{\theta(t)}$ and $\Xi_{\Pi_{\theta(t)},\rho(t)}$, respectively, we obtain

$$\begin{aligned} \Pi_{\vartheta(t)}\Big(t, x(t)\Big) &= \frac{1}{2}\Big(8t^2 + 8t + 9\Big)x^2(t); \\ \Xi_{\Pi_{\vartheta(t)},\varrho(t)}(t, x(t)) &= \frac{8t^2 + 8t + 9}{16t^2 + 16t + 20} x^2(t). \end{aligned}$$

Then

$$\begin{aligned} D_{\theta(t),\vartheta(t)}\Big(x(t)\Big) &= \Pi_{\theta(t)}\Big(t,x(t)\Big) - \Pi_{\vartheta(t)}\Big(t,x(t)\Big) \\ &= \frac{1}{4}\left(16t^2+16t+19\right)x^2(t) - \frac{1}{2}\left(8t^2+8t+9\right)x^2(t) = \frac{1}{4}x^2(t); \\ D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\Big(t,x(t)\Big) &= \Xi_{\Pi_{\theta(t)},\rho(t)}\Big(t,x(t)\Big) - \Xi_{\Pi_{\vartheta(t)},\varrho(t)}\Big(t,x(t)\Big) \\ &= \frac{128t^4+256t^3+424t^2+296t+173}{(16t^2+16t+20)(16t^2+16t+23)}x^2(t). \end{aligned}$$

Hence, $\Pi_{\theta(t)}, \Pi_{\vartheta(t)}, \Xi_{\Pi_{\theta(t)},\rho(t)}, \Xi_{\Pi_{\vartheta(t)},\varrho(t)}, D_{\theta(t),\vartheta}$ and $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}$ are gap functions for (RMEP).

4. Global error bounds

In this section, we establish error bounds for (RMEP) based on the gap functions studied in Section 3.

Let the mappings $\Omega, \mathcal{A}, H, K, F, Q, \widetilde{T}, T, \varphi, g$ be defined as in Section 2. To obtain the result of error bounds for (RMEP), let us introduce the following additional conditions:

(A_1) $g(t,\cdot)$ is $\kappa(t)$-strongly nonexpanding on K;
(A_2) $g(t,\cdot)$ is $\mu(t)$-Lipschitz continuous on K;
(A_3) $F(t,\cdot,\cdot)$ is $\alpha(t)$-strongly monotone on $K\times K$;
(A_4) For each $x\in K$, $F(t,x,\cdot)$ is $\beta(t)$-strongly convex uniformly on K;
(A_5) $\nabla_3 F(t,\cdot,\cdot)$ is $(L_1(t), L_2(t))$-mixed Lipschitz continuous;
(A_6) $Q(t,\cdot)$ is $\sigma(t)$-Lipschitz continuous on K;
(A_7) $Q(t,\cdot)$ is $\zeta(t)$-strongly g-monotone with respect to T on K;
(A_8) $T(t,\cdot)$ is $\mathcal{H}$-Lipschitz continuous with measurable function $\lambda(t)$ on K.

Using some necessary conditions above, we show that (RMEP) has a unique solution.

Lemma 4.1. *Let $(\Omega,\mathcal{A})$ be a measurable space. Suppose that the conditions (A_1), (A_3) and (A_7) hold. Then (RMEP) has a unique solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)\in\Sigma(t)$ are such that $x_1(t)\neq x_2(t)$, for all $t\in\Omega$. Then there exist $w_i(t)\in T(t,x_i(t))$, $i=1,2$, such that for any $t\in\Omega$, $y(t)\in K$,

$$F\Big(t,g(t,x_1(t)),y(t)\Big) + \Big\langle Q\Big(t,w_1(t)\Big), y(t)-g(t,x_1(t))\Big\rangle + \varphi\left(y(t)\right) - \varphi\left(g(t,x_1(t))\right) \geq 0, \tag{18}$$

$$F\Big(t,g(t,x_2(t)),y(t)\Big) + \Big\langle Q\Big(t,w_2(t)\Big), y(t)-g(t,x_2(t))\Big\rangle + \varphi\left(y(t)\right) - \varphi\left(g(t,x_2(t))\right) \geq 0. \tag{19}$$

Taking $y(t)=g(t,x_2(t))$ in (18) and $y(t)=g(t,x_1(t))$ in (19), adding the resultants, we have

$$\begin{aligned} F\Big(t,g(t,x_1(t)),g(t,x_2(t))\Big) &+ F\Big(t,g(t,x_2(t)),g(t,x_1(t))\Big) \\ &+ \Big\langle Q\Big(t,w_1(t)\Big)-Q\Big(t,w_2(t)\Big), g(t,x_2(t))-g(t,x_1(t))\Big\rangle \geq 0. \end{aligned} \tag{20}$$

Since $F(t,\cdot,\cdot)$ is $\alpha(t)$-strongly monotone and $g(t,\cdot)$ is $\kappa(t)$-strongly nonexpanding, we have

$$\begin{aligned} F\Big(t,g(t,x_1(t)),g(t,x_2(t))\Big) + F\Big(t,g(t,x_2(t)),g(t,x_1(t))\Big) &\leq -\alpha(t)\|g(t,x_1(t))-g(t,x_2(t))\|^2 \\ &\leq -\alpha(t)\kappa^2(t)\|x_1(t)-x_2(t)\|^2. \end{aligned}$$

