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Fractional Integral Identity, Estimation of Its Bounds and Some Applications to Trapezoidal Quadrature Rule

Artion Kashuri[a], Muhammad Uzair Awan[b], Muhammad Aslam Noor[c]

[a]*Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania*
[b]*Department of Mathematics, Government College University, Faisalabad, Pakistan*
[c]*Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan*

Abstract. The aim of this paper is to introduce a new extension of preinvexity called exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvexity. Some new integral inequalities of Hermite–Hadamard type for exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions via Riemann–Liouville fractional integral are established. Also, some new estimates with respect to trapezium-type integral inequalities for exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions via general fractional integrals are obtained. We show that the class of exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions includes several other classes of preinvex functions. We shown by two basic examples the efficiency of the obtained inequalities on the base of comparing those with the other corresponding existing ones. At the end, some new error estimates for trapezoidal quadrature formula are provided as well. This results may stimulate further research in different areas of pure and applied sciences.

1. Introduction

The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. *Let I be an interval in $\Re$. A function $f : I \to \Re$, is said to be convex on I if the inequality*

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b) \tag{1}$$

holds for all $a, b \in I$ and $t \in [0,1]$. Also, we say that f is concave, if the inequality in (1) holds in the reverse direction.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : I \subset \Re \to \Re$ be a convex function and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{2}$$

This inequality (2) is also known as trapezium inequality.

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Communicated by Miodrag Spalević
Corresponding Author: Muhammad Uzair Awan
Email addresses: artionkashuri@gmail.com (Artion Kashuri), awan.uzair@gmail.com (Muhammad Uzair Awan), noormaslam@gmail.com (Muhammad Aslam Noor)

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [4]-[12],[15]-[19],[22, 23, 28, 30, 31].

In [10], Dragomir and Agarwal proved the following results connected with the right part of (2).

Lemma 1.3. *Let $f : I^{\circ} \subset \Re \to \Re$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{3}$$

Theorem 1.4. *Let $f : I^{\circ} \subset \Re \to \Re$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $\left|f'\right|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{4}$$

Now, let us recall the following definitions.

Definition 1.5 ([22]). *A function: $f : I \subset \Re \to \Re$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0, 1)$, among $m \in (0, 1]$, satisfies the following inequality*

$$f\Big(tx + m(1-t)y\Big) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{5}$$

Definition 1.6 ([3]). *A set $K \subset \Re^n$ is said to be invex respecting the mapping $\eta : K \times K \to \Re^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.*

Definition 1.7 ([18]). *Let $h : [0, 1] \to \Re$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h–preinvex with respect to η, if*

$$f\Big(x + t\eta(y, x)\Big) \le h(1-t)f(x) + h(t)f(y) \tag{6}$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Definition 1.8 ([11]). *A set $K \subset \Re^n$ is named as m–invex with respect to the mapping $\eta : K \times K \to \Re^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

Remark 1.9. *Taking $m = 1$ in Definition 1.8, the mapping $\eta(y, mx)$ reduce to $\eta(y, x)$, and then we get Definition 1.6.*

Definition 1.10 ([25]). *Let $K \subset \Re$ be m–invex set respecting the mapping $\eta : K \times K \to \Re$ and $h_1, h_2 : [0, 1] \to [0, +\infty)$. A function $f : K \to \Re$ is said to be generalized (m, h_1, h_2)–preinvex, if*

$$f\Big(mx + t\eta(y, mx)\Big) \le mh_1(t)f(x) + h_2(t)f(y) \tag{7}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

Definition 1.11 ([16]). *Let $f \in L[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. *Here* $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
Note that $\alpha = 1$, *the fractional integral reduces to the classical integral.*

Furthermore, let us define a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t}dt < +\infty, \tag{8}$$

$$\frac{1}{A} \le \frac{\varphi(s)}{\varphi(r)} \le A \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2 \tag{9}$$

$$\frac{\varphi(r)}{r^2} \le B\frac{\varphi(s)}{s^2} \text{ for } s \le r \tag{10}$$

$$\left|\frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2}\right| \le C|r-s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2 \tag{11}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \ge 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \ge 0$, then φ satisfies (8)–(11), see reference [29]. Therefore, Sarikaya and Ertuğral [28] defined the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$${}_{a^+}I_{\varphi}f(x) = \int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt, \quad x > a \tag{12}$$

and

$${}_{b^-}I_{\varphi}f(x) = \int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt, \quad x < b. \tag{13}$$

This fractional integral operators are a new generalization of fractional integrals such as the Riemann–Liouville fractional integral, the k–Riemann–Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc. To read more about fractional analysis, see references [13, 14, 20, 27].

An important class of convex functions, which is called exponential convex functions, was introduced and studied by Antczak [2], Dragomir et al. [9] and Rashid et al. [26]. Alirezai and Mathar [1] have investigated their basic properties along with their potential applications in statistics and information theory. Awan et al. [5] and Pecarić and Jaksetić [24] defined another kind of exponential convex functions and have shown that the class of exponential convex functions unifies various unrelated concepts.

