Filomat 34:8 (2020), 2655–2665
https://doi.org/10.2298/FIL2008655D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Properties of the Generalized Fibonacci and Lucas Numbers

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Abstract. In this paper we consider the generalized Fibonacci numbers $F_{n,m}$ and the generalized Lucas numbers $L_{n,m}$. Also, we introduce new sequences of numbers $A_{n,m}$, $B_{n,m}$, $C_{n,m}$ and $D_{n,m}$. Further, we find the generating functions and some recurrence relations for these sequences of numbers.

1. Introduction

Generalized Fibonacci numbers $F_{n,m}$ ($m \geq 2$) are given by ([1], [2], for x=1)

$$F_{n,m} = F_{n-1,m} + F_{n-m,m}, \quad n \geq m, \tag{1}$$

with:

$$F_{0,m} = 0, \; F_{1,m} = \cdots = F_{m-1,m} = 1. \tag{2}$$

Generalized Lucas numbers $L_{n,m}$ are given by ([1], [2], for x=1)

$$L_{n,m} = L_{n-1,m} + L_{n-m,m}, \quad n \geq m, \tag{3}$$

with:

$$L_{0,m} = 2, \; L_{1,m} = \cdots = L_{m-1,m} = 1. \tag{4}$$

So, using the well-known method, we find the following generating functions for $F_{n,m}$ and $L_{n,m}$, respectively:

$$F_m(t) = \frac{t}{1-t-t^m} = \sum_{n=0}^{\infty} F_{n,m}\, t^n, \tag{5}$$

$$L_m(t) = \frac{2-t}{1-t-t^m} = \sum_{n=0}^{\infty} L_{n,m}\, t^n. \tag{6}$$

2010 *Mathematics Subject Classification*. Primary 11B83; Secondary 11B37, 11B39
Keywords. Recurrence relation, Convolution, Generating function, Explicit representation, Summation formula
Received: 19 August 2019; Accepted: 02 June 2020
Communicated by Miodrag Spalević
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Using the relations (5) and (6), we get the following explicit formulas, for the sequences $F_{n,m}$ and $L_{n,m}$, respectively:

$$F_{n+1,m} = \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k}, \tag{7}$$

$$L_{n,m} = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k}. \tag{8}$$

Remark 1.1. *The numbers $F_{n,m}$ are the special case of the Humbert's polynomials for $x = 1$, (see [6]), i.e., $F_{n+1,m} = P_n(m, 1/m, -1, -1, 1)$.*

It is easy to prove the following relations

$$\begin{aligned} L_{n,m} &= F_{n+1,m} + F_{n+1-m,m} \\ &= F_{n,m} + 2F_{n+1-m,m}. \end{aligned}$$

Next, from (1) and (2), we find the first $m+3$ members of $F_{n,m}$, and, by (3) and (4), the first $m+3$ members of $L_{n,m}$:

Table 1:

n	$F_{n,m}$	$L_{n,m}$
0	0	2
1	1	1
2	1	1
...	...	...
$m-1$	1	1
m	1	3
$m+1$	2	4
$m+2$	3	5
$m+3$	4	6
...	...	...

The particular cases of $F_{n,m}$ and $L_{n,m}$ are $F_{n,2}$, $L_{n,2}$, $F_{n,3}$ and $L_{n,3}$. The first 13 members of these numbers are given by Table 2. Namely, the numbers $F_{n,2}$ are the known Fibonacci numbers and $L_{n,2}$ are the known Lucas numbers.

Something more on the generalized Fibonacci and generalized Lucas polynomials and numbers can be seen in [3] and [4].

2. Summation formula

In this section we find a summation formula for the generalized Lucas numbers $L_{n,m}$.