Since $Q(t,\cdot)$ is $\zeta(t)$-strongly g-monotone with respect to T, we obtain

$$\Big\langle Q\Big(t,w_1(t)\Big)-Q\Big(t,w_2(t)\Big), g(t,x_2(t))-g(t,x_1(t))\Big\rangle \leq -\zeta(t)\|x_1(t)-x_2(t)\|^2.$$

Hence,

$$
\begin{aligned}
&F\Big(t, g(t, x_1(t)), g(t, x_2(t))\Big) + F\Big(t, g(t, x_2(t)), g(t, x_1(t))\Big) \\
&\quad + \Big\langle Q\Big(t, w_1(t)\Big) - Q\Big(t, w_2(t)\Big), g(t, x_2(t)) - g(t, x_1(t))\Big\rangle \leq -(\zeta(t) + \alpha(t)\kappa^2(t))\|x_1(t) - x_2(t)\|^2. \qquad (21)
\end{aligned}
$$

From (20) and (21), we get that $x_1(t) = x_2(t), \forall t \in \Omega$, the uniqueness of the solution of (RMEP). This completes the proof. □

Lemma 4.2. *Let* $(\Omega, \mathcal{A})$ *be a measurable space. Suppose that* $x^* : \Omega \to K$ *is a solution of* (RMEP) *and the conditions* (A_1)-(A_8) *hold. Then for any* $t \in \Omega$, $x(t) \in K$,

$$
\|x(t) - x^*(t)\| \leq \frac{(L_1(t) + 2L_2(t))\,\mu(t)\vartheta(t) + \sigma(t)\lambda(t)\vartheta(t)(1+\varepsilon) + \mu(t)}{\vartheta(t)\Big(\alpha(t)\kappa(t) + 2\beta(t)\kappa(t) + \zeta(t)\Big)} \Big\|g(t, x(t)) - y_{\vartheta(t)}\Big(x(t)\Big)\Big\|. \qquad (22)
$$

Proof. For each $t \in \Omega$, since $x^*(t) \in K$ is a solution of (RMEP) and $y_{\vartheta(t)}\Big(x(t)\Big) \in K$ for every $x(t) \in K$, we add (1) with $y(t) = y_{\vartheta(t)}\Big(x(t)\Big)$ and (7) with $\theta(t) = \vartheta(t)$, $y(t) = g\,(t, x^*(t))$ and get

$$
\begin{aligned}
0 &\leq \Big\langle \nabla_3 F\,(t, g(t, x^*(t)), g(t, x^*(t))) - \nabla_3 F\Big(t, g(t, x(t)), y_{\vartheta(t)}\,(x(t))\Big), y_{\vartheta(t)}\,(x(t)) - g(t, x^*(t))\Big\rangle \\
&\quad + \Big\langle Q\Big(t, w^*(t)\Big) - Q\Big(t, w(t)\Big), y_{\vartheta(t)}\,(x(t)) - g(t, x^*(t))\Big\rangle \\
&\quad + \frac{1}{\vartheta(t)} \Big\langle y_{\vartheta(t)}\,(x(t)) - g(t, x(t)), g(t, x^*(t)) - y_{\vartheta(t)}\,(x(t))\Big\rangle \\
&= \Big\langle \nabla_3 F\,(t, g(t, x^*(t)), g(t, x^*(t))) - \nabla_3 F\,(t, g(t, x(t)), g\,(t, x(t)))\,, y_{\vartheta(t)}\,(x(t)) - g(t, x^*(t))\Big\rangle \\
&\quad + \Big\langle \nabla_3 F\,(t, g(t, x(t)), g\,(t, x(t))) - \nabla_3 F\Big(t, g(t, x(t)), y_{\vartheta(t)}\,(x(t))\Big), y_{\vartheta(t)}\,(x(t)) - g(t, x^*(t))\Big\rangle \\
&\quad + \Big\langle Q\Big(t, w^*(t)\Big) - Q\Big(t, w(t)\Big), y_{\vartheta(t)}\,(x(t)) - g(t, x(t))\Big\rangle \\
&\quad + \Big\langle Q\Big(t, w^*(t)\Big) - Q\Big(t, w(t)\Big), g\,(t, x(t)) - g(t, x^*(t))\Big\rangle \\
&\quad + \frac{1}{\vartheta(t)} \Big\langle y_{\vartheta(t)}\,(x(t)) - g(t, x(t)), g(t, x^*(t)) - g(t, x(t))\Big\rangle \\
&\quad + \frac{1}{\vartheta(t)} \Big\langle y_{\vartheta(t)}\,(x(t)) - g(t, x(t)), g(t, x(t)) - y_{\vartheta(t)}\,(x(t))\Big\rangle. \qquad (23)
\end{aligned}
$$

It is clear that

$$
\frac{1}{\vartheta(t)} \Big\langle y_{\vartheta(t)}\,(x(t)) - g(t, x(t)), g(t, x(t)) - y_{\vartheta(t)}\,(x(t))\Big\rangle \leq 0. \qquad (24)
$$

