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On Riemann-Stieltjes Integral Boundary Value Problems of Caputo-Riemann-Liouville Type Fractional Integro-Differential Equations

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Abstract. Under different criteria, we prove the existence and uniqueness of solutions for a Riemann-Stieltjes integro-multipoint boundary value problem of Caputo-Riemann-Liouville type fractional integro-differential equations. Our results rely on the modern methods of functional analysis and are well-illustrated with the help of examples. Some interesting observations are also presented.

1. Introduction

Fractional order differential and integral operators play a significant role in the mathematical modeling of scientific and engineering phenomena with memory. An important feature of these operators is their nonlocal nature that helps to trace the hereditary characteristics of the related processes and materials under investigation. Thus fractional-order models provide more insight into the study of real-world problems. For application details, see [1]-[4] and the references cited therein. For the theoretical advancement of fractional calculus, we refer the reader to the books [5]-[8].

Fractional order boundary value problems involving classical, nonlocal, multipoint, and integral boundary conditions also received overwhelming attention, for instance, see [9]-[19] and the references cited therein. The role of integral boundary conditions is of significant importance in blood flow problems as these conditions provide a flexible mechanism to deal with the changing geometry of the blood vessels [20]. Also, integral boundary conditions are helpful in regularization of ill-posed problems [21].

In this paper, we study a new class of Riemann-Stieltjes integro-multipoint boundary value problems of Caputo-Riemann-Liouville type fractional integro-differential equations given by

$$ {}^cD^{p_2}\Big[({}^cD^{p_1}+\kappa)x(t)+\mu I^{\alpha}h(t,x(t))\Big]=g(t,x(t)),\ 1<p_1,p_2\le 2,\ t\in(a_1,a_2), \tag{1}$$

$$x(a_1)=\sum_{i=1}^{r-2}\lambda_i x(\omega_i)+\int_{a_1}^{a_2}x(s)dB(s),\ x'(a_1)=0,\ x(a_2)=0,\ x'(a_2)=0, \tag{2}$$

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where ${}^cD^w$ denotes the Caputo fractional differential operator of order w ($w = p_1, p_2$), $\kappa, \mu \in \mathbb{R}$, $\alpha > 0$, $h, g : [a_1, a_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ are a given continuous functions, B is a function of bounded variation, $a_1 < \omega_1 < \omega_2 < \cdots < \omega_{r-2} < a_2$, and $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots r-2$.

Here we remark that the existence and stability results for a fractional order differential equation of the form ${}^cD^q x(t) = f(t, x(t))$, $3 < q \leq 4$, subject to the boundary conditions (2) were obtained in [17].

The rest of the paper is arranged as follows. In section 2, we recall some basic concepts of fractional calculus and prove a new result related to a linear variant of the problem (1)-(2). In Section 3, we derive the existence results for the nonlinear boundary value problem (1)-(2) under different criteria by applying Leray-Schauder nonlinear alternative, Leray-Schauder degree theory, and Shaefer like fixed point theorem. A uniqueness result for the given problem, obtained by means of Banach contraction mapping principle, is presented in Section 4. Examples are constructed for illustrating the obtained results.

2. Basic result

Let us begin with some basic definitions of fractional calculus [5].

Definition 2.1. *The Riemann-Liouville fractional integral of order δ for a locally integrable real-valued function $q : [a, \infty) \longrightarrow \mathbb{R}$ is defined as*

$$I^\delta q(t) = \frac{1}{\Gamma(\delta)} \int_a^t \frac{q(y)}{(t-y)^{1-\delta}} dy, \ \delta > 0,$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *For an $(n-1)$-times absolutely continuous function $q : [a, \infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order $\delta \in (n-1, n]$ is defined as*

$${}^cD^\delta q(t) = \frac{1}{\Gamma(n-\delta)} \int_a^t (t-y)^{n-\delta-1} q^{(n)}(y) dy.$$

Lemma 2.3. *[5] For $n-1 < \delta \leq n$, the general solution of the fractional differential equation: ${}^cD^\delta y(\xi) = 0$, $\xi \in [a, b]$, is*

$$y(\xi) = e_0 + e_1(\xi - a) + e_2(\xi - a)^2 + ... + e_{n-1}(\xi - a)^{n-1},$$

where $e_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$. Furthermore,

$$I^\delta \, {}^cD^\delta y(\xi) = y(\xi) + \sum_{i=0}^{n-1} e_i (\xi - a)^i.$$

In the following lemma, we solve a linear variant of the problem (1)-(2).

Lemma 2.4. *For $\sigma, \rho \in C([a_1, a_2])$, the linear problem consisting of the fractional integro-differential equation*

$${}^cD^{p_2}\Big[({}^cD^{p_1} + \kappa)x(t) + \mu I^\alpha \sigma(t)\Big] = \rho(t), \ 1 < p_1, p_2 \leq 2, \ t \in (a_1, a_2), \tag{3}$$

and the boundary conditions (2) is equivalent to the integral equation

$$\begin{aligned}
x(t) &= -\kappa \int_{a_1}^{t} \frac{(t-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \mu \int_{a_1}^{t} \frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(s)ds \\
&+ \int_{a_1}^{t} \frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(s)ds + \chi_1(t)\Big[\kappa \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\
&+ \mu \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(s)ds - \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(s)ds\Big] \\
&+ \chi_2(t)\Big[\kappa \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)} x(s)ds + \mu \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)} \sigma(s)ds \\
&- \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)} \rho(s)ds\Big] + \chi_3(t)\Big[-\kappa \sum_{i=1}^{r-2} \lambda_i \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\
&- \mu \sum_{i=1}^{r-2} \lambda_i \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(s)ds + \sum_{i=1}^{r-2} \lambda_i \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(s)ds \\
&+ \int_{a_1}^{a_2} \Big(-\kappa \int_{a_1}^{s} \frac{(s-u)^{p_1-1}}{\Gamma(p_1)} x(u)du - \mu \int_{a_1}^{s} \frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(u)du \\
&+ \int_{a_1}^{s} \frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(u)du\Big) dB(s)\Big], \qquad (4)
\end{aligned}$$

where

$$\begin{cases}
\chi_1(t) = \beta_1 \frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)} + \beta_4 \frac{(t-a_1)^{p_1+1}}{\Gamma(p_1+2)} + \beta_7, \ \chi_2(t) = \beta_2 \frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)} + \beta_5 \frac{(t-a_1)^{p_1+1}}{\Gamma(p_1+2)} + \beta_8, \\
\chi_3(t) = \beta_3 \frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)} + \beta_6 \frac{(t-a_1)^{p_1+1}}{\Gamma(p_1+2)} + \beta_9,
\end{cases} \qquad (5)$$