Theorem 2.1. *The numbers $L_{n,m}$ and $F_{n,m}$ satisfy the following relations*

$$\sum_{i=0}^{[(n-1)/m]} (-1)^i L_{n-mi,m} = \begin{cases} F_{n+1,m} \pm 1, & n = mk + l, \quad l = 1, 2, \ldots, m-1, \\ F_{n+1,m}, & n = mk + m. \end{cases} \tag{9}$$

Table 2:

n	$F_{n,2}$	$L_{n,2}$	$F_{n,3}$	$L_{n,3}$	$F_{n,4}$	$L_{n,4}$
0	0	2	0	2	0	2
1	1	1	1	1	1	1
2	1	3	1	1	1	1
3	2	4	1	3	1	1
4	3	7	2	4	1	3
5	5	11	3	5	2	4
6	8	18	4	8	3	5
7	13	29	6	12	4	6
8	21	47	9	17	5	9
9	34	76	13	25	7	13
10	55	123	19	37	10	18
11	89	199	28	54	14	24
12	144	322	41	79	19	33

Proof. Here we use mathematical induction on n.
It is easy to check the formula (9) for $n = 1, 2, \cdots, m-1$.
Suppose that (9) is correct for $n \geq 1$. Then:

1° For $n = mk + m$, we find:

$$\sum_{i=0}^{k} (-1)^i L_{n-mi,m} =$$
$$L_{n,m} - L_{n-m,m} + L_{n-2m,m} + \cdots + (-1)^{k-1} L_{2m,m} + (-1)^k L_{m,m}$$
$$= (F_{n+1,m} + F_{n+1-m,m}) - (F_{n+1-m,m} + F_{n+1-2m,m}) + \cdots + (-1)^{k-1}(F_{2m+1,m} + F_{m+1,m}) + (-1)^k (F_{m+1,m} + F_{1,m})$$
$$= F_{n+1,m} \pm F_{1,m} = F_{n+1,m} \pm 1;$$

2° For $n = mk + l$, $l = 1, 2, \ldots, m-1$, we get:

$$L_{n,m} - L_{n-m,m} + L_{n-2m,m} + \cdots + (-1)^{k-1} L_{l+m,m} + (-1)^k L_{l,m}$$
$$= (F_{n+1,m} + F_{n+1-m,m}) - (F_{n+1-m,m} + F_{n+1-2m,m}) + \cdots + (-1)^{k-1}(F_{l+1+m,m} + F_{l+1,m}) + (-1)^k F_{l+1,m} (L_{l,m} = F_{l+1,m})$$
$$= F_{n+1,m}.$$

Remark 2.2. *For $m = 2$, by (9), we get*

$$\sum_{i=0}^{[(n-1)/2]} (-1)^i L_{n-2i} = \begin{cases} F_{n+1} \pm 1, & n = 2k+2, \\ F_{n+1}, & n = 2k+1. \end{cases}$$

and, for $m = 3$, again by (9), we find

$$\sum_{i=0}^{[(n-1)/3]} (-1)^i L_{n-3i,3} = \begin{cases} F_{n+1,3} \pm 1, & n = 3k+3, \\ F_{n+1,3}, & n = 3k+l, \ l = 1, 2, \end{cases}$$

where $k \in \mathbb{N}$.

3. New sequences of numbers

In this section we introduce and we consider new sequences of numbers: $A_{n,m}$, $B_{n,m}$, $C_{n,m}$, and $D_{n,m}$. Namely, these numbers are the convolutions of the generalized Fibonacci numbers and the generalized Lucas numbers. For these sequences of numbers we find the generating functions and some interesting properties.

Definition 3.1. *The sequence of numbers $A_{n,m}$ ($m \geq 2$) is defined by*

$$A_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} F_{i,m} F_{n-m(i-1),m}. \tag{10}$$

where $n = mp + l$, $l = 0, 1, 2, \ldots, m-1$.

Some first members of the sequences $A_{n,2}$, $A_{n,3}$ and $A_{n,m}$, are given in the following table.

Table 3:

n	$A_{n,2}$	$A_{n,3}$	n	$A_{n,m}$
1	1	1	1	1
2	1	1	2	1
3	3	...	...	...
5	9	4	$m-1$	1
6	13	5	m	1
7	25	9	$m+1$	3
8	38	13	$m+2$	4
9	68	18	$m+3$	5
...	...	...	...	...