Using (A_4), we have

$$
\begin{aligned}
&F\,(t, g(t, x(t)), g(t, x^*(t))) - F\,(t, g(t, x(t)), g(t, x(t))) \\
&\quad \geq \langle \nabla_3 F\,(t, g(t, x(t)), g(t, x(t)))\,, g(t, x^*(t)) - g(t, x(t))\rangle + \beta(t)\|g(t, x(t)) - g(t, x^*(t))\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&F\,(t, g(t, x^*(t)), g(t, x(t))) - F\,(t, g(t, x^*(t)), g(t, x^*(t))) \\
&\quad \geq \langle \nabla_3 F\,(t, g(t, x^*(t)), g(t, x^*(t)))\,, g(t, x(t)) - g(t, x^*(t))\rangle + \beta(t)\|g(t, x(t)) - g(t, x^*(t))\|^2.
\end{aligned}
$$

Note that $F(t, x, x) = 0, \forall x \in K$, adding the above inequalities together, we have

$$
\begin{aligned}
&F\,(t, g(t, x(t)), g(t, x^*(t))) + F\,(t, g(t, x^*(t)), g(t, x(t))) \geq 2\beta(t)\|g(t, x(t)) - g(t, x^*(t))\|^2 \\
&\quad + \langle \nabla_3 F\,(t, g(t, x(t)), g(t, x(t))) - \nabla_3 F\,(t, g(t, x^*(t)), g(t, x^*(t)))\,, g(t, x^*(t)) - g(t, x(t))\rangle. \qquad (25)
\end{aligned}
$$

As $F(t,\cdot,\cdot)$ is $\alpha(t)$-strongly monotone on $K\times K$, we get

$$F\left(t, g(t,x(t)), g(t,x^*(t))\right) + F\left(t, g(t,x^*(t)), g(t,x(t))\right) \leq -\alpha(t)\|g(t,x(t)) - g(t,x^*(t))\|^2.$$

Hence, it follows from (25) that

$$\begin{aligned}
&(\alpha(t){+}2\beta(t))\|g(t,x(t)) - g(t,x^*(t))\|^2 \\
&\quad\leq\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g(t,x(t))\right), g(t,x^*(t)) - g(t,x(t))\rangle \\
&\quad=\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g(t,x(t))\right), g(t,x^*(t)) - y_{\vartheta(t)}\left(x(t)\right)\rangle \\
&\qquad+\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g(t,x(t))\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\rangle
\end{aligned}$$

or

$$\begin{aligned}
&\left\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x^*(t))\right\rangle \\
&\leq\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g(t,x(t))\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\rangle \\
&\quad-(\alpha(t)+2\beta(t))\|g(t,x(t)) - g(t,x^*(t))\|^2.
\end{aligned} \tag{26}$$

Moreover, we have

$$\begin{aligned}
&\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x^*(t))\right\rangle \\
&\quad=\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), g\left(t,x(t)\right) - g(t,x^*(t))\right\rangle \\
&\qquad+\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), y_{\vartheta(t)}\left(x(t)\right) - g\left(t,x(t)\right)\right\rangle
\end{aligned}$$

Since $\nabla_3 F\left(t,x(t),\cdot\right)$ is monotone on K,

$$\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), y_{\vartheta(t)}\left(x(t)\right) - g\left(t,x(t)\right)\right\rangle \leq 0.$$

Hence,

$$\begin{aligned}
&\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x^*(t))\right\rangle \\
&\quad\leq\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), g\left(t,x(t)\right) - g(t,x^*(t))\right\rangle
\end{aligned} \tag{27}$$

Applying (A_7), we have

$$\left\langle Q\left(t,w^*(t)\right) - Q\left(t,w(t)\right), g\left(t,x(t)\right) - g(t,x^*(t))\right\rangle \leq -\zeta(t)\|x^*(t) - x(t)\|^2. \tag{28}$$

It follows from (23) – (28) that

$$\begin{aligned}
&\left(\alpha(t)+2\beta(t)\right)\|g(t,x(t)) - g(t,x^*(t))\|^2 + \zeta(t)\|x^*(t) - x(t)\|^2 \\
&\quad\leq\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g(t,x(t))\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\rangle \\
&\qquad+\left\langle\nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), g\left(t,x(t)\right) - g(t,x^*(t))\right\rangle \\
&\qquad+\left\langle Q\left(t,w^*(t)\right) - Q\left(t,w(t)\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\rangle \\
&\qquad+\frac{1}{\vartheta(t)}\left\langle y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t)), g(t,x^*(t)) - g(t,x(t))\right\rangle.
\end{aligned} \tag{29}$$

By (A_2) and (A_5), we obtain

$$\begin{aligned}
&\langle\nabla_3 F\left(t, g(t,x^*(t)), g(t,x^*(t))\right) - \nabla_3 F\left(t, g(t,x(t)), g(t,x(t))\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\rangle \\
&\leq(L_1(t)+L_2(t))\left\|g(t,x^*(t)) - g(t,x(t))\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t)))\right\| \\
&\leq(L_1(t)+L_2(t))\mu(t)\left\|x^*(t) - x(t)\right\|\left\|y_{\vartheta(t)}(x(t)) - g(t,x(t))\right\|
\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\left\langle \nabla_3 F\left(t, g(t,x(t)), g\left(t,x(t)\right)\right) - \nabla_3 F\left(t, g(t,x(t)), y_{\vartheta(t)}\left(x(t)\right)\right), g\left(t,x(t)\right) - g(t,x^*(t))\right\rangle \\ &\leq L_2(t)\left\|g(t,x^*(t)) - g(t,x(t))\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\| \\ &\leq L_2(t)\mu(t)\left\|x^*(t) - x(t)\right\|\left\|y_{\vartheta(t)}(x(t)) - g(t,x(t))\right\|. \end{aligned} \tag{31}$$