$$\begin{cases}
\beta_1 = -\frac{E_1E_6}{\gamma_1}, \ \beta_2 = \frac{E_1^2E_6 - E_1E_4 + \gamma_1}{E_3\gamma_1}, \ \beta_3 = \frac{E_1}{\gamma_1}, \ \beta_4 = \frac{E_3E_6}{\gamma_1}, \ \beta_5 = \frac{E_4 - E_1E_6}{\gamma_1}, \ \beta_6 = -\frac{E_3}{\gamma_1}, \\
\beta_7 = 1 + \frac{E_1^2E_6 - E_2E_3E_6}{\gamma_1}, \ \beta_8 = \frac{E_1^2E_4 - E_1^3E_6 - E_1\gamma_1}{E_3\gamma_1} + \frac{E_1E_2E_6 - E_2E_4}{\gamma_1}, \ \beta_9 = \frac{-E_1^2 + E_2E_3}{\gamma_1},
\end{cases} \qquad (6)$$

$$\begin{cases}
E_1 = \frac{(a_2-a_1)^{p_1}}{\Gamma(p_1+1)}, \ E_2 = \frac{(a_2-a_1)^{p_1+1}}{\Gamma(p_1+2)}, \ E_3 = \frac{(a_2-a_1)^{p_1-1}}{\Gamma(p_1)}, \ E_4 = -\sum_{i=1}^{r-2} \lambda_i \frac{(\omega_i-a_1)^{p_1}}{\Gamma(p_1+1)} - \int_{a_1}^{a_2} \frac{(s-a_1)^{p_1}}{\Gamma(p_1+1)} dB(s), \\
E_5 = -\sum_{i=1}^{r-2} \lambda_i \frac{(\omega_i-a_1)^{p_1+1}}{\Gamma(p_1+2)} - \int_{a_1}^{a_2} \frac{(s-a_1)^{p_1+1}}{\Gamma(p_1+2)} dB(s), \ E_6 = 1 - \sum_{i=1}^{r-2} \lambda_i - \int_{a_1}^{a_2} dB(s),
\end{cases} \qquad (7)$$

and it is assumed that

$$\gamma_1 = E_2E_3E_6 - E_3E_5 - E_1^2E_6 + E_1E_4 \neq 0. \qquad (8)$$

Proof. Applying the integral operator I^{p_2} to both sides of (3) and using Lemma 2.3, we get

$$({}^cD^{p_1} + \kappa)x(t) + \mu I^{\alpha}\sigma(t) = I^{p_2}\rho(t) + c_1 + c_2(t-a_1), \qquad (9)$$

where c_1 and c_2 are unknown arbitrary constants. Now operating the integral operator I^{p_1} to (9), we obtain

$$x(t) = -\kappa I^{p_1}x(t) - \mu I^{\alpha+p_1}\sigma(t) + I^{p_1+p_2}\rho(t) + I^{p_1}c_1 + I^{p_1}c_2(t-a_1) + c_3 + c_4(t-a_1). \qquad (10)$$

From (10), we have

$$\begin{aligned} x(t) &= -\kappa \int_{a_1}^{t} \frac{(t-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \mu \int_{a_1}^{t} \frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(s)ds + \int_{a_1}^{t} \frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(s)ds \\ &+ c_1 \frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)} + c_2 \frac{(t-a_1)^{p_1+1}}{\Gamma(p_1+2)} + c_3 + c_4(t-a_1), \end{aligned} \tag{11}$$

$$\begin{aligned} x'(t) &= -\kappa \int_{a_1}^{t} \frac{(t-s)^{p_1-2}}{\Gamma(p_1-1)} x(s)ds - \mu \int_{a_1}^{t} \frac{(t-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)} \sigma(s)ds \\ &+ \int_{a_1}^{t} \frac{(t-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)} \rho(s)ds + c_1 \frac{(t-a_1)^{p_1-1}}{\Gamma(p_1)} + c_2 \frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)} + c_4. \end{aligned} \tag{12}$$

Using the boundary conditions (2) in (11) and (12) together with the notations (7), we obtain $c_4 = 0$ and a system of algebraic equations in c_1, c_2 and c_3 given by

$$E_1 c_1 + E_2 c_2 + c_3 = I_1, \tag{13}$$

$$E_3 c_1 + E_1 c_2 = I_2, \tag{14}$$

$$E_4 c_1 + E_5 c_2 + E_6 c_3 = I_3, \tag{15}$$

where I_i $(i = 1, 2, 3)$ are given by

$$\begin{aligned} I_1 &= \kappa \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds + \mu \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(s)ds - \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(s)ds, \\ I_2 &= \kappa \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)} x(s)ds + \mu \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)} \sigma(s)ds - \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)} \rho(s)ds, \\ I_3 &= -\kappa \sum_{i=1}^{r-2} \lambda_i \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \mu \sum_{i=1}^{r-2} \lambda_i \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(s)ds \\ &+ \sum_{i=1}^{r-2} \lambda_i \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(s)ds + \int_{a_1}^{a_2} \Big(-\kappa \int_{a_1}^{s} \frac{(s-u)^{p_1-1}}{\Gamma(p_1)} x(u)du \\ &- \mu \int_{a_1}^{s} \frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \sigma(u)du + \int_{a_1}^{s} \frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \rho(u)du \Big) dB(s). \end{aligned} \tag{16}$$

Eliminating c_3 from (13) and (15), we get

$$(E_1 E_6 - E_4) c_1 + (E_2 E_6 - E_5) c_2 = E_6 I_1 - I_3. \tag{17}$$

Solving (14) and (17) for c_1 and c_2, we find that

$$c_1 = -\frac{E_1 E_6}{\gamma_1} I_1 + \frac{E_1^2 E_6 - E_1 E_4 + \gamma_1}{E_3 \gamma_1} I_2 + \frac{E_1}{\gamma_1} I_3, \ \gamma_1 \neq 0, \tag{18}$$

$$c_2 = \frac{E_3 E_6}{\gamma_1} I_1 + \frac{E_4 - E_1 E_6}{\gamma_1} I_2 - \frac{E_3}{\gamma_1} I_3, \ \gamma_1 \neq 0, \tag{19}$$

where γ_1 is defined by (8). Using (18) and (19) in (13), we get

$$c_3 = \Big[1+\frac{E_1^2E_6-E_2E_3E_6}{\gamma_1}\Big]I_1+\Big[\frac{E_1^2E_4-E_1^3E_6-E_1\gamma_1}{E_3\gamma_1}+\frac{E_1E_2E_6-E_2E_4}{\gamma_1}\Big]I_2+\Big[\frac{-E_1^2+E_2E_3}{\gamma_1}\Big]I_3.$$