Theorem 3.2. *The generating function for the sequence of numbers $A_{n,m}$ is given by*

$$A_m(t) = \frac{t}{(1-t-t^m)(1-t^m-t^{m^2})}. \tag{11}$$

Proof. Since

$$A_m(t) = A_{1,m}t + A_{2,m}t^2 + A_{3,m}t^3 + \cdots + A_{n,m}t^n + \ldots,$$

and by (10), we get

$$\begin{aligned}
A_m(t) &= (F_{1,m}F_{1,m})t + (F_{1,m}F_{2,m})t^2 + (F_{1,m}F_{3,m})t^3 + \ldots \\
&\quad + (F_{1,m}F_{m,m})t^m + (F_{1,m}F_{m+1,m} + F_{2,m}F_{1,m})t^{m+1} + \ldots \\
&= (F_{1,m}t + F_{2,m}t^{m+1} + F_{3,m}t^{2m+1} + F_{4,m}t^{3m+1} + \ldots) \cdot (F_{1,m} + F_{2,m}t + F_{3,m}t^2 + F_{4,m}t^3 + \ldots) \\
&= \frac{1}{t^{m-1}} \left[F_{1,m}t^m + F_{2,m}t^{2m} + F_{3,m}t^{3m} + F_{4,m}t^{4m} + \ldots\right] \cdot \frac{1}{t} \left[F_{1,m}t + F_{2,m}t^2 + F_{3,m}t^3 + F_{4,m}t^4 + \ldots\right] \\
&= \frac{1}{t^{m-1}} \cdot \frac{t^m}{1-t^m-t^{m^2}} \cdot \frac{1}{t} \cdot \frac{t}{1-t-t^m} \\
&= \frac{t}{(1-t-t^m)(1-t^m-t^{m^2})}.
\end{aligned}$$

Corollary 3.3. *For $m = 2$ (see [5]), by (10), we get*

$$A_2(t) = \frac{t}{(1-t-t^2)(1-t^2-t^4)} = \sum_{n=0}^{\infty} A_n t^n.$$

Theorem 3.4. *The numbers $A_{n,m}$ satisfy the following relations*

$$A_{n,m} = \begin{cases} A_{n-1,m} + A_{n-m,m}, & \text{if } l \neq 1, \\ A_{n-1,m} + A_{n-m,m} + F_{p+1,m}, & \text{if } l = 1, \end{cases} \tag{12}$$

where $n = mp + l$ and $l = 0, 1, \cdots, m-1$.

Proof. Let $m \geq 2$.

1° If $l = 0$, then we get:

$$\begin{aligned} &n = mp, \;\; [(mp+m-1)/m] = p, \\ &n-1 = mp-1, \;\; [(mp-1+m-1)/m] = p, \\ &n-m = mp-m, \;\; [(mp-m+m-1)/m] = p-1, \end{aligned}$$

and

$$\begin{aligned} A_{n,m} &= F_{1,m}F_{n,m} + F_{2,m}F_{n-m,m} + F_{3,m}F_{n-2m,m} + \cdots + F_{p,m}F_{m,m}, \\ A_{n-1,m} &= F_{1,m}F_{n-1,m} + F_{2,m}F_{n-1-m,m} + F_{3,m}F_{n-1-2m,m} + \cdots + F_{p,m}F_{m-1,m}; \\ A_{n-m,m} &= F_{1,m}F_{n-m,m} + F_{2,m}F_{n-2m,m} + F_{3,m}F_{n-3m,m} + \cdots + F_{p-1,m}F_{m,m}. \end{aligned}$$

Since

$$A_{n-1,m} + A_{n-m,m} = F_{1,m}F_{n,m} + F_{2,m}F_{n-m,m} + F_{3,m}F_{n-2m,m} + \cdots + F_{p,m}F_{m-1,m}, \;\; (F_{m,m} = F_{m-1,m} = 1),$$

we conclude that

$$A_{n,m} = A_{n-1,m} + A_{n-m,m}.$$

2° If $l = 1$, then we find:

$$\begin{aligned} &n = mp+1, \;\; [(mp+1+m-1)/m] = p+1, \\ &n-1 = mp, \;\; [(mp+1-1+m-1)/m] = p, \\ &n-m = mp+1-m, \;\; [(mp+1-m+m-1)/m] = p, \end{aligned}$$

and

$$\begin{aligned} A_{n,m} &= F_{1,m}F_{n,m} + F_{2,m}F_{n-m,m} + \cdots + F_{p,m}F_{m+1,m} + F_{p+1,m}F_{1,m}; \\ A_{n-1,m} &= F_{1,m}F_{n-1,m} + F_{2,m}F_{n-1-m,m} + F \cdots + F_{p,m}F_{m,m}; \\ A_{n-m,m} &= F_{1,m}F_{n-m,m} + F_{2,m}F_{n-2m,m} + \cdots + F_{p,m}F_{1,m}. \end{aligned}$$