Moreover,

$$\begin{aligned}&\left\langle Q\left(t, w^*(t)\right) - Q\left(t, w(t)\right), y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\rangle \\ &\qquad\leq \left\|Q\left(t,w^*(t)\right) - Q\left(t,w(t)\right)\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\| \\ &\qquad\leq \sigma(t)\left\|w^*(t) - w(t)\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\| \quad \text{(by (A}_6\text{))} \\ &\qquad\leq \sigma(t)(1+\varepsilon)\mathcal{H}\left(T(t,x^*(t)), T(t,x(t))\right)\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\| \quad \text{(by Lemma 2.9)} \\ &\qquad\leq \sigma(t)\lambda(t)(1+\varepsilon)\left\|x^*(t) - x(t)\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\| \quad \text{(by (A}_8\text{))}. \end{aligned} \tag{32}$$

Also, it is clear that from (A_2)

$$\begin{aligned}&\frac{1}{\vartheta(t)}\left\langle y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t)), g(t,x^*(t)) - g(t,x(t))\right\rangle \\ &\qquad\leq \frac{1}{\vartheta(t)}\left\|g(t,x^*(t)) - g(t,x(t))\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\| \\ &\qquad\leq \frac{\mu(t)}{\vartheta(t)}\left\|x^*(t) - x(t)\right\|\left\|y_{\vartheta(t)}(x(t)) - g(t,x(t))\right\|. \end{aligned} \tag{33}$$

From (29) – (33), we have

$$\begin{aligned}&\left(\alpha(t) + 2\beta(t)\right)\|g(t,x(t)) - g(t,x^*(t))\|^2 + \zeta(t)\|x^*(t) - x(t)\|^2 \\ &\quad\leq \left((L_1(t) + 2L_2(t))\,\mu(t) + \sigma(t)\lambda(t)(1+\varepsilon) + \frac{\mu(t)}{\vartheta(t)}\right)\left\|x^*(t) - x(t)\right\|\left\|y_{\vartheta(t)}\left(x(t)\right) - g(t,x(t))\right\|. \end{aligned}$$

By (A_1), we have

$$\kappa(t)\|x(t) - x^*(t)\| \leq \|g(t,x(t)) - g(t,x^*(t))\|.$$

Therefore,

$$\|x(t) - x^*(t)\| \leq \frac{(L_1(t) + 2L_2(t))\,\mu(t)\vartheta(t) + \sigma(t)\lambda(t)\vartheta(t)(1+\varepsilon) + \mu(t)}{\vartheta(t)\left(\alpha(t)\kappa(t) + 2\beta(t)\kappa(t) + \zeta(t)\right)}\left\|g(t,x(t)) - y_{\vartheta(t)}\left(x(t)\right)\right\|$$

i.e., inequality (22) holds. □

From Lemma 4.2, we get the following global error bound for (RMEP) by using the regularized gap function of Fukushima type $\Pi_{\theta(t)}$.

Theorem 4.3. *Suppose that $x^* : \Omega \to K$ is a solution of* (RMEP) *and the conditions* (A_1)-(A_8) *hold. Then for each $t \in \Omega$, we can get the following global error bound by $\Pi_{\theta(t)}\left(t, x(t)\right)$ for* (RMEP)*:*

$$\|x(t) - x^*(t)\| \leq \mathcal{E}_1(t)\sqrt{\Pi_{\theta(t)}\left(t, x(t)\right)} \tag{34}$$

where

$$\mathcal{E}_1(t) = \frac{\sqrt{\theta(t)}\left((L_1(t) + 2L_2(t))\,\mu(t)\theta(t) + \sigma(t)\lambda(t)\theta(t)(1+\varepsilon) + \mu(t)\right)}{\theta(t)\left(\alpha(t)\kappa(t) + 2\beta(t)\kappa(t) + \zeta(t)\right)\sqrt{\beta(t)\theta(t) - 1}} \text{ and } \beta(t)\theta(t) > 1, \tag{35}$$

for all $x(t) \in K$.

Proof. For any $x(t) \in K$, taking $y(t) = g(t, x(t))$ and $w(t) = w_\theta(t)$ in (7), we have

$$\Big\langle \nabla_3 F\Big(t, g(t,x(t)), y_{\theta(t)}\big(x(t)\big)\Big) + Q\big(t, w_\theta(t)\big) + \frac{1}{\theta(t)}\Big(y_{\theta(t)}\big(x(t)\big) - g(t,x(t))\Big), g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\rangle + \varphi\big(g(t,x(t))\big) - \varphi\Big(y_{\theta(t)}\big(x(t)\big)\Big) \geq 0$$

or

$$\Big\langle Q\big(t, w_\theta(t)\big), g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\rangle - \varphi\Big(y_{\theta(t)}\big(x(t)\big)\Big) + \varphi\left(g(t,x(t))\right) - \frac{1}{\theta(t)}\Big\|g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\|^2 \geq \Big\langle \nabla_3 F\Big(t, g(t,x(t)), y_{\theta(t)}\big(x(t)\big)\Big), y_{\theta(t)}\big(x(t)\big) - g(t,x(t))\Big\rangle.$$