Definition 1.12 ([2, 9, 26]). *A function* $f : K \subset \Re \to \Re$ *is said to be exponentially convex, if*

$$e^{f((1-t)a+tb)} \le (1-t)e^{f(a)} + te^{f(b)} \tag{14}$$

holds for all $a, b \in K$, $t \in [0,1]$, *where* f *is positive.*

For the applications of exponentially convex functions in different field of statistics, information theory and mathematical sciences, see [1, 2, 5, 21] and the references therein.

Definition 1.13 ([26]). *A function $f : K \subset \Re \to \Re$ is said to be exponentially m–convex, where $m \in (0, 1]$, if*

$$e^{f((1-t)a+mtb)} \leq (1-t)e^{f(a)} + mte^{f(b)} \tag{15}$$

holds for all $a, b \in K$, $t \in [0, 1]$, where f is positive.

Motivated by the above literatures, the main objective of this article is to establish in Section 2 fractional integral inequalities using a new class of preinvex functions called exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex function. Also, using a new identity pertaining differentiable functions defined on m–invex set as auxiliary result, some new Hermite–Hadamard inequalities for exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions via Riemann–Liouville fractional integral will be obtained. Also, some new estimates with respect to trapezium-type integral inequalities for exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions via general fractional integrals will be given. Various special cases will be discussed. At the end of this section we will demonstrate by two basic examples the efficiency of the obtained inequalities on the base of comparing those with the other corresponding existing ones. In Section 3, some new error estimates for trapezoidal quadrature formula will be given. This results may stimulate further research in different areas of pure and applied sciences. In Section 4, a briefly conclusion is provided as well.

2. Main results

Now we introduce exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions.

Definition 2.1. *Let $K \subset \Re$ be m–invex set with respect to the mapping $\eta : K \times K \to \Re$ for some fixed $m \in (0, 1]$ and $h_1, h_2 : [0, 1] \to [0, +\infty)$. A function $f : K \to (0, +\infty)$ is called exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex, if*

$$e^{f(mx+t\eta(y,mx))} \leq mh_1(t)e^{\omega_1 f(x)} + h_2(t)e^{\omega_2 f(y)} \tag{16}$$

holds for all $x, y \in K$, $t \in [0, 1]$ and ω_1, $\omega_2 \in \Re$.

Remark 2.2. *In definition 2.1, if we choose $\omega_1 = \omega_2 = 1$, $h_1(t) = 1 - t$, $h_2(t) = t$ and $\eta(y, mx) = y - mx$, this definition reduce to the definition 1.13.*

Remark 2.3. *Under the conditions of remark 2.2, taking $m = 1$, we get definition 1.12.*

Remark 2.4. *Let us discuss some special cases of definition 2.1 as follows:*

(I) Taking $h_1(t) = h(1 - t)$, $h_2(t) = h(t)$, then we get exponentially $(m, \omega_1, \omega_2, h)$–preinvex functions.
(II) Taking $h_1(t) = h_2(t) = t(1 - t)$, then we get exponentially $(m, \omega_1, \omega_2, tgs)$–preinvex functions.
(III) Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get exponentially (m, ω_1, ω_2)–MT–preinvex functions.

In this section, we obtain Hermite–Hadamard type inequalities for exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex function via Riemann–Liouville fractional integral.

Theorem 2.5. *Let $K = [ma, ma + \eta(b, ma)] \subset \Re$ be m–invex set with respect to the mapping $\eta : K \times K \to \Re$ for some fixed $m \in (0, 1]$, where $a < b$ and $\eta(b, ma) > 0$. Suppose $h_1, h_2 : [0, 1] \to [0, +\infty)$ be continuous functions. Let $f, g : K \to (0, +\infty)$ be exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions. If $f, g \in L(K)$, then for ω_1, $\omega_2 \in \Re$ and $\alpha > 0$, the following inequality holds:*

$$\begin{aligned} &\frac{\Gamma(\alpha)}{\eta^{\alpha}(b, ma)} \times \left\{ J^{\alpha}_{(ma+\eta(b,ma))^-} e^{f(ma)} + J^{\alpha}_{(ma+\eta(b,ma))^-} e^{g(ma)} \right\} \\ &\leq m\left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right) H_1(\alpha) + \left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right) H_2(\alpha), \end{aligned} \tag{17}$$

where

$$H_i(\alpha) = \int_0^1 t^{\alpha-1} h_i(t)dt, \quad \forall\, i = 1, 2. \tag{18}$$

Proof. From exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvexity of f and g for all $t \in [0, 1]$, we have

$$e^{f(ma+t\eta(b,ma))} \leq mh_1(t)e^{\omega_1 f(a)} + h_2(t)e^{\omega_2 f(b)}$$

and

$$e^{g(ma+t\eta(b,ma))} \leq mh_1(t)e^{\omega_1 g(a)} + h_2(t)e^{\omega_2 g(b)}.$$