So, we see that the next relation holds:

$$A_{n,m} = A_{n-1,m} + A_{n-m,m} + F_{p+1,m}.$$

3° If $l \geq 2$, then

$$\begin{aligned} &n = mp+l, \;\; [(mp+l+m-1)/m] = p+1, \\ &n-1 = mp+l-1, \;\; [(mp+l-1+m-1)/m] = p+1, \\ &n-m = mp+l-m, \;\; [(mp+l-m+m-1)/m] = p, \end{aligned}$$

and

$$\begin{aligned} A_{n,m} &= F_{1,m}F_{n,m} + F_{2,m}F_{n-m,m} + \cdots + F_{p,m}F_{m+l,m} + F_{p+1,m}F_{l,m};\\ A_{n-1,m} &= F_{1,m}F_{n-1,m} + F_{2,m}F_{n-1-m,m} + \cdots + F_{p,m}F_{m+l-1,m} + F_{p+1,m}F_{l-1,m};\\ A_{n-m,m} &= F_{1,m}F_{n-m,m} + F_{2,m}F_{n-2m,m} + \cdots + F_{p,m}F_{l,m}. \end{aligned}$$

So

$$A_{n,m} = A_{n-1,m} + A_{n-m,m}, \quad (F_{l,m} = F_{l-1,m}).$$

Example 3.5. *Let* $m = 3$ *and* $n = 3 \cdot 2$*, then*

$$\begin{aligned} &A_{6,3} = F_{1,3}F_{6,3} + F_{2,3}F_{3,3} = 5,\\ &A_{5,3} = F_{1,3}F_{5,3} + F_{2,3}F_{2,3} = 4,\\ &A_{3,3} = F_{1,3}F_{3,3} = 1,\\ &A_{6,3} = A_{5,3} + A_{3,3} = 5. \end{aligned}$$

If $n = 3 \cdot 2 + 1$*, then we get*

$$\begin{aligned} &A_{7,3} = F_{1,3}F_{7,3} + F_{2,3}F_{4,3} + F_{3,3}F_{1,3} = 9,\\ &A_{4,3} = F_{1,3}F_{4,3} + F_{2,3}F_{1,3} = 3,\\ &A_{7,3} = A_{6,3} + A_{4,3} + F_{3,3} = 9. \end{aligned}$$

If $n = 3 \cdot 2 + 2$*, then*

$$\begin{aligned} &A_{8,3} = F_{1,3}F_{8,3} + F_{2,3}F_{5,3} + F_{3,3}F_{2,3} = 13,\\ &A_{8,9} = A_{7,3} + A_{5,3} = 13. \end{aligned}$$

Definition 3.6. *The sequence* $B_{n,m}$ *is defined by*

$$B_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} F_{i,m}L_{n-m(i-1),m}, \tag{13}$$

where $n = mp + l$*,* $l = 0, 1, 2, \ldots, m-1$*.*

Theorem 3.7. *For* $m \geq 2$ *and* $n = mp + l$*, where* $l = 0, 1, \ldots, m-1$*, we get*

$$B_{n,m} = \begin{cases} B_{n-1,m} + B_{n-m,m} + F_{p+1,m}, & l = 1,\\ B_{n-1,m} + B_{n-m,m}, & l \neq 0, 1,\\ B_{n-1,m} + B_{n-m,m} + 2F_{p,m}, & l = 0. \end{cases} \tag{14}$$

Proof. 1° If $l = 0$, then we get:

$$\begin{aligned} &n = mp, \quad [(mp + m - 1)/m] = p,\\ &n - 1 = mp - 1, \quad [(mp - 1 + m - 1)/m] = p,\\ &n - m = mp - m, \quad [(mp - m + m - 1)/m] = p - 1, \end{aligned}$$

and

$$\begin{aligned} B_{n,m} &= F_{1,m}L_{n,m} + F_{2,m}L_{n-m,m} + \cdots + F_{p,m}L_{n-m(p-1),m},\\ B_{n-1,m} &= F_{1,m}L_{n-1,m} + F_{2,m}L_{n-1-m,m} + \cdots + F_{p-1,m}L_{n-1-m(p-2),m} + F_{p,m}L_{n-1-m(p-1),m},\\ B_{n-m,m} &= F_{1,m}L_{n-m,m} + F_{2,m}L_{n-2m,m} + \cdots + F_{p-1,m}L_{n-m-m(p-2),m}. \end{aligned}$$