This implies

$$\begin{aligned}\Pi_{\theta(t)}(t,x(t)) \geq & - F\Big(t, g(t,x(t)), y_{\theta(t)}\big(x(t)\big)\Big) \\ & + \Big\langle \nabla_3 F\Big(t, g(t,x(t)), y_{\theta(t)}\big(x(t)\big)\Big), y_{\theta(t)}\big(x(t)\big) - g(t,x(t))\Big\rangle \\ & - \frac{1}{\theta(t)}\Big\|g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\|^2.\end{aligned}$$

As $F(t, g(t,x(t)), g(t,x(t))) = 0$, from (A_4), we have

$$\Pi_{\theta(t)}(t,x(t)) \geq \beta(t)\Big\|g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\|^2 - \frac{1}{\theta(t)}\Big\|g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\|^2.$$

Hence,

$$\Big\|g(t,x(t)) - y_{\theta(t)}\big(x(t)\big)\Big\| \leq \sqrt{\frac{\theta(t)}{\beta(t)\theta(t) - 1}}\sqrt{\Pi_{\theta(t)}(t,x(t))}. \tag{36}$$

From taking $\vartheta(t) = \theta(t)$ in (22) and (36), we obtain the inequality (34). □

Without using the Lipschitz continuities of $\nabla_3 F, Q$ and T, we can also derive the global error bound for (RMEP).

Theorem 4.4. *Suppose that $x^* : \Omega \to K$ is a solution of* (RMEP) *and the conditions* (A_1)-(A_3) *and* (A_7) *hold. Then for each $t \in \Omega$, we can get the following global error bound by $\Pi_{\theta(t)}\big(t, x(t)\big)$ for* (RMEP)*:*

$$\|x(t) - x^*(t)\| \leq \mathcal{E}_2(t)\sqrt{\Pi_{\theta(t)}\big(t, x(t)\big)} \tag{37}$$

where

$$\mathcal{E}_2(t) = \frac{1}{\sqrt{\alpha(t)\kappa^2(t) + \zeta(t) - \dfrac{\mu^2(t)}{2\theta(t)}}} \text{ and } \alpha(t)\kappa^2(t) + \zeta(t) > \frac{\mu^2(t)}{2\theta(t)}, \tag{38}$$

for all $x(t) \in K$.

Proof. For each $t \in \Omega$, fix an arbitrary $x(t) \in K$. Since $x^*(t) \in \Sigma(t)$, $x^*(t) \in K$. From (5), for $w(t) \in T(t, x(t))$ we have

$$\begin{aligned}\Pi_{\theta(t)}\big(t, x(t)\big) \geq & \,\Phi_{\theta(t)}\big(t, x(t), g(t, x^*(t)), w(t)\big) \\ & = -F\big(t, g(t,x(t)), g(t, x^*(t))\big) + \Big\langle Q\big(t, w(t)\big), g(t,x(t)) - g(t,x^*(t))\Big\rangle \\ & - \varphi\big(g(t, x^*(t))\big) + \varphi\big(g(t,x(t))\big) - \frac{1}{2\theta(t)}\Big\|g(t,x(t)) - g(t,x^*(t))\Big\|^2.\end{aligned}$$

Moreover, since $x^*(t) \in \Sigma(t)$, there exists $w^*(t) \in T(t, x^*(t))$ such that

$$F\Big(t, g(t, x^*(t)), g(t, x(t))\Big) + \Big\langle Q\Big(t, w^*(t)\Big), g(t, x(t)) - g(t, x^*(t))\Big\rangle + \varphi\Big(g(t, x(t))\Big) - \varphi\Big(g(t, x^*(t))\Big) \geq 0.$$

Thus

$$\begin{aligned} F\Big(t, g(t, x(t)), g(t, x^*(t))\Big) &+ F\Big(t, g(t, x^*(t)), g(t, x(t))\Big) \\ &+ \Big\langle Q\Big(t, w^*(t)\Big) - Q\Big(t, w(t)\Big), g(t, x(t)) - g(t, x^*(t))\Big\rangle \\ &+ \frac{1}{2\theta(t)}\Big\|g(t, x(t)) - g(t, x^*(t))\Big\|^2 \geq -\Pi_{\theta(t)}\Big(t, x(t)\Big). \end{aligned}$$

Using (A_3) and (A_7), we have

$$-\alpha(t)\Big\|g(t, x(t)) - g(t, x^*(t))\Big\|^2 - \zeta(t)\Big\|x(t) - x^*(t)\Big\|^2 + \frac{1}{2\theta(t)}\Big\|g(t, x(t)) - g(t, x^*(t))\Big\|^2 \geq -\Pi_{\theta(t)}\Big(t, x(t)\Big).$$

Moreover, since $g(t, \cdot)$ is $\kappa(t)$-strongly nonexpanding and $\mu(t)$-Lipschitz continuous, i.e.

$$\kappa(t)\|x(t) - x^*(t)\| \leq \|g(t, x(t)) - g(t, x^*(t))\| \leq \mu(t)\|x(t) - x^*(t)\|.$$

Hence, we get

$$\left(\alpha(t)\kappa^2(t) + \zeta(t) - \frac{\mu^2(t)}{2\theta(t)}\right)\Big\|x(t) - x^*(t)\Big\|^2 \leq \Pi_{\theta(t)}\Big(t, x(t)\Big).$$

Therefore, we obtain inequality (37). □

Next, we give two global error bounds for (RMEP) by using the regularized gap function of the Moreau-Yosida type $\Xi_{\Pi_{\theta(t)},\rho(t)}$ and the results of Theorems 4.3 and 4.4.