Inserting the values of c_1, c_2, c_3 and c_4 in (11) together with notations (5), we obtain the solution (4). The converse of the lemma follows by direct computation. □

3. Existence results

Let $\mathcal{B}$ denote the Banach space of all continuous functions from $[a_1,a_2]\longrightarrow\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup\{|x(t)|,t\in[a_1,a_2]\}$. In view of Lemma 2.4, we transform the problem (1)-(2) into an equivalent fixed point problem as

$$x=\mathcal{G}x, \tag{20}$$

where $\mathcal{G}:\mathcal{B}\longrightarrow\mathcal{B}$ is defined by

$$\begin{aligned}
(\mathcal{G}x)(t) &= -\kappa\int_{a_1}^{t}\frac{(t-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds-\mu\int_{a_1}^{t}\frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}h(s,x(s))ds\\
&+\int_{a_1}^{t}\frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}g(s,x(s))ds+\chi_1(t)\Big[\kappa\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds\\
&+\mu\int_{a_1}^{a_2}\frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}h(s,x(s))ds-\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}g(s,x(s))ds\Big]\\
&+\chi_2(t)\Big[\kappa\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)}x(s)ds+\mu\int_{a_1}^{a_2}\frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)}h(s,x(s))ds\\
&-\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)}g(s,x(s))ds\Big]+\chi_3(t)\Big[-\kappa\sum_{i=1}^{r-2}\lambda_i\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds\\
&-\mu\sum_{i=1}^{r-2}\lambda_i\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}h(s,x(s))ds\\
&+\sum_{i=1}^{r-2}\lambda_i\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1+p_1-1}}{\Gamma(p_1+p_1)}g(s,x(s))ds+\int_{a_1}^{a_2}\Big(-\kappa\int_{a_1}^{s}\frac{(s-u)^{p_1-1}}{\Gamma(p_1)}x(u)du\\
&-\mu\int_{a_1}^{s}\frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}h(u,x(u))du+\int_{a_1}^{s}\frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}g(u,x(u)))du\Big)dB(s)\Big].
\end{aligned}\tag{21}$$

Observe that the problem (1)-(2) has solutions if the operator $\mathcal{G}$ has fixed points. For computational convenience, we set

$$A_0 = \sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{p_1}}{\Gamma(p_1+1)}+|\chi_2(t)|\frac{(a_2-a_1)^{p_1-1}}{\Gamma(p_1)}$$

$$
\begin{aligned}
&+ \quad |\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{p_1}}{\Gamma(p_1+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{p_1}}{\Gamma(p_1+1)}dB(s)\Big)\Big\},\\
A_1 \quad &= \sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}+|\chi_2(t)|\frac{(a_2-a_1)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}\\
&+ \quad |\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}dB(s)\Big)\Big\},\\
A_2 \quad &= \sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+|\chi_2(t)|\frac{(a_2-a_1)^{p_1+p_1-1}}{\Gamma(p_1+p_1)}\\
&+ \quad |\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}dB(s)\Big)\Big\}. \qquad (22)
\end{aligned}
$$

Our first existence result for the problem (1)-(2) is based on Leray-Schauder alternative criterion [22].

Theorem 3.1. *Let $h, g : [a_1, a_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions and the following conditions hold:*

(H_1) *there exist functions $\eta_1, \eta_2 \in C([a_1, a_2] \times \mathbb{R}^+)$, and nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $|h(t, x)| \leq \eta_1(t)\psi_1(\|x\|)$, $|g(t, x)| \leq \eta_2(t)\psi_2(\|x\|)$, $\forall\ (t, x) \in [a_1, a_2] \times \mathbb{R}$;*

(H_2) *there exists a positive constant M such that*

$$M\Big(|\kappa|MA_0+|\mu|\|\eta_1\|\psi_1(M)A_1+\|\eta_2\|\psi_2(M)A_2\Big)^{-1}>1,$$

where A_i $(i = 0, 1, , 2)$ are defined by (22).

Then the problem (1)-(2) has at least one solution on $[a_1, a_2]$.

Proof. We complete the proof in several steps. Firstly, we show that the operator $\mathcal{G} : \mathcal{B} \longrightarrow \mathcal{B}$ defined by (21) is bounded. For $b > 0$, let $S_b = \{x \in \mathcal{B} : \|x\| \leq b\}$ be a bounded set in $\mathcal{B}$. Then, for $x \in S_b$, we have

$$
\begin{aligned}
\|\mathcal{G}x\| \quad &= \sup_{t\in[a_1,a_2]}|(\mathcal{G}x)(t)|\\
&\leq \sup_{t\in[a_1,a_2]}\Big\{|\kappa|\int_{a_1}^{t}\frac{(t-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds+|\mu|\int_{a_1}^{t}\frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds\\
&+ \int_{a_1}^{t}\frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(s,x(s))|ds+|\chi_1(t)|\Big[|\kappa|\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds\\
&+ |\mu|\int_{a_1}^{a_2}\frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds+\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(s,x(s))|ds\Big]\\
&+ |\chi_2(t)|\Big[|\kappa|\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)}|x(s)|ds+|\mu|\int_{a_1}^{a_2}\frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)}|h(s,x(s))|ds\\
&+ \int_{a_1}^{a_2}\frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)}|g(s,x(s))|ds\Big]+|\chi_3(t)|\Big[|\kappa|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\quad |\mu|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds\\
&+\quad \sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(s,x(s))|ds+\int_{a_1}^{a_2}\Big(|\kappa|\int_{a_1}^{s}\frac{(s-u)^{p_1-1}}{\Gamma(p_1)}|x(u)|du\\
&+\quad |\mu|\int_{a_1}^{s}\frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(u,x(u))|du+\int_{a_1}^{s}\frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(u,x(u))|du\Big)dB(s)\Big]\Big\}\\
&\le\quad |\kappa|\|x\|\sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{p_1}}{\Gamma(p_1+1)}+|\chi_2(t)|\frac{(a_2-a_1)^{p_1-1}}{\Gamma(p_1)}\\
&+\quad |\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{p_1}}{\Gamma(p_1+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{p_1}}{\Gamma(p_1+1)}dB(s)\Big)\Big\}\\
&+\quad |\mu|\|\eta_1\|\psi_1(\|x\|)\sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}\\
&+\quad |\chi_2(t)|\frac{(a_2-a_1)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}+|\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}dB(s)\Big)\Big\}\\
&+\quad \|\eta_2\|\psi_2(\|x\|)\sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\\
&+\quad |\chi_2(t)|\frac{(a_2-a_1)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}+|\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}dB(s)\Big)\Big\},
\end{aligned}
$$

which in view of (22), leads to the following estimate:

$$
\begin{aligned}
\|\mathcal{G}x\| &\le |\kappa|\|x\|A_0+|\mu|\|\eta_1\|\psi_1(\|x\|)A_1+\|\eta_2\|\psi_2(\|x\|)A_2\\
&\le |\kappa|bA_0+|\mu|\|\eta_1\|\psi_1(b)A_1+\|\eta_2\|\psi_2(b)A_2.
\end{aligned}
$$