Adding both sides of the above inequalities, we get

$$e^{f(ma+t\eta(b,ma))} + e^{g(ma+t\eta(b,ma))} \leq m\left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right)h_1(t) + \left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right)h_2(t). \tag{19}$$

Multiplying both sides of inequality (19) with $t^{\alpha-1}$ and integrating over $[0, 1]$, we obtain

$$\int_0^1 t^{\alpha-1}\left[e^{f(ma+t\eta(b,ma))} + e^{g(ma+t\eta(b,ma))}\right]dt$$

$$\leq m\left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right)\int_0^1 t^{\alpha-1}h_1(t)dt + \left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right)\int_0^1 t^{\alpha-1}h_2(t)dt.$$

Using definition 1.11, we get the required result. □

Corollary 2.6. *In Theorem 2.5, if we choose $m = 1$ and $\eta(b, ma) = b - ma$, we get*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \times \left\{J^{\alpha}_{b^-}e^{f(a)} + J^{\alpha}_{b^-}e^{g(a)}\right\}$$

$$\leq \left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right)H_1(\alpha) + \left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right)H_2(\alpha). \tag{20}$$

Corollary 2.7. *In Theorem 2.5, if we choose $\alpha = 1$, we obtain*

$$\frac{1}{\eta(b,ma)}\int_{ma}^{ma+\eta(b,ma)}\left[e^{f(t)} + e^{g(t)}\right]dt$$

$$\leq m\left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right)H_1 + \left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right)H_2, \tag{21}$$

where

$$H_i = \int_0^1 h_i(t)dt, \quad \forall\, i = 1, 2. \tag{22}$$

Theorem 2.8. *Let $K = [ma, ma + \eta(b, ma)] \subset \mathfrak{R}$ be m–invex set with respect to the mapping $\eta : K \times K \to \mathfrak{R}$ for some fixed $m \in (0, 1]$, where $a < b$ and $\eta(b, ma) > 0$. Suppose $h_1, h_2 : [0, 1] \to [0, +\infty)$ be continuous functions. Let $f, g : K \to (0, +\infty)$ be exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex functions. If $f, g \in L(K)$, then for ω_1, $\omega_2 \in \mathfrak{R}$ and $\alpha > 0$, the following inequality holds:*

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,ma)} \times \left\{J^{\alpha}_{(ma)^+}e^{f(ma+\eta(b,ma))} + J^{\alpha}_{(ma+\eta(b,ma))^-}e^{g(ma)}\right\}$$

$$\leq m\left(e^{\omega_1 f(a)}C_1(\alpha) + e^{\omega_1 g(a)}H_1(\alpha)\right) + e^{\omega_2 f(b)}C_2(\alpha) + e^{\omega_2 g(b)}H_2(\alpha), \tag{23}$$

where

$$C_i(\alpha) = \int_0^1 (1-t)^{\alpha-1}h_i(t)dt, \quad \forall\, i = 1, 2, \tag{24}$$

and $H_1(\alpha)$, $H_2(\alpha)$ are defined as in Theorem 2.5.

Proof. From exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvexity of f and g for all $t \in [0, 1]$, we have

$$e^{f(ma+t\eta(b,ma))} \le mh_1(t)e^{\omega_1 f(a)} + h_2(t)e^{\omega_2 f(b)}$$

and

$$e^{g(ma+t\eta(b,ma))} \le mh_1(t)e^{\omega_1 g(a)} + h_2(t)e^{\omega_2 g(b)}.$$

Multiplying first above inequality with $(1-t)^{\alpha-1}$, the second with $t^{\alpha-1}$ and adding both sides, we get

$$(1-t)^{\alpha-1}e^{f(ma+t\eta(b,ma))} + t^{\alpha-1}e^{g(ma+t\eta(b,ma))} \le (1-t)^{\alpha-1}\Big[mh_1(t)e^{\omega_1 f(a)} + h_2(t)e^{\omega_2 f(b)}\Big] \tag{25}$$

$$+t^{\alpha-1}\Big[mh_1(t)e^{\omega_1 g(a)} + h_2(t)e^{\omega_2 g(b)}\Big].$$

Integrating over $[0, 1]$ both sides of inequality (25) and using definition 1.11, we obtain the required result. □

Corollary 2.9. *In Theorem 2.8, if we choose $m = 1$ and $\eta(b, ma) = b - ma$, we get*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} \times \Big\{J^\alpha_{a^+} e^{f(b)} + J^\alpha_{b^-} e^{g(a)}\Big\}$$

$$\le e^{\omega_1 f(a)}C_1(\alpha) + e^{\omega_1 g(a)}H_1(\alpha) + e^{\omega_2 f(b)}C_2(\alpha) + e^{\omega_2 g(b)}H_2(\alpha). \tag{26}$$

Corollary 2.10. *In Theorem 2.8, if we choose $\alpha = 1$, we obtain Corollary 2.7.*

Remark 2.11. *Under the conditions of Theorems 2.5 and 2.8, using remark 2.4, we can derive several new integral inequalities. The details are left to the interested reader.*

For establishing some new results regarding generalizations of Hermite–Hadamard type integral inequalities associated with exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvexity via general fractional integrals, we need the following lemma.