Theorem 4.5. *Assume that $x^* : \Omega \to K$ is a solution of* (RMEP) *and all the conditions of Theorem 4.3 hold. Then for any measurable function $\rho : \Omega \to (0, +\infty)$, we have*

$$\|x(t) - x^*(t)\Big\| \leq \frac{\sqrt{2}}{\sqrt{\min\left\{\dfrac{1}{\mathcal{E}_1^2(t)}, \rho(t)\kappa^2(t)\right\}}}\sqrt{\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t))}, \quad \forall t \in \Omega, x(t) \in K. \tag{39}$$

Proof. It follows from Theorem 4.3 that

$$\begin{aligned} \Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t)) &= \min_{z(t)\in K}\left\{\Pi_{\theta(t)}(t, z(t)) + \rho(t)\Big\|g(t, x(t)) - g(t, z(t))\Big\|^2\right\} \\ &\geq \min_{z(t)\in K}\left\{\frac{1}{\mathcal{E}_1^2(t)}\|x^*(t) - z(t)\Big\|^2 + \rho(t)\kappa^2(t)\Big\|x(t) - z(t)\Big\|^2\right\} \\ &\geq \min\left\{\frac{1}{\mathcal{E}_1^2(t)}, \rho(t)\kappa^2(t)\right\}\min_{z(t)\in K}\left\{\|x^*(t) - z(t)\|^2 + \|x(t) - z(t)\|^2\right\} \\ &\geq \frac{1}{2}\min\left\{\frac{1}{\mathcal{E}_1^2(t)}, \rho(t)\kappa^2(t)\right\}\|x(t) - x^*(t)\Big\|^2, \end{aligned}$$

where the following inequality is used:

$$\|x^* - z\|^2 + \|x - z\|^2 \geq \frac{(\|x^* - z\| + \|x - z\|)^2}{2} \geq \frac{\|x - x^*\|^2}{2}, \forall x, x^*, z \in H.$$

Therefore,

$$\|x(t) - x^*(t)\|^2 \leq \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{1}{\mathcal{E}_1^2(t)}, \rho(t)\kappa^2(t)\right\}}} \sqrt{\Xi_{\Pi_{\theta(t),\rho(t)}}(t, x(t))},$$

which implies that the proof is completed. □

Theorem 4.6. *Assume that $x^* : \Omega \to K$ is a solution of* (RMEP) *and all the conditions of Theorem 4.4 hold. Then for any measurable function $\rho : \Omega \to (0, +\infty)$, we have*

$$\|x(t) - x^*(t)\| \leq \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{1}{\mathcal{E}_2^2(t)}, \rho(t)\kappa^2(t)\right\}}} \sqrt{\Xi_{\Pi_{\theta(t),\rho(t)}}(t, x(t))}, \quad \forall t \in \Omega, x(t) \in K. \tag{40}$$

Proof. From the result of Theorem 4.4, by the same proof as in Theorem 4.5, we obtain Theorem 4.6. □

Finaly, we get the global error bounds for (RMEP) by using the D-gap functions in Section 3.

Theorem 4.7. *Suppose that $x^* : \Omega \to K$ is a solution of* (RMEP) *and the conditions* (A_1)-(A_8) *hold. Then for $\theta(t) > \vartheta(t)$, we can get the following global error bound by $D_{\theta(t),\vartheta(t)}\big(t, x(t)\big)$ for the* (RMEP)*:*

$$\|x(t) - x^*(t)\| \leq \mathcal{E}_3(t) \sqrt{D_{\theta(t),\vartheta(t)}\big(t, x(t)\big)} \tag{41}$$

where

$$\mathcal{E}_3(t) = \frac{\sqrt{2\theta(t)\vartheta(t)}\left((L_1(t) + 2L_2(t))\,\mu(t)\vartheta(t) + \sigma(t)\lambda(t)\vartheta(t)(1+\varepsilon) + \mu(t)\right)}{\vartheta(t)\big(\alpha(t)\kappa(t) + 2\beta(t)\kappa(t) + \zeta(t)\big)\sqrt{\theta(t) - \vartheta(t)}} \tag{42}$$

for all $t \in \Omega$, $x(t) \in K$.