This shows that the operator $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{B}$. Next, we show that $\mathcal{G}$ is equicontinuous on $\mathcal{B}$. Let $t_1,t_2\in[a_1,a_2]$ with $a_1<t_1<t_2<a_2$, and $x\in S_b$, where S_b is a bounded set in $\mathcal{B}$. Then

$$
\begin{aligned}
&|(\mathcal{G}x)(t_2)-(\mathcal{G}x)(t_1)|\\
&\le\quad |\kappa|\Big[\int_{a_1}^{t_1}\frac{|(t_2-s)^{p_1-1}-(t_1-s)^{p_1-1}|}{\Gamma(p_1)}|x(s)|ds+\int_{t_1}^{t_2}\frac{|(t_2-s)^{p_1-1}|}{\Gamma(p_1)}|x(s)|ds\Big]\\
&+\quad |\mu|\Big[\int_{a_1}^{t_1}\frac{|(t_2-s)^{\alpha+p_1-1}-(t_1-s)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{t_1}^{t_2} \frac{|(t_2-s)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds\Big] + \int_{a_1}^{t_1} \frac{|(t_2-s)^{p_1+p_2-1}-(t_1-s)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(s,x(s))|ds \\
&+ \int_{t_1}^{t_2} \frac{|(t_2-s)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(s,x(s))|ds \\
&+ |\chi_1(t_2)-\chi_1(t_1)|\Big[|\kappa|\int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1-1}|}{\Gamma(p_1)}|x(s)|ds + |\mu|\int_{a_1}^{a_2} \frac{|(a_2-s)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds \\
&+ \int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(s,x(s))|ds\Big] + |\chi_2(t_2)-\chi_2(t_1)|\Big[|\kappa|\int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1-2}|}{\Gamma(p_1-1)}|x(s)|ds \\
&+ |\mu|\int_{a_1}^{a_2} \frac{|(a_2-s)^{\alpha+p_1-2}|}{\Gamma(\alpha+p_1-1)}|h(s,x(s))|ds + \int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1+p_2-2}|}{\Gamma(p_1+p_2-1)}|g(s,x(s))|ds\Big] \\
&+ |\chi_3(t_2)-\chi_3(t_1)|\Big[|\kappa|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i} \frac{|(\omega_i-s)^{p_1-1}|}{\Gamma(p_1)}|x(s)|ds \\
&+ |\mu|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i} \frac{|(\omega_i-s)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds + \sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i} \frac{|(\omega_i-s)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(s,x(s))|ds \\
&+ \int_{a_1}^{a_2}\Big(|\kappa|\int_{a_1}^{s} \frac{|(s-u)^{p_1-1}|}{\Gamma(p_1)}|x(u)|du \\
&+ |\mu|\int_{a_1}^{s} \frac{|(s-u)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(u,x(u))|)du + \int_{a_1}^{s} \frac{|(s-u)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(u,x(u))|du\Big)dB(s)\Big] \\
&\le \frac{|\kappa|\|x\|}{\Gamma(p_1+1)}\Big(|(t_2-a_1)^{p_1}-(t_1-a_1)^{p_1}| + 2(t_2-t_1)^{p_1}\Big) \\
&+ \frac{|\mu|\|\eta_1\|\psi_1(\|x\|)}{\Gamma(\alpha+p_1+1)}\Big(|(t_2-a_1)^{\alpha+p_1}-(t_1-a_1)^{\alpha+p_1}| + 2(t_2-t_1)^{\alpha+p_1}\Big) \\
&+ \frac{\|\eta_2\|\psi_2(\|x\|)}{\Gamma(p_1+p_2+1)}\Big(|(t_2-a_1)^{p_1+p_2}-(t_1-a_1)^{p_1+p_2}| + 2(t_2-t_1)^{p_1+p_2}\Big) \\
&+ |\chi_1(t_2)-\chi_1(t_1)|\Big[|\kappa|\int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1-1}|}{\Gamma(p_1)}|x(s)|ds \\
&+ |\mu|\int_{a_1}^{a_2} \frac{|(a_2-s)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds + \int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(s,x(s))|ds\Big] \\
&+ |\chi_2(t_2)-\chi_2(t_1)|\Big[|\kappa|\int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1-2}|}{\Gamma(p_1-1)}|x(s)|ds + |\mu|\int_{a_1}^{a_2} \frac{|(a_2-s)^{\alpha+p_1-2}|}{\Gamma(\alpha+p_1-1)}|h(s,x(s))|ds \\
&+ \int_{a_1}^{a_2} \frac{|(a_2-s)^{p_1+p_2-2}|}{\Gamma(p_1+p_2-1)}|g(s,x(s))|ds\Big] + |\chi_3(t_2)-\chi_3(t_1)|\Big[\\
&\quad |\kappa|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i} \frac{|(\omega_i-s)^{p_1-1}|}{\Gamma(p_1)}|x(s)|ds + |\mu|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i} \frac{|(\omega_i-s)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds
\end{aligned}
$$

$$+ \sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{|(\omega_i-s)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(s,x(s))|ds+\int_{a_1}^{a_2}\Big(|\kappa|\int_{a_1}^{s}\frac{|(s-u)^{p_1-1}|}{\Gamma(p_1)}|x(u)|du$$

$$+ |\mu|\int_{a_1}^{s}\frac{|(s-u)^{\alpha+p_1-1}|}{\Gamma(\alpha+p_1)}|h(u,x(u))|)du+\int_{a_1}^{s}\frac{|(s-u)^{p_1+p_2-1}|}{\Gamma(p_1+p_2)}|g(u,x(u))|du\Big)dB(s)\Big].$$