Lemma 2.12. *Let $K = [ma, ma + \eta(b, ma)] \subset \Re$ be m–invex set with respect to the mapping $\eta : K \times K \to \Re$ for some fixed $m \in (0, 1]$, where $a < b$ and $\eta(b, ma) > 0$. If $f : K \to \Re$ is a differentiable function on K° such that $f' \in L(K)$, then the following identity for generalized fractional integrals holds:*

$$\frac{e^{f(ma)} + e^{f(ma+\eta(b,ma))}}{2} - \frac{1}{2\Lambda_m(1)\eta(b, ma)} \times \Big\{ {}_{(ma)^+}I_\varphi\, e^{f(ma+\eta(b,ma))} + {}_{(ma+\eta(b,ma))^-}I_\varphi\, e^{f(ma)}\Big\}$$

$$= \frac{\eta(b, ma)}{2\Lambda_m(1)} \int_0^1 \Big[\Lambda_m(t) - \Lambda_m(1-t)\Big]e^{f(ma+t\eta(b,ma))}f'(ma + t\eta(b, ma))dt, \tag{27}$$

where

$$\Lambda_m(t) = \int_0^t \frac{\varphi\left(\eta(b, ma)u\right)}{u}du < +\infty. \tag{28}$$

We denote

$$\Xi_{f,\Lambda_m}(a, b) = \frac{\eta(b, ma)}{2\Lambda_m(1)} \int_0^1 \Big[\Lambda_m(t) - \Lambda_m(1-t)\Big]e^{f(ma+t\eta(b,ma))}f'(ma + t\eta(b, ma))dt. \tag{29}$$

Proof. From (29), we have

$$\Xi_{f,\Lambda_m}(a,b) = \frac{\eta(b,ma)}{2\Lambda_m(1)} \times \left[\int_0^1 \Lambda_m(t) e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt \right.$$

$$\left. - \int_0^1 \Lambda_m(1-t) e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt \right]$$

$$= \frac{\eta(b,ma)}{2} \times \left[\Xi^{(1)}_{f,\Lambda_m}(a,b) - \Xi^{(2)}_{f,\Lambda_m}(a,b) \right], \tag{30}$$

where

$$\Xi^{(1)}_{f,\Lambda_m}(a,b) = \int_0^1 \Lambda_m(t) e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt \tag{31}$$

and

$$\Xi^{(2)}_{f,\Lambda_m}(a,b) = \int_0^1 \Lambda_m(1-t) e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt. \tag{32}$$

Now, integrating by parts (31), changing the variable $u = ma + t\eta(b,ma)$ and using definition 12, we get

$$\Xi^{(1)}_{f,\Lambda_m}(a,b) = \frac{\Lambda_m(t) e^{f(ma+t\eta(b,ma))}}{\eta(b,ma)} \Big|_0^1 - \frac{1}{\eta(b,ma)} \int_0^1 \frac{\varphi\left(\eta(b,ma)t\right)}{t} e^{f(ma+t\eta(b,ma))} dt$$

$$= \frac{\Lambda_m(1) e^{f(ma+\eta(b,ma))}}{\eta(b,ma)} - \frac{1}{\eta^2(b,ma)} \, {}_{(ma+\eta(b,ma))^-}I_\varphi \, e^{f(ma)}. \tag{33}$$

Similarly, using (32), we obtain

$$\Xi^{(2)}_{f,\Lambda_m}(a,b) = -\frac{\Lambda_m(1) e^{f(ma)}}{\eta(b,ma)} + \frac{1}{\eta^2(b,ma)} \, {}_{(ma)^+}I_\varphi \, e^{f(ma+\eta(b,ma))}. \tag{34}$$

Substituting (33) and (34) in (30), we get (27). The completes the proof. □

Remark 2.13. *In Lemma 2.12, if we choose* $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ *for* $\alpha > 0$, *we get the following identity for fractional integrals:*

$$\frac{e^{f(ma)} + e^{f(ma+\eta(b,ma))}}{2} - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,ma)} \times \left\{ J^\alpha_{(ma)^+} e^{f(ma+\eta(b,ma))} + J^\alpha_{(ma+\eta(b,ma))^-} e^{f(ma)} \right\}$$

$$= \frac{\eta(b,ma)}{2} \int_0^1 \left[t^\alpha - (1-t)^\alpha \right] e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt. \tag{35}$$

Using Lemma 2.12, we now state the following theorems.