Proof. From (16) and (22), we obtain inequality (41). □

Theorem 4.8. *Suppose that $x^* : \Omega \to K$ is a solution of* (RMEP) *and the conditions* (A_1)-(A_8) *hold. Then for $\theta(t) > \vartheta(t)$ and $\rho(t) > \varrho(t)$, we can get the following global error bound by $D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t, x(t)\big)$ for the* (RMEP)*:*

$$\|x(t) - x^*(t)\| \leq \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{1}{\mathcal{E}_3^2(t)}, (\rho(t) - \varrho(t))\kappa^2(t)\right\}}} \sqrt{D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t, x(t)\big)}, \quad \forall t \in \Omega, x(t) \in K. \tag{43}$$

Proof. We rewrite (17) as follows

$$\begin{aligned} D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t, x(t)\big) &\geq D_{\theta(t),\vartheta(t)}\big(t, z_{\theta(t),\rho(t)}(x(t))\big) \\ &\quad + (\rho(t) - \varrho(t))\kappa^2(t)\big\|x(t) - z_{\theta(t),\rho(t)}(x(t))\big\|^2. \end{aligned}$$

From the error bound of the D-gap function $D_{\theta(t),\vartheta(t)}$ in (41), we obtain

$$D_{\theta(t),\vartheta(t)}\big(t, z_{\theta(t),\rho(t)}(x(t))\big) \geq \frac{1}{\mathcal{E}_3^2(t)}\big\|z_{\theta(t),\rho(t)}(x(t)) - x^*(t)\big\|^2.$$

Hence,

$$
\begin{aligned}
D^{*}_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\Big(t,x(t)\Big) &\geq \frac{1}{\mathcal{E}_3^2(t)}\Big\|z_{\theta(t),\rho(t)}\left(x(t)\right)-x^{*}(t)\Big\|^2 \\
&\quad + (\rho(t)-\varrho(t))\kappa^2(t)\Big\|x(t)-z_{\theta(t),\rho(t)}\left(x(t)\right)\Big\|^2 \\
&\geq \min\left\{\frac{1}{\mathcal{E}_3^2(t)}, (\rho(t)-\varrho(t))\kappa^2(t)\right\} \\
&\quad \times\left(\Big\|z_{\theta(t),\rho(t)}\left(x(t)\right)-x^{*}(t)\Big\|^2+\Big\|x(t)-z_{\theta(t),\rho(t)}\left(x(t)\right)\Big\|^2\right) \\
&\geq \frac{1}{2}\min\left\{\frac{1}{\mathcal{E}_3^2(t)}, (\rho(t)-\varrho(t))\kappa^2(t)\right\}\Big\|x(t)-x^{*}(t)\Big\|^2.
\end{aligned}
$$

Therefore,

$$
\|x(t)-x^{*}(t)\| \leq \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{1}{\mathcal{E}_3^2(t)}, (\rho(t)-\varrho(t))\kappa^2(t)\right\}}}\sqrt{D^{*}_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\Big(t,x(t)\Big)}.
$$

The proof is completed. □

Remark 4.9. (a) Error bounds in Theorems 4.3 and 4.4 are established by using the regularized gap functions of Fukushima type in Theorem 3.5. Hence, Theorem 4.3 extends Lemma 4.1 in [35] (see Remark 3.6).

(b) From Remark 3.12, the regularized gap functions of Moreau-Yosida type, D-gap functions for mixed equilibrium problems in random environment are new. On the other hand, error bounds in Theorems 4.5, 4.6, 4.7 and 4.8 are established by using the regularized gap functions of Moreau-Yosida type, D-gap functions. Thus, Theorems 4.5, 4.6, 4.7 and 4.8 are new.

The following example shows a case where all assumptions imposed in Theorems 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8 are satisfied.

Example 4.10. Let $\Omega, \mathcal{A}, H, K, F, Q, T, \varphi, g$ and the gap functions $\Pi_{\theta(t)}, \Pi_{\vartheta(t)}, \Xi_{\Pi_{\theta(t)},\rho(t)}, \Xi_{\Pi_{\vartheta(t)},\varrho(t)}$, $D_{\theta(t),\vartheta}$ and $D^{*}_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}$ be defined as in Example 3.13. By Example 3.13, $\Sigma(t)=\{0\}$.

It is easy to verify that the conditions (A_1), (A_2), (A_6) - (A_8) hold with $\kappa(t)=\mu(t)=1$, $\sigma(t)=2t$, $\zeta(t)=4t$ and $\lambda(t)=2$. Now, we check the conditions (A_3) - (A_5).

• $\forall x(t), y(t) \in [0,\frac{1}{2}]$, we have

$$
\begin{aligned}
F\left(t,x(t),y(t)\right)+F\left(t,y(t),x(t)\right) &= (t^2+1)(y^2(t)+3x(t)y(t)-4x^2(t))+(t^2+1)(x^2(t)+3x(t)y(t)-4y^2(t)) \\
&= -(3t^2+3)(x(t)-y(t))^2.
\end{aligned}
$$

Hence, $F(t,\cdot,\cdot)$ is $\alpha(t)$-strongly monotone with $\alpha(t)=3t^2+3$.

• $\forall x(t), y_1(t), y_2(t) \in [0,\frac{1}{2}]$, we have

$$
\begin{aligned}
&F\Big(t,x(t),y_1(t)\Big)-F(t,x(t),y_2(t)\Big)-\langle\nabla_3 F\Big(t,x(t),y_2(t)\Big),y_1(t)-y_2(t)\rangle \\
&\quad = (t^2+1)(y_1^2(t)+3x(t)y_1(t)-4x^2(t))-(t^2+1)(y_2^2(t)+3x(t)y_2(t)-4x^2(t)) \\
&\qquad -\langle(t^2+1)(2y_2(t)+3x(t)),y_1(t)-y_2(t)\rangle \\
&\quad = (t^2+1)(y_1(t)-y_2(t))^2.
\end{aligned}
$$

Hence, $F(t,x(t),\cdot)$ is $\beta(t)$-strongly convex uniformly in the second argument with $\beta(t)=t^2+1$.