Hence

$$\begin{aligned} B_{n-1,m} + B_{n-m,m} &= F_{1,m}L_{n,m} + F_{2,m}L_{n-m,m} + \cdots + F_{p-1,m}L_{2m-1,m} + F_{p,m}L_{m-1,m} \\ &= F_{1,m}L_{n,m} + F_{2,m}L_{n-m,m} + \cdots + F_{p,m}(L_{m,m} - L_{0,m}) \\ &= B_{n,m} - 2F_{p,m}. \end{aligned}$$

2° If $l = 1$, then:

$$\begin{aligned} &n = mp + 1, \;\; [(mp + 1 + m - 1)/m] = p + 1, \\ &n - 1 = mp, \;\; [(mp + m - 1)/m] = p, \\ &n - m = mp + 1 - m, \;\; [(mp + 1 - m + m - 1)/m] = p, \end{aligned}$$

and

$$\begin{aligned} B_{n,m} &= F_{1,m}L_{n,m} + F_{2,m}L_{n-m,m} + \cdots + F_{p,m}L_{m+1,m} + F_{p+1,m}L_{1,m}, \\ B_{n-1,m} &= F_{1,m}L_{n-1,m} + F_{2,m}L_{n-1-m,m} + \cdots + F_{p,m}L_{m,m}, \\ B_{n-m,m} &= F_{1,m}L_{n-m,m} + F_{2,m}L_{n-2m,m} + \cdots + F_{p,m}L_{1,m}. \end{aligned}$$

So

$$B_{n,m} = B_{n-1,m} + B_{n-m,m} + F_{p+1,m}.$$

3° Let $l \geq 2$. Now we get:

$$\begin{aligned} &n = mp + l, \;\; [(mp + l + m - 1)/m] = p + 1, \\ &n - 1 = mp + l - 1, \;\; [(mp + l - 1 + m - 1)/m] = p + 1, \\ &n - m = mp - m + l, \;\; [(mp + l - m + m - 1)/m] = p, \end{aligned}$$

and

$$\begin{aligned} B_{n,m} &= F_{1,m}L_{n,m} + F_{2,m}L_{n-m,m} + \cdots + F_{p,m}L_{m+l,m} + F_{p+1,m}L_{l,m}, \\ B_{n-1,m} &= F_{1,m}L_{n-1,m} + F_{2,m}L_{n-1-m,m} + \cdots + F_{p,m}L_{l+m-1,m} + F_{p+1,m}L_{l-1,m}, \\ B_{n-m,m} &= F_{1,m}L_{n-m,m} + F_{2,m}L_{n-2m,m} + \cdots + F_{p,m}L_{l,m}. \end{aligned}$$

So

$$B_{n,m} = B_{n-1,m} + B_{n-m,m}, \;\; (L_{l-1,m} = L_{l,m}).$$

Example 3.8. *Let $m = 4$ and $n = 4 \cdot 2$, then we have*

$$\begin{aligned} &B_{8,4} = F_{1,4}L_{8,4} + F_{2,4}L_{4,4} = 12, \\ &B_{7,4} = F_{1,4}L_{7,4} + F_{2,4}L_{3,4} = 7, \\ &B_{4,4} = F_{1,4}L_{4,4} = 3, \\ &B_{8,4} = B_{7,4} + B_{4,4} + 2F_{2,4} = 12. \end{aligned}$$

Let $n = 4 \cdot 2 + 1$, then

$$\begin{aligned} &B_{9,4} = F_{1,4}L_{9,4} + F_{2,4}L_{5,4} + F_{3,4}L_{1,4} = 18, \\ &B_{5,4} = F_{1,4}L_{5,4} + F_{2,4}L_{1,4} = 5, \\ &B_{9,4} = B_{8,5} + B_{5,4} + F_{3,4} = 18. \end{aligned}$$