Theorem 2.14. *Let* $K = [ma, ma+\eta(b,ma)] \subset \mathfrak{R}$ *be m–invex set with respect to the mapping* $\eta : K \times K \to \mathfrak{R}$ *for some fixed* $m \in (0,1]$, *where* $a < b$ *and* $\eta(b,ma) > 0$. *Suppose* $h_1, h_2 : [0,1] \to [0,+\infty)$ *be continuous functions. Let* $f : K \to (0,+\infty)$ *be a differentiable exponentially* $(m, \omega_1, \omega_2, h_1, h_2)$*–preinvex function on* K° *such that* $f' \in L(K)$ *and* $\omega_1, \omega_2 \in \mathfrak{R}$. *If* $|f'|^q$ *is generalized* (m, h_1, h_2)*–preinvex function, then for* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *the following inequality holds:*

$$\left| \Xi_{f,\Lambda_m}(a,b) \right| \leq \frac{\eta(b,ma)}{2\Lambda_m(1)} \sqrt[p]{B_{\Lambda_m}(p)} \tag{36}$$

$$\times \sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q G_{h_2}},$$

where

$$B_{\Lambda_m}(p) = \int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right|^p dt, \tag{37}$$

$$F_{h_1,h_2} = \int_0^1 h_1(t)h_2(t)dt, \quad G_{h_i} = \int_0^1 [h_i(t)]^2 dt, \quad \forall\, i = 1,2 \tag{38}$$

and

$$\Delta_f(q;\omega_1,\omega_2,a,b) = e^{q\omega_1 f(a)}\left|f'(b)\right|^q + e^{q\omega_2 f(b)}\left|f'(a)\right|^q. \tag{39}$$

Proof. From Lemma 2.12, exponentially $(m,\omega_1,\omega_2,h_1,h_2)$–preinvexity of f, generalized (m,h_1,h_2)–preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|\Xi_{f,\Lambda_m}(a,b)\right| \le \frac{\eta(b,ma)}{2\Lambda_m(1)} \int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right|\left|e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))\right|dt$$

$$\le \frac{\eta(b,ma)}{2\Lambda_m(1)} \sqrt[p]{B_{\Lambda_m}(p)} \left(\int_0^1 e^{qf(ma+t\eta(b,ma))}\left|f'(ma+t\eta(b,ma))\right|^q dt\right)^{\frac{1}{q}}$$

$$\le \frac{\eta(b,ma)}{2\Lambda_m(1)} \sqrt[p]{B_{\Lambda_m}(p)}$$

$$\times \left(\int_0^1 \left[mh_1(t)e^{q\omega_1 f(a)} + h_2(t)e^{q\omega_2 f(b)}\right]\left[mh_1(t)\left|f'(a)\right|^q + h_2(t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{\eta(b,ma)}{2\Lambda_m(1)} \sqrt[p]{B_{\Lambda_m}(p)}$$

$$\times \sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q G_{h_2}}.$$

The proof of Theorem 2.14 is completed. □

We point out some special cases of Theorem 2.14.

Corollary 2.15. *In Theorem 2.14, if we choose $\varphi(t) = t$, we get*

$$\left|\frac{e^{f(ma)} + e^{f(ma+\eta(b,ma))}}{2} - \frac{1}{\eta(b,ma)} \int_{ma}^{ma+\eta(b,ma)} e^{f(t)}dt\right| \le \frac{\eta(b,ma)}{2\sqrt[p]{p+1}} \tag{40}$$

$$\times \sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q G_{h_2}}.$$

Corollary 2.16. *In Theorem 2.14, if we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$ and $\eta(b,ma) = b - ma$, we obtain*

$$\left|\frac{e^{f(ma)} + e^{f(b)}}{2} - \frac{\Gamma(\alpha+1)}{2(b-ma)^\alpha} \times \left\{J^\alpha_{(ma)^+} e^{f(b)} + J^\alpha_{b^-} e^{f(ma)}\right\}\right| \le \frac{(b-ma)}{2} \sqrt[p]{B(p,\alpha)} \tag{41}$$

$$\times \sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q G_{h_2}},$$

where

$$B(p,\alpha) = \int_0^1 \left|t^\alpha - (1-t)^\alpha\right|^p dt.$$

Theorem 2.17. *Let $K = [ma, ma + \eta(b, ma)] \subset \Re$ be m–invex set with respect to the mapping $\eta : K \times K \to \Re$ for some fixed $m \in (0, 1]$, where $a < b$ and $\eta(b, ma) > 0$. Suppose $h_1, h_2 : [0, 1] \to [0, +\infty)$ be continuous functions. Let $f : K \to (0, +\infty)$ be a differentiable exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvex function on K° such that $f' \in L(K)$ and $\omega_1, \omega_2 \in \Re$. If $|f'|^q$ is generalized (m, h_1, h_2)–preinvex function, then for $q \geq 1$, the following inequality holds:*

$$\left|\Xi_{f,\Lambda_m}(a,b)\right| \leq \frac{\eta(b,ma)}{2\Lambda_m(1)}\Big[B_{\Lambda_m}(1)\Big]^{1-\frac{1}{q}} \tag{42}$$

$$\times\sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q P_{\Lambda_m,h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)S_{\Lambda_m,h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q P_{\Lambda_m,h_2}},$$

where

$$S_{\Lambda_m,h_1,h_2} = \int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right| h_1(t)h_2(t)dt, \tag{43}$$

$$P_{\Lambda_m,h_i} = \int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right|[h_i(t)]^2 dt, \quad \forall\, i = 1,2 \tag{44}$$

and $\Delta_f(q;\omega_1,\omega_1,a,b)$, $B_{\Lambda_m}(1)$ are defined as in Theorem 2.14.