- $\forall x_1(t), x_2(t), y_1(t), y_2(t) \in [0, \frac{1}{2}]$, we have

$$\|\nabla_3 F\Big(t, x_1(t), y_1(t)) - F(t, x_1(t), y_2(t)\Big)\| \leq 3(t^2+1)\|x_1(t) - x_2(t)\| + 2(t^2+1)\|y_1(t) - y_2(t)\|.$$

Hence, $\nabla_3 F(t, \cdot, \cdot)$ is $(L_1(t), L_2(t))$-mixed Lipschitz continuous with $L_1(t) = 3t^2+3$ and $L_2(t) = 2t^2+2$. Therefore, the conditions (A_3) - (A_5) hold. Thus, all the conditions (A_1) - (A_8) hold, and so the inequalities (34), (37), (39), (40), (41) and (43) hold.

For example, let $\theta(t) = 2, \vartheta(t) = 1, \rho(t) = 1, \varrho(t) = \frac{1}{2}, \epsilon \to 0$ for all $t \in [0, 1]$. Then with $\mathcal{E}_1(t), \mathcal{E}_2(t)$ and $\mathcal{E}_3(t)$ defined by (35), (38) and (42), respectively, we have

$$\mathcal{E}_1(t) = \frac{14t^2+8t+15}{\Big(5t^2+4t+5\Big)\sqrt{4t^2+2}};$$
$$\mathcal{E}_2(t) = \frac{2}{\sqrt{12t^2+16t+11}};$$
$$\mathcal{E}_3(t) = \frac{14t^2+8t+16}{5t^2+4t+5}.$$

Then, for any $t \in [0, 1], x(t) \in [0, \frac{1}{2}]$, we get $\|x(t) - 0\| = |x(t) - 0| = x(t)$ and

- $$\mathcal{E}_1(t)\sqrt{\Pi_{\theta(t)}\Big(t, x(t)\Big)} = \frac{14t^2+8t+15}{(5t^2+4t^2+5)\sqrt{4t^2+2}}\sqrt{\frac{1}{4}(16t^2+16t+19)\,x^2(t)} \geq \frac{37\sqrt{34}}{56}x(t) \approx 3.85x(t) \geq x(t).$$
- $$\mathcal{E}_2(t)\sqrt{\Pi_{\theta(t)}\Big(t, x(t)\Big)} = \frac{2}{\sqrt{12t^2+16t+11}}\sqrt{\frac{1}{4}(16t^2+16t+19)\,x^2(t)} \geq \sqrt{\frac{17}{13}}x(t) \approx 1.14x(t) \geq x(t).$$
- $$\frac{\sqrt{2}}{\sqrt{\min\left\{\frac{1}{\mathcal{E}_1^2(t)}, \rho(t)\kappa^2(t)\right\}}}\sqrt{\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t))}$$
$$= \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{(5t^2+4t+5)^2(4t^2+2)}{(14t^2+8t+15)^2}, 1\right\}}}\sqrt{\frac{16t^2+16t+19}{16t^2+16t+23}x^2(t)}$$
$$\geq \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{10}{3}, 1\right\}}}\sqrt{\frac{19}{23}}x(t) \approx 1.29x(t) \geq x(t)$$
- $$\frac{\sqrt{2}}{\sqrt{\min\left\{\frac{1}{\mathcal{E}_2^2(t)}, \rho(t)\kappa^2(t)\right\}}}\sqrt{\Xi_{\Pi_{\theta(t)},\rho(t)}(t, x(t))}$$
$$\geq \frac{\sqrt{2}}{\sqrt{\min\left\{\frac{11}{4}, 1\right\}}}\sqrt{\frac{19}{23}}x(t) \approx 1.29x(t) \geq x(t).$$

- $\mathcal{E}_3(t)\sqrt{D_{\theta(t),\vartheta(t)}\big(t,x(t)\big)} = \dfrac{14t^2+8t+16}{5t^2+4t+5}\sqrt{\dfrac{1}{4}x^2(t)} \geq \dfrac{19}{14}x(t) \geq x(t).$

- $$\frac{\sqrt{2}}{\sqrt{\min\left\{\dfrac{1}{\mathcal{E}_3^2(t)},(\rho(t)-\varrho(t))\kappa^2(t)\right\}}}\sqrt{D^*_{\theta(t),\rho(t),\vartheta(t),\varrho(t)}\big(t,x(t)\big)}$$
$$\geq \frac{\sqrt{2}(14t^2+8t+16)}{5t^2+4t+5}\sqrt{\frac{128t^4+256t^3+424t^2+296t+173}{(16t^2+16t+20)(16t^2+16t+23)}}x(t)$$
$$\geq \frac{19}{7}\sqrt{\frac{173}{230}} \approx 2.35x(t) \geq x(t).$$

Thus, the inequalities (34), (37), (39), (40), (41) and (43) hold.

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