Now, let $n = 4 \cdot 2 + 2$, *and*

$$B_{10,4} = F_{1,4}L_{10,4} + F_{2,4}L_{6,4} + F_{3,4}L_{2,4} = 24,$$
$$B_{6,4} = F_{1,4}L_{6,4} + F_{2,4}L_{2,4} = 6,$$
$$B_{10,4} = B_{9,4} + B_{6,4} = 24,$$

Finally, let $n = 4 \cdot 2 + 3$, *then*

$$B_{11,4} = F_{1,4}L_{11,4} + F_{2,4}L_{7,4} + F_{3,4}L_{3,4} = 31,$$
$$B_{11,4} = B_{10,4} + B_{7,4} = 24 + 7 = 31.$$

Theorem 3.9. *The generating function of the sequence* $B_{n,m}$ *is given by*

$$B_m(t) = \frac{t + 2t^m}{(1 - t - t^m)(1 - t^m - t^{m^2})}. \tag{15}$$

Proof.

$$\begin{aligned} B_m(t) &= \sum_{n=1}^{\infty} B_{n,m}\, t^n = B_{1,m}t + B_{2,m}t^2 + \cdots + B_{m-1,m}t^{m-1} + \ldots \\ &= (F_{1,m}L_{1,m})t + (F_{1,m}L_{2,m})t^2 + \cdots + (F_{1,m}L_{m-1,m})t^{m-1} + \ldots \\ &= (L_{1,m} + L_{2,m}t + L_{3,m}t^2 + \cdots + L_{m-1,m}t^{m-2} + L_{m,m}t^{m-1} + \ldots)\times \\ &\quad (F_{1,m}t + F_{2,m}t^{m+1} + F_{3,m}t^{2m+1} + F_{4,m}t^{3m+1} + \ldots) \\ &= \frac{1}{t}(L_{1,m}t + L_{2,m}t^2 + L_{3,m}t^3 + \cdots + L_{m,m}t^m + L_{0,m} - L_{0,m}) \cdot \frac{1}{t^{m-1}}(F_{1,m}t^m + F_{2,m}t^{2m} + F_{3,m}t^{3m} + \ldots) \\ &= \frac{1}{t^m}\left(\frac{2-t}{1-t-t^m} - 2\right) \cdot \frac{t^m}{1 - t^m - t^{m^2}} \\ &= \frac{t + 2t^m}{(1 - t - t^m)(1 - t^m - t^{m^2})}. \end{aligned}$$

Corollary 3.10.

$$B_2(t) = \sum_{n=1}^{\infty} B_{n,2}t^n = \frac{t + 2t^2}{(1 - t - t^2)(1 - t^2 - t^4)}.$$

Definition 3.11. *The sequence of numbers* $C_{n,m}$ *is given by*

$$C_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} L_{i,m}F_{n-m(i-1),m}. \tag{16}$$

where $n = mp + l, \;\; l = 0, 1, 2, \ldots, m - 1.$

Theorem 3.12. *The generating function of the sequence* $C_{n,m}$ *is*

$$C_m(t) = \frac{t^m + 2t^{m^2}}{t^{m-1}(1 - t - t^m)(1 - t^m - t^{m^2})}. \tag{17}$$

Proof. The relation (17) can be proved in a similar way as the relation (15), or the relation (11).
It is easy to prove the following statement.

Theorem 3.13. *The sequence of numbers* $C_{n,m}$*, for* $n = mp + l$*, where* $l = 0, 1, \ldots, m-1$*, and* $m \geq 2$*, satisfies the following properties*

$$C_{n,m} = \begin{cases} C_{n-1,m} + C_{n-m,m}, & l \neq 1, \\ C_{n-1,m} + C_{n-m,m} + L_{p+1,m}, & l = 1. \end{cases} \tag{18}$$

Definition 3.14.

$$D_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} L_{i,m} L_{n-m(i-1),m}, \tag{19}$$

where $n = mp + l$, $l = 0, 1, 2, \ldots, m-1$.

One can easily prove the following statements.

Theorem 3.15. *The formula*

$$D_m(t) = \frac{(1 + 2t^{m-1})(t^m + 2t^{m^2})}{t^{m-1}(1 - t - t^m)(1 - t^m - t^{m^2})} \tag{20}$$

is the generating function of the sequence $D_{n,m}$.