Proof. From Lemma 2.12, exponentially $(m, \omega_1, \omega_2, h_1, h_2)$–preinvexity of f, generalized (m, h_1, h_2)–preinvexity of $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$\left|\Xi_{f,\Lambda_m}(a,b)\right| \leq \frac{\eta(b,ma)}{2}\int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right|\left|e^{f(ma+t\eta(b,ma))}f'(ma+t\eta(b,ma))\right|dt$$

$$\leq \frac{\eta(b,ma)}{2\Lambda_m(1)}\Big[B_{\Lambda_m}(1)\Big]^{1-\frac{1}{q}}\left(\int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right|e^{qf(ma+t\eta(b,ma))}\left|f'(ma+t\eta(b,ma))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(b,ma)}{2\Lambda_m(1)}\Big[B_{\Lambda_m}(1)\Big]^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\Lambda_m(t) - \Lambda_m(1-t)\right|\Big[mh_1(t)e^{q\omega_1 f(a)} + h_2(t)e^{q\omega_2 f(b)}\Big]\Big[mh_1(t)\left|f'(a)\right|^q + h_2(t)\left|f'(b)\right|^q\Big]dt\right)^{\frac{1}{q}}$$

$$= \frac{\eta(b,ma)}{2\Lambda_m(1)}\Big[B_{\Lambda_m}(1)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q P_{\Lambda_m,h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)S_{\Lambda_m,h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q P_{\Lambda_m,h_2}}.$$

The proof of Theorem 2.17 is completed. □

We point out some special cases of Theorem 2.17.

Corollary 2.18. *In Theorem 2.17, if we choose $\varphi(t) = t$, we get*

$$\left|\frac{e^{f(ma)} + e^{f(ma+\eta(b,ma))}}{2} - \frac{1}{\eta(b,ma)}\int_{ma}^{ma+\eta(b,ma)} e^{f(t)}dt\right| \leq 2^{\frac{1-2q}{q}}\eta(b,ma) \tag{45}$$

$$\times\sqrt[q]{m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q \Omega_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)\Theta_{h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q \Omega_{h_2}},$$

where

$$\Theta_{h_1,h_2} = \int_0^1 \left|2t-1\right|h_1(t)h_2(t)dt, \quad \Omega_{h_i} = \int_0^1 \left|2t-1\right|[h_i(t)]^2 dt, \quad \forall\, i = 1,2. \tag{46}$$

Corollary 2.19. *In Theorem 2.17, if we choose* $\varphi(t) = \dfrac{t^{\alpha}}{\Gamma(\alpha)}$ *for* $\alpha > 0$ *and* $\eta(b, ma) = b - ma$*, we obtain*

$$\left|\frac{e^{f(ma)} + e^{f(b)}}{2} - \frac{\Gamma(\alpha+1)}{2(b-ma)^{\alpha}} \times \left\{J^{\alpha}_{(ma)^+} e^{f(b)} + J^{\alpha}_{b^-} e^{f(ma)}\right\}\right| \le \frac{(b-ma)}{2}\Big[B_{\Lambda_m}(1)\Big]^{1-\frac{1}{q}} \tag{47}$$

$$\times \sqrt[q]{m^2 e^{q\omega_1 f(a)} \left|f'(a)\right|^q P_{\Lambda_m, h_1} + m\Delta_f(q; \omega_1, \omega_2, a, b) S_{\Lambda_m, h_1, h_2} + e^{q\omega_2 f(b)} \left|f'(b)\right|^q P_{\Lambda_m, h_2}}.$$

Remark 2.20. *Under the conditions of Theorems 2.14 and 2.17, using Remark 2.4, for the appropriate choices of function* $\varphi(t) = t$, $\dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi(t) = \dfrac{t}{\alpha} \exp\left[\left(-\dfrac{1-\alpha}{\alpha}\right)t\right]$*, where* $\alpha \in (0, 1)$*, we can get several new integral inequalities. The details are left to the interested reader.*