In the above inequality, notice that the right hand side tends to zero as $t_2 - t_1 \longrightarrow 0$ independent of $x \in S_b$, which shows that $\mathcal{G}$ is equicontinuous. Thus, by the Arzela-Ascoli theorem, the operator $\mathcal{G} : \mathcal{B} \longrightarrow \mathcal{B}$ is completely continuous. In the following step, we show the boundedness of the set of all solutions to equation $x = \beta\mathcal{G}x$ for $0 \le \beta \le 1$. For $t \in [a_1, a_2]$, employing the computations used in proving that $\mathcal{G}$ is bounded, we have

$$\begin{aligned}
|x(t)| &= |\beta(\mathcal{G}x)(t)| \\
&\le |\kappa|\|x\|\sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{p_1}}{\Gamma(p_1+1)}+|\chi_2(t)|\frac{(a_2-a_1)^{p_1-1}}{\Gamma(p_1)} \\
&+ |\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{p_1}}{\Gamma(p_1+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{p_1}}{\Gamma(p_1+1)}dB(s)\Big)\Big\} \\
&+ |\mu|\|\eta_1\|\psi_1(\|x\|)\sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}+|\chi_1(t)|\frac{(a_2-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)} \\
&+ |\chi_2(t)|\frac{(a_2-a_1)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}+|\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)} \\
&+ \int_{a_1}^{a_2}\frac{(s-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)}dB(s)\Big)\Big\}+\|\eta_2\|\psi_2(\|x\|)\sup_{t\in[a_1,a_2]}\Big\{\frac{(t-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} \\
&+ |\chi_1(t)|\frac{(a_2-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+|\chi_2(t)|\frac{(a_2-a_1)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \\
&+ |\chi_3(t)|\Big(\sum_{i=1}^{r-2}|\lambda_i|\frac{(\omega_i-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\int_{a_1}^{a_2}\frac{(s-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)}dB(s)\Big)\Big\},
\end{aligned}$$

which, on taking the norm for $t \in [a_1, a_2]$, yields

$$\|x\| \le |\kappa|\|x\|A_0 + |\mu|\|\eta_1\|\psi_1(\|x\|)A_1 + \|\eta_2\|\psi_2(\|x\|)A_2.$$

Rewriting the above inequality, we get

$$\frac{\|x\|}{|\kappa|\|x\|A_0 + |\mu|\|\eta_1\|\psi_1(\|x\|)A_1 + \|\eta_2\|\psi_2(\|x\|)A_2} \le 1.$$

From (H_2), there exists M such that $\|x\| \ne M$. Let us consider the set $\mathcal{X} = \{x \in \mathcal{B} : \|x\| < M\}$. Notice that the operator $\mathcal{G} : \mathcal{X} \longrightarrow C([a_1, a_2], \mathbb{R})$ is continuous and completely continuous. Thus, from the choice of $\mathcal{X}$, there is no $x \in \partial\mathcal{X}$ such that $x = \beta\mathcal{G}x$ for some $0 < \beta < 1$. Therefore, by Leray-Schauder nonlinear alternative [22], the operator $\mathcal{G}$ has a fixed point $x \in \bar{\mathcal{X}}$, which implies that there exists a solution of the problem (1)-(2) on $[a_1, a_2]$. □

Example 3.2. *Consider the following fractional integro-differential supplemented with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions:*

$$\begin{cases} {}^cD^{\frac{3}{2}}\Big[\Big({}^cD^{\frac{5}{4}} + \frac{3}{70}\Big)x(t) + \frac{1}{90}I^{\frac{1}{2}}h(t,x)\Big] = g(t,x)\ ,\ t \in (0, \frac{4}{3}), \\ x(0) = \sum_{i=1}^{5} \lambda_i x(\omega_i) + \int_0^{\frac{4}{3}} x(s)dB(s),\ x'(0) = 0,\ x(\frac{4}{3}) = 0,\ x'(\frac{4}{3}) = 0, \end{cases} \tag{23}$$

where $a_1 = 0,\ a_2 = \frac{4}{3},\ p_2 = \frac{3}{2},\ p_1 = \frac{5}{4},\ \alpha = \frac{1}{2},\ \kappa = \frac{3}{70},\ \mu = \frac{1}{90},\ \lambda_1 = \frac{-1}{6},\ \lambda_2 = 0,\ \lambda_3 = \frac{-1}{2},\ \lambda_4 = \frac{1}{2},\ \lambda_5 = 1,\ \omega_1 = \frac{1}{8},\ \omega_2 = \frac{1}{4},\ \omega_3 = \frac{1}{2},\ \omega_4 = \frac{3}{4},\ \omega_5 = 1,\ h(t,x) = \frac{(t+1)^2}{24}\Big(\frac{|x+3|}{15(1+|x+3|)} + \cos x\Big)$, *and* $g(t,x) = \dfrac{2}{\sqrt{t^2+1600}}\Big(tan^{-1}x + \sin x\Big)$. *Let us take* $B(s) = \frac{s^2-45}{2}$. *Using the given data, we find that* $A_0 \simeq 6.02737, A_1 \simeq 4.76728$ *and* $A_2 \simeq 2.27540$.

Evidently, $|h(t,x)| \le \frac{(t+1)^2}{24}(\frac{1}{15} + 1)$ *and* $|g(t,x)| \le \frac{2}{\sqrt{t^2+1600}}(\frac{\pi}{2} + \|x\|)$. *Let us fix* $\eta_1(t) = \frac{(t+1)^2}{24},\ \psi_1(\|x\|) = \frac{16}{15}$, $\eta_2(t) = \frac{2}{\sqrt{t^2+1600}}$ *and* $\psi_2(\|x\|) = \frac{\pi}{2} + \|x\|$. *By the assumption:* $M\Big(|\kappa|MA_0 + |\mu|\|\eta_1\|\psi_1(M)A_1 + \|\eta_2\|\psi_2(M)A_2\Big)^{-1} > 1$, *we have that* $M > M_1$ *with* $M_1 \simeq 0.305019$. *Therefore the hypotheses of Theorem 3.1 hold and hence there exists at least one solution for the problem (23) on* $[0, \frac{4}{3}]$.

In the following result, we make use of Leray-Schauder degree theory to establish the existence of solutions for the problem (1)-(2)

Theorem 3.3. *Suppose that there exist positive constants* $\epsilon_i,\ \rho_i\ (i = 1, 2)$ *such that* $|h(t,x)| \le \epsilon_1|x| + \rho_1,\ |g(t,x)| \le \epsilon_2|x| + \rho_2,\ \forall\ (t,x) \in [a_1,a_2] \times \mathbb{R}$ *and*

$$0 < \Big(|\kappa|A_0 + |\mu|\epsilon_1 A_1 + \epsilon_2 A_2\Big) < 1. \tag{24}$$

Then the problem (1)-(2) has at least one solution on $[a_1, a_2]$.