Theorem 3.16. *Numbers* $D_{n,m}$ *satisfy the following formulas*

$$D_{n,m} = \begin{cases} D_{n-1,m} + D_{n-m,m} + 2L_{p,m}, & n = mp, \\ D_{n-1,m} + D_{n-m,m} + L_{p+1,m}, & n = mp + 1, \\ D_{n-1,m} + D_{n,m}, & n = mp + l, \ l = 2, 3, \ldots, m-1. \end{cases} \tag{21}$$

Example 3.17. *Let* $m = 4$ *and*

1° $n = 4 \cdot 2$

$$\begin{aligned} D_{8,4} &= L_{1,4}L_{8,4} + L_{2,4}L_{4,4} = 12, \\ D_{7,4} &= L_{1,4}L_{7,4} + L_{2,4}L_{3,4} = 7, \\ D_{4,4} &= L_{1,4}L_{4,4} = 3, \\ D_{8,4} &= D_{7,4} + D_{4,4} + 2L_{2,4} = 12. \end{aligned}$$

2° $n = 4 \cdot 2 + 1$

$$\begin{aligned} D_{9,4} &= L_{1,4}L_{9,4} + L_{2,4}L_{5,4} + L_{3,4}L_{1,4} = 18, \\ D_{5,4} &= L_{1,4}L_{5,4} + L_{2,4}L_{1,4} = 5, \\ D_{9,4} &= D_{8,4} + D_{5,4} + L_{3,4} = 18. \end{aligned}$$

3° *Let* $n = 4 \cdot 2 + 2$

$$\begin{aligned} D_{10,4} &= L_{1,4}L_{10,4} + L_{2,4}L_{6,4} + L_{3,4}L_{2,4} = 24, \\ D_{6,4} &= L_{1,4}L_{6,4} + L_{2,4}L_{2,4} = 6. \\ D_{10,4} &= D_{9,4} + D_{6,4} = 18 + 6 = 24. \end{aligned}$$

4° $n = 4 \cdot 2 + 3$

$$\begin{aligned} D_{11,4} &= L_{1,4}L_{11,4} + L_{2,4}L_{7,4} + L_{3,4}L_{3,4} = 31, \\ D_{11,4} &= D_{10,4} + D_{7,4} = 24 + 7 = 31. \end{aligned}$$

Remark 3.18. *We took the values of* $F_{n,m}$ *and* $L_{n,m}$ *from Table 2.*

4. Some properties of the generalized Fibonacci and Lucas numbers

In this section we give some properties of the generalized Fibonacci numbers and the generalized Lucas numbers.

Theorem 4.1. *The numbers $F_{n,m}$ and $L_{n,m}$ satisfy the following relations*

$$\sum_{i=0}^{n} F_{i,m} = \sum_{j=0}^{m-1} F_{n+2-m+j,m} - 1. \tag{22}$$

$$\sum_{i=0}^{n} L_{i,m} = \sum_{j=0}^{m-1} L_{n+2-m+j,m} - 1. \tag{23}$$

Proof. It is easy to prove (23) for $n = 1$.
Suppose that (23) is correct for $n \geq 1$.
Then, for $n + 1$, we get

$$\begin{aligned} \sum_{i=0}^{n} L_{i,m} + L_{n+1,m} &= \sum_{j=0}^{m-1} L_{n+2-m+j,m} - 1 + L_{n+1,m} \\ &= L_{n+2-m,m} + L_{n+3-m,m} + \cdots + L_{n,m} + L_{n+1,m} + L_{n+1,m} - 1 \\ &= L_{n+3-m,m} + L_{n+4-m,m} + \cdots + L_{n+1,m} + L_{n+2,m} - 1 \\ &= \sum_{j=0}^{m-1} L_{n+3-m+j,m} - 1, \quad (L_{n+1,m} + L_{n+2-m,m} = L_{n+2,m}). \end{aligned}$$

The formula (22) can be similarly proved.