We now demonstrate the sharpness of the obtained inequalities by comparing with the other existing ones. For simplicity, let us take, respectively, $h_1(t) = 1 - t$, $h_2(t) = t$ and $\eta(y, mx) = y - mx$ for all $m \in (0, 1]$, where ω_1, $\omega_2 \in (-\infty, 1)$. From exponentially $(m, \omega_1, \omega_2, 1-t, t)$–convexity of function f, follows the definition 1.13. Indeed

$$e^{f(mx+t(y-mx))} \le m(1-t)e^{\omega_1 f(x)} + te^{\omega_2 f(y)} \le m(1-t)e^{f(x)} + te^{f(y)}. \tag{48}$$

From inequality (48), we have that, if the function f is exponentially $(m, \omega_1, \omega_2, 1-t, t)$–convex then it is exponentially m-convex as well. Also, if in addition, we take special value $m = 1$, we get

$$e^{f(x+t(y-x))} \le (1-t)e^{\omega_1 f(x)} + te^{\omega_2 f(y)} \le (1-t)e^{f(x)} + te^{f(y)}. \tag{49}$$

Inequality (49), show that, if the function f is exponentially $(1, \omega_1, \omega_2, 1-t, t)$–convex then it is exponentially convex.

The above inequalities (48) and (49) confirms the efficiency of the obtained inequalities on the base of comparing those with the other corresponding existing ones in known literatures about convexity and exponentially convexity, see [1, 2, 5, 9, 21, 26], which is the most important of all in the applied mathematics. Let see the following interesting example.

Example 2.21 ([1]). *The error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $erf(\sqrt{x})$, $x \ge 0$*, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. But from the reverse of the left-side of above inequality (49), if the function* f *is exponentially* $(1, \omega_1, \omega_2, 1-t, t)$*–concave then it is exponentially concave. This shows that the exponentially* $(1, \omega_1, \omega_2, 1-t, t)$*–concave functions can play important part in communication theory and information theory.*

Let us do another interpretation of the above error function. Without lost of the generality we take, respectively, $x = 0$, $y = 1$, $m = 1$ *and* ω_1 *an arbitrary real number for all* $t \in (0, 1)$*. We define* $erf(1) = \xi > 0$*. Since* $erf(\sqrt{x})$ *for* $x \ge 0$*, is an exponentially concave function and* $erf(0) = 0$*, we have*

$$(1-t)e^{\omega_1 f(0)} + te^{\omega_2 f(1)} \ge (1-t)e^{f(0)} + te^{f(1)}.$$

Hence

$$1 - t + te^{\omega_2 \xi} \ge 1 - t + te^{\xi}.$$

From the last inequality it is evident that for all $\omega_2 \ge 1$ *and* $t \in (0, 1)$ *this inequality is satisfied.*

Now we recall the concept of θ–exponentially convex functions, which is mainly due to Awan et al. [5] and after we will give another example in order to show efficiency of our new definition.

Definition 2.22 ([5]). *Let $\theta \in \Re$. Then a real–valued function $f : [0,+\infty) \to \Re$ is said to be θ–exponentially convex, if*

$$f\Big(tx + (1-t)y\Big) \leq te^{\theta x} f(x) + (1-t)e^{\theta y} f(y) \tag{50}$$

is valid for all $x, y \in [0,+\infty)$ and $t \in [0,1]$.

Example 2.23. *The function $f : \Re \to \Re$, defined by $f(t) = -t^2$ is a concave function, thus this is an θ–exponentially convex for all $\theta > 0$. Without lost of the generality, we choose, respectively, $x = 0$, $y = 1$, $m = 1$, ω_1 an arbitrary real number and $\omega_2 \geq 1$ for all $t \in (0,1)$. It is clear that f it is exponentially $(1, \omega_1, \omega_2, 1-t, t)$–convex function as well. Moreover it is evident that every exponentially $(1, \omega_1, \omega_2, 1-t, t)$–convex function is θ–exponentially convex for all $\theta > 0$ and $\omega_2 \geq 1$, where ω_1 is an arbitrary real number for all $t \in (0,1)$. This shows the efficiency of our new definition.*

3. Applications

In this section, we provide some new error estimates for trapezoidal quadrature formula. Let Q be the partition of the points $a = x_0 < x_1 < \ldots < x_k = b$ of the interval $[a,b]$. Let consider the following quadrature formula:

$$\int_a^b e^{f(x)} dx = T(f,Q) + E(f,Q),$$

where

$$T(f,Q) = \sum_{i=0}^{k-1} \frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2}(x_{i+1} - x_i)$$

is the trapezoidal version and $E(f,Q)$ is denote the associated approximation error.