Proof. In order to show that there exists $x \in \mathbb{R}$ satisfying (20), let us introduce a set $S_K = \{x \in \mathcal{B} : \max_{t\in[a_1,a_2]} |x(t)| < K\}$, where K is a positive constant to be fixed later. Then, it is enough to show that $\mathcal{G} : \bar{S}_K \longrightarrow C([a_1,a_2])$ is such that

$$x \ne \Lambda\mathcal{G}x, \forall\ x \in \partial S_K,\ \forall\ 0 \le \Lambda \le 1. \tag{25}$$

Define

$$\Phi(\Lambda, x) = \Lambda\mathcal{G}x,\ x \in C(\mathbb{R}),\ 0 \le \Lambda \le 1,$$

and note that $\phi_\Lambda(x) = x - \Phi(\Lambda, x) = x - \Lambda\mathcal{G}x$ is completely continuous. In case (25) holds, then the following Leray-Schauder degrees are well defined and the homotopy invariance of topological degree implies that

$$\begin{aligned} \deg(\phi_\Lambda, S_K, 0) &= \deg(I - \Lambda\mathcal{G}, S_K, 0) = \deg(\phi_1, S_K, 0) \\ &= \deg(\phi_0, S_K, 0) = \deg(I, S_K, 0) = 1 \ne 0,\ 0 \in S_K, \end{aligned}$$

where I is the unit operator. So $\phi_1(t) = x - \Lambda\mathcal{G}x = 0$ for at least one $x \in S_K$ by the nonzero property of Leray-Schauder degrees. Next we establish (25). For that, we suppose that $x = \Lambda\mathcal{G}x$ for some $0 \le \Lambda \le 1$

and for all $t \in [a_1, a_2]$. Then

$$\begin{aligned}
|x(t)| &= |\Lambda \mathcal{G} x(t)| \\
&\le |\kappa| \|x\| \sup_{t\in[a_1,a_2]} \Big\{ \frac{(t-a_1)^{p_1}}{\Gamma(p_1+1)} + |\chi_1(t)| \frac{(a_2-a_1)^{p_1}}{\Gamma(p_1+1)} + |\chi_2(t)| \frac{(a_2-a_1)^{p_1-1}}{\Gamma(p_1)} \\
&+ |\chi_3(t)| \Big(\sum_{i=1}^{r-2} |\lambda_i| \frac{(\omega_i-a_1)^{p_1}}{\Gamma(p_1+1)} + \int_{a_1}^{a_2} \frac{(s-a_1)^{p_1}}{\Gamma(p_1+1)} dB(s) \Big) \Big\} \\
&+ |\mu|(\epsilon_1 |x| + \rho_1) \sup_{t\in[a_1,a_2]} \Big\{ \frac{(t-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)} + |\chi_1(t)| \frac{(a_2-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)} \\
&+ |\chi_2(t)| \frac{(a_2-a_1)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} + |\chi_3(t)| \Big(\sum_{i=1}^{r-2} |\lambda_i| \frac{(\omega_i-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)} \\
&+ \int_{a_1}^{a_2} \frac{(s-a_1)^{\alpha+p_1}}{\Gamma(\alpha+p_1+1)} dB(s) \Big) \Big\} + (\epsilon_2 |x| + \rho_2) \sup_{t\in[a_1,a_2]} \Big\{ \frac{(t-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} \\
&+ |\chi_1(t)| \frac{(a_2-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + |\chi_2(t)| \frac{(a_2-a_1)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} \\
&+ |\chi_3(t)| \Big(\sum_{i=1}^{r-2} |\lambda_i| \frac{(\omega_i-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \int_{a_1}^{a_2} \frac{(s-a_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} dB(s) \Big) \Big\} \\
&\le \Big(|\kappa| A_0 + |\mu| \epsilon_1 A_1 + \epsilon_2 A_2 \Big) |x| + \Big(|\mu| \rho_1 A_1 + \rho_2 A_2 \Big),
\end{aligned}$$

which, on taking norm and solving for $\|x\|$, yields

$$\|x\| \le \frac{\tau_2}{1-\tau_1},$$

where $\tau_1 = |\kappa| A_0 + |\mu| \epsilon_1 A_1 + \epsilon_2 A_2$, $\tau_2 = |\mu| \rho_1 A_1 + \rho_2 A_2$. In view of (24), letting $K = \frac{\tau_2}{1-\tau_1} + 1$, (25) holds. □

Example 3.4. *Consider the same problem (23) in Example 3.2 with*

$$h(t,x) = \Big(\frac{2}{65+e^t} \Big) x + \frac{2}{15}, \text{ and } g(t,x) = \frac{7}{(\sqrt{t^3+3600})} x + \frac{|x+\frac{1}{8}|}{\frac{1}{8} + |x+\frac{1}{8}|}. \tag{26}$$

Clearly, $|h(t,x)| \le \frac{1}{33} \|x\| + \frac{2}{15}$ *and* $|g(t,x)| \le \frac{7}{60} \|x\| + 1$. *So,* $\epsilon_1 = \frac{1}{33}$, $\epsilon_2 = \frac{7}{60}$, $\rho_1 = \frac{2}{15}$, $\rho_2 = 1$ *and* $\tau_1 = |\kappa| A_0 + |\mu| \epsilon_1 A_1 + \epsilon_2 A_2 \simeq 0.525384 < 1$. *Hence, by the conclusion of Theorem 3.3, there exists at least one solution for the problem (23) with* $h(t,x)$ *and* $g(t,x)$ *given by (26) on* $[0, \frac{4}{3}]$.

Next we prove an existence result with the aid of Shaefer like fixed point theorem, which is stated below [23].

Theorem 3.5. *Let* $\mathcal{K}$ *be a Banach space. Suppose that* $\mathcal{U} : \mathcal{K} \longrightarrow \mathcal{K}$ *is a completely continuous operator and the set* $\mathcal{M} = \{z \in \mathcal{K} | z = \gamma \mathcal{U} z, 0 < \gamma < 1\}$ *is bounded. Then* $\mathcal{U}$ *has a fixed point in* $\mathcal{K}$.

Theorem 3.6. *Assume that there exists* $\nu_1, \nu_2 \in C([a_1,a_2], \mathbb{R}^+)$ *such that* $|h(t,x(t))| \le \nu_1(t)$, $|g(t,x(t))| \le \nu_2(t)$, $\forall t \in [a_1,a_2]$, $x \in \mathcal{B}$, *with* $\sup_{t\in[a_1,a_2]} |\nu_1(t)| = \|\nu_1\|$, $\sup_{t\in[a_1,a_2]} |\nu_2(t)| = \|\nu_2\|$. *Then there exists at least one solution for the problem (1)-(2) on* $[a_1,a_2]$.