Theorem 4.2. *Let $h_{n,m} = F_{n,m}$ or $h_{n,m} = L_{n,m}$. Then the following relations*

$$\sum_{i=0}^{n} \binom{n}{i} h_{r+(m-1)i,m} = h_{r+mn,m}, \tag{24}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i h_{r+mi,m} = (-1)^n h_{r+(m-1)n,m}, \quad r \in \mathbb{N}_0 \tag{25}$$

are correct, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. It is easy to prove the relation (24) for $n = 1$.
Suppose that (24) is correct for $n \geq 1$.
Then, for $n + 1$, we get

$$\begin{aligned} h_{r+m(n+1),m} &= h_{r+mn+m,m} \\ &= h_{r+mn+m-1,m} + h_{r+mn,m} \\ &= h_{r+mn+m-1,m} + \sum_{i=0}^{n} \binom{n}{i} h_{r+(m-1)i,m} \end{aligned}$$

$$= \sum_{i=0}^{n} \binom{n}{i} h_{r+(m-1)i,m} + \sum_{i=1}^{n+1} \binom{n}{i-1} h_{r+(m-1)i,m}$$
$$= \sum_{i=1}^{n} \left(\binom{n}{i} + \binom{n}{i-1} \right) h_{r+(m-1)i,m} + \binom{n}{0} h_{r+(m-1)\cdot 0,m} + \binom{n+1}{n+1} h_{r+(m-1)(n+1),m}$$
$$= \sum_{i=0}^{n+1} \binom{n+1}{i} h_{r+(m-1)i,m}.$$

The formula (25) can be similarly proved.

Theorem 4.3. *The sequence of numbers $F_{n,m}$ satisfies the following relation*

$$F_{n+s,m} = F_{s,m}F_{n+1,m} + F_{s+1-m,m}F_{n,m} + F_{s+2-m,m}F_{n-1,m} + \cdots + F_{s-1,m}F_{n+2-m,m}, \tag{26}$$

where $s \geq m-1$, $n \geq m$, $m \geq 2$.

Proof. Since

$$(1 - t - t^m) \cdot \sum_{n=0}^{\infty} F_{n+s,m}\, t^n = F_{s,m} + F_{s+1,m}\, t + F_{s+2,m}\, t2 + \cdots + F_{s+n,m}\, t^n + \cdots + F_{s+n+m,m}\, t^{n+m} + \ldots$$
$$- F_{s,m}\, t - F_{s+1,m}\, t^2 - F_{s+2,m}\, t^3 - \cdots - F_{n-1+s,m}\, t^n - \cdots - F_{n+s,m}\, t^{n+1} - \cdots - \sum_{n=0}^{\infty} F_{n+s,m}\, t^{n+m}$$
$$= F_{s,m} + t(F_{s+1,m} - F_{s,m}) + t^2(F_{s+2,m} - F_{s+1}) + \cdots + t^{m-1}(F_{m-1+s,m} - F_{m-2+s,m})$$
$$= F_{s,m} + F_{s+1-m,m}\, t + F_{s+2-m,m}\, t^2 + \cdots + F_{s-1,m}\, t^{m-1},$$

hence

$$\sum_{n=0}^{\infty} F_{n+s,m}\, t^n = \frac{1}{1-t-t^m} \cdot F_{s,m} + \frac{t}{1-t-t^m} \cdot F_{s+1-m,m} + \frac{t^2}{1-t-t^m} \cdot F_{s+2-m,m} + \cdots + \frac{t^{m-1}}{1-t-t^m} \cdot F_{s-1,m}$$
$$= F_{s,m}F_{n+1,m} + F_{s+1-m,m}F_{n,m} + F_{s+2-m,m}F_{n-1,m} + \cdots + F_{s-1,m}F_{n+2-m,m}.$$

Corollary 4.4. *For $m = 2$ and $m = 4$, from (26), respectively, we get:*

$$F_{n+s} = F_sF_{n+1} + F_{s-1}F_n;$$
$$F_{n+s,4} = F_{s,4}F_{n+1,4} + F_{s-3,4}F_{n,4} + F_{s-2,4}F_{n-1,4} + F_{s-1,4}F_{n-2,4}.$$

Example 4.5. *For $m = 4$, $n = 8$ and $s = 4$, from Table 2, we find*

$$F_{8+4,4} = F_{4,4}F_{9,4} + F_{1,4}F_{8,4} + F_{2,4}F_{7,4} + F_{3,4}F_{6,4} = 19 = F_{12,4}.$$

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