Proposition 3.1. *Let $f : [a,b] \to (0,+\infty)$ be a differentiable exponentially $(\omega_1, \omega_2, h_1, h_2)$–convex function on (a,b), where $a < b$ and $\omega_1, \omega_2 \in \Re$. Suppose $h_1, h_2 : [0,1] \to [0,+\infty)$ be continuous functions. If $|f'|^q$ is generalized (h_1, h_2)–convex on $[a,b]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\Big|E(f,Q)\Big| \leq \frac{1}{2\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \tag{51}$$

$$\times \sqrt[q]{e^{q\omega_1 f(x_i)} \Big|f'(x_i)\Big|^q G_{h_1} + \Delta_f(q; \omega_1, \omega_2, x_i, x_{i+1}) F_{h_1,h_2} + e^{q\omega_2 f(x_{i+1})} \Big|f'(x_{i+1})\Big|^q G_{h_2}},$$

where

$$\Delta_f(q; \omega_1, \omega_2, x_i, x_{i+1}) = e^{q\omega_1 f(x_i)} \Big|f'(x_{i+1})\Big|^q + e^{q\omega_2 f(x_{i+1})} \Big|f'(x_i)\Big|^q \tag{52}$$

and F_{h_1,h_2}, G_{h_1}, G_{h_2} are defined as in Theorem 2.14.

Proof. Applying Theorem 2.14 for $m = 1$, $\eta(b, ma) = b - ma$ and $\varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ ($i = 0, \ldots, k-1$) of the partition Q, we have

$$\left|\frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} e^{f(x)} dx\right| \leq \frac{(x_{i+1} - x_i)}{2\sqrt[p]{p+1}} \tag{53}$$

$$\times\sqrt[q]{e^{q\omega_1 f(x_i)}\left|f'(x_i)\right|^q G_{h_1}+\Delta(q;\omega,x_i,x_{i+1})F_{h_1,h_2}+e^{q\omega_2 f(x_{i+1})}\left|f'(x_{i+1})\right|^q G_{h_2}}.$$

Hence from (53), we get

$$\left|E(f,Q)\right|=\left|\int_a^b e^{f(x)}dx-T(f,Q)\right|$$

$$\leq\left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}}e^{f(x)}dx-\frac{e^{f(x_i)}+e^{f(x_{i+1})}}{2}(x_{i+1}-x_i)\right\}\right|$$

$$\leq\sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}}e^{f(x)}dx-\frac{e^{f(x_i)}+e^{f(x_{i+1})}}{2}(x_{i+1}-x_i)\right\}\right|$$

$$\leq\frac{1}{2\sqrt[p]{p+1}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2$$

$$\times\sqrt[q]{e^{q\omega_1 f(x_i)}\left|f'(x_i)\right|^q G_{h_1}+\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})F_{h_1,h_2}+e^{q\omega_2 f(x_{i+1})}\left|f'(x_{i+1})\right|^q G_{h_2}}.$$

The proof of Proposition (3.1) is completed. □

Proposition 3.2. *Let $f:[a,b]\to(0,+\infty)$ be a differentiable exponentially $(\omega_1,\omega_2,h_1,h_2)$–convex function on (a,b), where $a<b$ and $\omega_1,\omega_2\in\Re$. Suppose $h_1,h_2:[0,1]\to[0,+\infty)$ be continuous functions. If $|f'|^q$ is generalized (h_1,h_2)–convex on $[a,b]$ for $q\geq 1$, then the following inequality holds:*

$$\left|E(f,Q)\right|\leq 2^{\frac{1-2q}{q}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{54}$$

$$\times\sqrt[q]{e^{q\omega_1 f(x_i)}\left|f'(x_i)\right|^q\Omega_{h_1}+\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})\Theta_{h_1,h_2}+e^{q\omega_2 f(x_{i+1})}\left|f'(x_{i+1})\right|^q\Omega_{h_2}},$$

where $\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})$ is defined from (52) and Θ_{h_1,h_2}, Ω_{h_1}, Ω_{h_2} are defined from (46).

Proof. The proof is analogous as to that of Proposition 3.1, taking $m=1$, $\eta(b,ma)=b-ma$ and $\varphi(t)=t$ in Theorem 2.17. □

Remark 3.3. *Under the conditions of Theorems 2.14 and 2.17, using Remark 2.4, for the appropriate choices of function $\varphi(t)=t$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$, where $\alpha\in(0,1)$, we can deduce several new bounds for the trapezoidal quadrature formula using above idea and technique. The details are left to the interested reader.*

4. Conclusion

We have established some fractional integral inequalities using a new class of preinvex functions called exponentially $(m,\omega_1,\omega_2,h_1,h_2)$–preinvex. By applying the new identity pertaining to differentiable functions, some new Hermite–Hadamard inequalities via exponentially $(m,\omega_1,\omega_2,h_1,h_2)$–preinvex functions involving Riemann–Liouville fractional integral are obtained. Also, some new estimates with respect to trapezium-type integral inequalities for exponentially $(m,\omega_1,\omega_2,h_1,h_2)$–preinvex functions via general fractional integrals are given. In order to show the unification of our main results various special cases were also discussed. Using two interesting examples we have also shown the efficiency of the obtained results. At the end, some new error estimates for trapezoidal quadrature formula were provided as well. Exponentially $(m,\omega_1,\omega_2,h_1,h_2)$–preinvex functions can be employed for statistical analysis, recurrent neural networks, and experimental designs. We believe that this general class will be very useful and will be explored due to its dominant characteristics.

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