Proof. It has already been shown in Theorem 3.1 that the operator $\mathcal{G} : \mathcal{B} \longrightarrow \mathcal{B}$ is completely continuous. Next we consider the set $\mathcal{X} = \{x \in \mathcal{B} | x = \gamma \mathcal{G}x,\ 0 < \gamma < 1\}$, and show that it is bounded. Let $x \in \mathcal{X}$, then $x = \gamma\mathcal{G}x$, $0 < \gamma < 1$. For any $t \in [a_1, a_2]$, we get

$$
\begin{aligned}
|x(t)| &= \gamma |(\mathcal{G}x)(t)| \\
&\le \sup_{t\in[a_1,a_2]} \Big\{ |\kappa| \int_{a_1}^{t} \frac{(t-s)^{p_1-1}}{\Gamma(p_1)} |x(s)| ds + |\mu| \int_{a_1}^{t} \frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} |h(s,x(s))| ds \\
&+ \int_{a_1}^{t} \frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} |g(s,x(s))| ds + |\chi_1(t)| \Big[|\kappa| \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)} |x(s)| ds \\
&+ |\mu| \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} |h(s,x(s))| ds + \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} |g(s,x(s))| ds \Big] \\
&+ |\chi_2(t)| \Big[|\kappa| \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)} |x(s)| ds + |\mu| \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)} |h(s,x(s))| ds \\
&+ \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)} |g(s,x(s))| ds \Big] + |\chi_3(t)| \Big[|\kappa| \sum_{i=1}^{r-2} |\lambda_i| \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)} |x(s)| ds \\
&+ |\mu| \sum_{i=1}^{r-2} |\lambda_i| \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} |h(s,x(s))| ds \\
&+ \sum_{i=1}^{r-2} |\lambda_i| \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} |g(s,x(s))| ds + \int_{a_1}^{a_2} \Big(|\kappa| \int_{a_1}^{s} \frac{(s-u)^{p_1-1}}{\Gamma(p_1)} |x(u)| du \\
&+ |\mu| \int_{a_1}^{s} \frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} |h(u,x(u))| du + \int_{a_1}^{s} \frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)} |g(u,x(u))| du \Big) dB(s) \Big] \Big\} \\
&\le |\kappa| \|x\| A_0 + |\mu| \|\nu_1\| A_1 + \|\nu_2\| A_2,
\end{aligned}
$$

where A_0, A_1 and A_2 given by (22). Thus, $\|x\| \le |\kappa|\|x\|A_0 + |\mu|\|\nu_1\|A_1 + \|\nu_2\|A_2$ for any $t \in [a_1, a_2]$. So, the set $\mathcal{X}$ is bounded. Thus the conclusion of Theorem 3.5 applies and the operator $\mathcal{G}$ has at least one fixed point. In consequence, there exists at least one solution for the problem (1)-(2) on $[a_1, a_2]$. □

Example 3.7. *Consider the problem (23) with*

$$
h(t,x) = \frac{15e^{38+t^7}}{(\sqrt{t^5}+4)^2} \Big(\frac{|2\cos x|}{2+|2\cos x|} \Big) \text{ and } g(t,x) = \Big(\frac{\sqrt{t^2+23}}{23} \Big) \tan^{-1} x + \ln(t^9+45). \tag{27}
$$

Clearly, $|h(t,x)| \le \frac{15e^{38+t^7}}{(\sqrt{t^5}+4)^2} = \nu_1(t) > 0$ *and* $|g(t,x)| \le \dfrac{\pi\sqrt{t^2+23}}{46} + \ln(t^9+45) = \nu_2(t) > 0$. *Hence, by the conclusion of Theorem 3.6, there exists at least one solution for the problem (23) with* $h(t,x)$ *and* $g(t,x)$ *given by (27) on* $[0, \frac{4}{3}]$.

4. Uniqueness of solution

In this section, we study the uniqueness of solutions for the problem (1)-(2) by means of Banach contraction mapping principle.

Theorem 4.1. *Assume that $h, g : [a_1, a_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions satisfying the Lipschitz condition:*

$$(H_3) : |h(t,x) - h(t,y)| \leq L_1|x-y|, \ |g(t,x) - g(t,y)| \leq L_2|x-y|, \ \forall t \in [a_1, a_2], \ \ x, y \in \mathbb{R},$$

where L_i $(i = 1, 2)$ are Lipschitz constants. Then there exists a unique solution for the problem (1)-(2) on $[a_1, a_2]$ if

$$\bar{\Delta} = |\kappa|A_0 + |\mu|L_1A_1 + L_2A_2 < 1, \tag{28}$$

where A_i $(i = 0, 1, 2)$ are given by (22).

Proof. Let us set $N_1 = \sup_{t\in[a_1,a_2]} |h(t,0)|$, $N_2 = \sup_{t\in[a_1,a_2]} |g(t,0)|$, and choose

$$\theta \geq \frac{|\mu|N_1A_1 + N_2A_2}{1 - |\kappa|A_0 - |\mu|L_1A_1 - L_2A_2}.$$

We define $S_\theta = \{x \in \mathcal{B} : \|x\| \leq \theta\}$ and show that $\mathcal{G}S_\theta \subset S_\theta$, where the operator $\mathcal{G}$ is defined by (21). For any $x \in S_\theta$, $t \in [a_1, a_2]$, we have

$$\begin{aligned} |h(t,x(t))| &= |h(t,x(t)) - h(t,0) + h(t,0)| \leq |h(t,x(t)) - h(t,0)| + |h(t,0)| \\ &\leq L_1|x(t)| + N_1 \leq L_1\|x\| + N_1 \leq L_1\theta + N_1, \\ |g(t,x(t))| &= |g(t,x(t)) - g(t,0) + g(t,0)| \leq |g(t,x(t)) - g(t,0)| + |g(t,0)| \\ &\leq L_2|x(t)| + N_2 \leq L_2\|x\| + N_2 \leq L_2\theta + N_2. \end{aligned}$$

Then we have

$$\begin{aligned} \|\mathcal{G}x\| \leq & \sup_{t\in[a_1,a_2]} \Big\{ |\kappa| \int_{a_1}^{t} \frac{(t-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds + |\mu| \int_{a_1}^{t} \frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds \\ & + \int_{a_1}^{t} \frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(s,x(s))|ds + |\chi_1(t)|\Big[|\kappa| \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds \\ & + |\mu| \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds + \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(s,x(s))|ds\Big] \\ & + |\chi_2(t)|\Big[|\kappa| \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)}|x(s)|ds + |\mu| \int_{a_1}^{a_2} \frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)}|h(s,x(s))|ds \\ & + \int_{a_1}^{a_2} \frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)}|g(s,x(s))|ds\Big] + |\chi_3(t)|\Big[|\kappa| \sum_{i=1}^{r-2} |\lambda_i| \int_{a_1}^{\omega_i} \frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad |\mu|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(s,x(s))|ds\\
&+ \quad \sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(s,x(s))|ds+\int_{a_1}^{a_2}\Big(|\kappa|\int_{a_1}^{s}\frac{(s-u)^{p_1-1}}{\Gamma(p_1)}|x(u)|du\\
&+ \quad |\mu|\int_{a_1}^{s}\frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}|h(u,x(u))|du+\int_{a_1}^{s}\frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}|g(u,x(u))|du\Big)dB(s)\Big]\Big\}\\
&\le \quad |\kappa|\|x\|A_0+|\mu|(L_1\|x\|+N_1)A_1+(L_2\|x\|+N_2)A_2\\
&\le \quad |\kappa|\theta A_0+|\mu|(L_1\theta+N_1)A_1+(L_2\theta+N_2)A_2\le\theta.
\end{aligned}
$$

This shows that $\mathcal{G}x\in S_\theta$ for any $x\in S_\theta$. Therefore, $\mathcal{G}S_\theta\subset S_\theta$. Now, we show that $\mathcal{G}$ is a contraction. For $x,y\in\mathcal{B}$ and $t\in[a_1,a_2]$, we obtain

$$
\begin{aligned}
&\|(\mathcal{G}x)-(\mathcal{G}y)\| = \sup_{t\in[a_1,a_2]}\Big|(\mathcal{G}x)(t)-(\mathcal{G}y)(t)\Big|\\
&\le \quad \sup_{t\in[a_1,a_2]}\Big\{|\kappa|\int_{a_1}^{t}\frac{(t-s)^{p_1-1}}{\Gamma(p_1)}|x(s)-y(s)|ds\\
&+ \quad |\mu|\int_{a_1}^{t}\frac{(t-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}\Big|h(s,x(s))-h(s,y(s))\Big|ds+\int_{a_1}^{t}\frac{(t-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}\Big|g(s,x(s))\\
&- \quad g(s,y(s))\Big|ds+|\chi_1(t)|\Big[|\kappa|\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1-1}}{\Gamma(p_1)}|x(s)-y(s)|ds\\
&+ \quad |\mu|\int_{a_1}^{a_2}\frac{(a_2-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}\Big|h(s,x(s))-h(s,y(s))\Big|ds\\
&+ \quad \int_{a_1}^{a_2}\frac{(a_2-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}\Big|g(s,x(s))-g(s,y(s))\Big|ds\Big]\\
&+ \quad |\chi_2(t)|\Big[|\kappa|\int_{a_1}^{a_2}\frac{(a_2-s)^{p_1-2}}{\Gamma(p_1-1)}|x(s)-y(s)|ds+|\mu|\int_{a_1}^{a_2}\frac{(a_2-s)^{\alpha+p_1-2}}{\Gamma(\alpha+p_1-1)}\Big|h(s,x(s))-h(s,y(s))\Big|ds\\
&+ \quad \int_{a_1}^{a_2}\frac{(a_2-s)^{p_1+p_2-2}}{\Gamma(p_1+p_2-1)}\Big|g(s,x(s))-g(s,y(s))\Big|ds\Big]\\
&+ \quad |\chi_3(t)|\Big[|\kappa|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1-1}}{\Gamma(p_1)}|x(s)-y(s)|ds\\
&+ \quad |\mu|\sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}\Big|h(s,x(s))-h(s,y(s))\Big|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=1}^{r-2}|\lambda_i|\int_{a_1}^{\omega_i}\frac{(\omega_i-s)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}\Big|g(s,x(s))-g(s,y(s))\Big|ds+\int_{a_1}^{a_2}\Big(|\kappa|\int_{a_1}^{s}\frac{(s-u)^{p_1-1}}{\Gamma(p_1)}|x(u)-y(u)|du\\
&+ |\mu|\int_{a_1}^{s}\frac{(s-u)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}\Big|h(u,x(u))-h(u,y(u))\Big|du+\int_{a_1}^{s}\frac{(s-u)^{p_1+p_2-1}}{\Gamma(p_1+p_2)}\Big|g(u,x(u))-g(u,y(u))\Big|du\Big)dB(s)\Big]\Big\}\\
&\le |\kappa|\|x-y\|A_0+|\mu|L_1\|x-y\|A_1+L_2\|x-y\|A_2\\
&= \bar{\Delta}\|x-y\|,
\end{aligned}
$$

which, in view of (28), implies that the operator $\mathcal{G}$ is a contraction. Hence, we deduce by the conclusion of contraction mapping principle that the problem (1)-(2) has a unique solution on $[a_1, a_2]$. □

Example 4.2. *Let us consider the problem (23) with*

$$h(t,x)=\frac{\sin^2 t}{\sqrt{t^2+4900}}\,arc\ tan\ x+\frac{2x}{\sqrt{t^2+6400}}+e^{-t},$$

and

$$g(t,x)=\frac{1}{\sqrt{t+400}}\Big(\frac{|x|}{6+|x|}+\frac{\cos x}{12}\Big)+27\ln(t+1),\ t\in(0,\frac{4}{3})$$

Obviously, $|h(t,x)-h(t,y)|\le L_1\|x-y\|$ *with* $L_1=\frac{11}{280}$ *and* $|g(t,x)-g(t,y)|\le L_2\|x-y\|$ *with* $L_2=\frac{13}{240}$. *Using the given data, we find that* $\bar{\Delta}\simeq 0.383648<1$, *where* $\bar{\Delta}$ *is defined by (28). Clearly the hypothesis of Theorem 4.1 is satisfied. Hence it follows by the conclusion of Theorem 4.1 that there is a unique solution for the problem (23) on* $[0,\frac{4}{3}]$.

Special cases. For $\mu = 0$, our results correspond to the problem consisting of Langevin equation with two fractional orders and Riemann-Stieltjes integro-multipoint boundary conditions. Letting $\kappa = 0$ in the results of this paper, we obtain the ones for fractional integro-differential equation:

$${}^cD^{p_2}\Big({}^cD^{p_1}x(t)+\mu I^{\alpha}h(t,x(t))\Big)=g(t,x(t)),$$

subject to the boundary conditions (2). Moreover, our results specialize to any fixed domain by taking the specific values of a_1 and a_2. Thus our results are not only new in the given configuration but also specialize to some new cases.

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