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A Numerical Study of a Coupled System of Fractional Differential Equations

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Abstract. We consider a certain class of coupled systems of fractional differential equations involving ψ-Caputo fractional derivatives. A numerical approach is provided for solving this class of systems. The method is based on operational matrix of fractional integration of an arbitrary ψ-polynomial basis. A theoretical study related to the numerical scheme and the convergence of the method is presented. Next, several numerical examples are given using different types of polynomials aiming to confirm the efficiency of our approach.

1. Introduction

We consider the coupled system of fractional differential equations

$$\begin{cases} {}^{C}D_a^{\alpha,\psi} x(t) = A_{11}x(t) + A_{12}y(t) + p_1(t) \\ {}^{C}D_a^{\beta,\psi} y(t) = A_{21}x(t) + A_{22}y(t) + p_2(t) \end{cases} \quad ; \quad a < t < b \tag{1}$$

subject to the initial conditions

$$(\delta_\psi)^i x(a) = x_i, \quad (\delta_\psi)^i y(a) = y_i, \quad i \in \{0, 1, \cdots, m-1\}, \tag{2}$$

where x and y are the unknown functions, A_{ij}, x_i, y_i are given constants, $p_i : [a, b] \to \mathbb{R}$ are given functions, m is a positive integer and $\alpha, \beta \in \left(\max\{\frac{1}{2}, m-1\}, m\right)$. Here ψ is a C^1 function in $[a, b]$, $\psi([a, b]) = [0, 1]$, $\psi' > 0$, ${}^{C}D_a^{r,\psi}$, $r \in \{\alpha, \beta\}$, is the ψ-Caputo fractional derivative of order r, and

$$(\delta_\psi)^0 z(t) = z(t), \quad (\delta_\psi)^i z(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^i z(t), \quad i = 1, 2, \cdots m-1,$$

for $z \in \{x, y\}$. Systems of type (1) were used as fractional models of different real world phenomena, such as oscillator theory [19], pollution [6, 14, 29], circuit simulations [7], etc. Our aim is to provide a numerical method for solving (1)–(2), as well as a rigorous justification of its convergence.

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After discovering the importance of fractional calculus in applications (see, e.g.[5, 12, 13, 15, 20, 21, 24, 25]), this theory attracted much attention from researchers both in mathematics and in other disciplines. In particular, several contributions related to the development of numerical techniques for solving fractional differential equations were published. One of the popular methods is the operational matrix approach, which consists to transform the problem to an equation of algebraic-type by projecting it on an adequate polynomial basis. This method has been first used for solving standard differential equations (see e.g. [22, 23]). Next, due to the properties of fractional operators, it was shown that this technique is still useful for solving large classes of fractional differential equations (see e.g. [1, 3, 8, 9, 11, 14, 17, 19, 26, 28] and the references therein). In particular, in [3], the authors investigated a certain class of fractional differential equations involving ψ-Caputo fractional derivatives. Namely, in order to obtain numerical solutions, they used the operational matrix approach by introducing ψ-shifted Legendre polynomials.

Motivated by the above cited works, we propose in this paper a numerical approach based on the operational matrix technique for solving (1)–(2). The convergence is established for an arbitrary ψ-polynomial basis. Moreover, several numerical experiments using different types of polynomials are provided.

The rest of the paper is organized as follows. In Section 2, we establish some preliminary results. In Section 3, some properties related to ψ-polynomials are derived. In Section 4, a general formula of the ψ-fractional integral of ψ-polynomials is obtained. The proof of the convergence is given in Section 5. Next, numerical tests are presented using two different types of polynomials: ψ-shifted Legendre polynomials and ψ-shifted Jacobi polynomials with parameters $(0, -\frac{1}{2})$.

2. Preliminaries

First, let us fix some notations. We denote by $\mathbb{N}$ the set of positive integers. Let J be a finite interval in $\mathbb{R}$ and $\omega : J \to [0, \infty)$ be a measurable function. For $p > 1$, let $L^p(J, \omega(\sigma)\, d\sigma)$ be the weighted Lebesgue space of measurable functions $\mathcal{R}$ in J satisfying

$$\int_J |\mathcal{R}(\sigma)|^p \omega(\sigma)\, d\sigma < \infty$$

If $\omega \equiv 1$, then $L^p(J, \omega(\sigma)\, d\sigma) = L^p(J)$, is the standard Lebesgue space. The space $L^p(J, \omega(\sigma)\, d\sigma)$ is equipped with the norm

$$\|\mathcal{R}\|_{L^p(J,\omega(\sigma)d\sigma)} = \left(\int_J |\mathcal{R}(\sigma)|^p \omega(\sigma)\, d\sigma\right)^{\frac{1}{p}}, \quad \forall\, \mathcal{R} \in L^p(J, \omega(\sigma)d\sigma).$$

The scalar product in $L^2(J, \omega(\sigma)\, d\sigma)$ is defined by

$$(\mathcal{R}, \mathcal{S})_{L^2(J,\omega(\sigma)d\sigma)} = \int_J \mathcal{R}(\sigma)\mathcal{S}(\sigma)\omega(\sigma)\, d\sigma, \quad \forall\, \mathcal{R}, \mathcal{S} \in L^2(J, \omega(\sigma)\, d\sigma).$$

Further, let ψ be a C^1 function in $I = [a, b]$ such that

(i) $\psi' > 0$.

(ii) $\psi(a) = 0$ and $\psi(b) = 1$.

Let $\omega \in L^1([0, 1])$ be a positive function such that $\omega^{-1} \in C([0, 1])$. Denote $\mu(\sigma) = \psi'(\sigma)\omega(\psi(\sigma))$, $\forall \sigma \in I$. For $\mathcal{R} : I \to \mathbb{R}$, let

$$T\mathcal{R} = \mathcal{R} \circ \psi^{-1}.$$

Lemma 2.1. *For all $\mathcal{R}, \mathcal{S} \in L^2(I, \mu(\sigma)\, d\sigma)$, it holds*

$$(\mathcal{R}, \mathcal{S})_{L^2(I,\mu(\sigma)d\sigma)} = (T\mathcal{R}, T\mathcal{S})_{L^2([0,1],\omega(s)ds)}.$$

Proof. Let $\mathcal{R}, \mathcal{S} \in L^2(I, \mu(\sigma)\, d\sigma)$. Then

$$\begin{aligned}(\mathcal{R}, \mathcal{S})_{L^2(I,\mu(\sigma)d\sigma)} &= \int_a^b \mathcal{R}(\sigma)\mathcal{S}(\sigma)\mu(\sigma)\, d\sigma \\ &= \int_a^b \mathcal{R}(\sigma)\mathcal{S}(\sigma)\psi'(\sigma)\omega((\psi(\sigma))\, d\sigma.\end{aligned}$$

Using the change of variable $s = \psi(\sigma)$, by (i)–(ii), we get

$$\begin{aligned}(\mathcal{R}, \mathcal{S})_{L^2(I,\mu(\sigma)d\sigma)} &= \int_0^1 \mathcal{R}\left(\psi^{-1}(s)\right)\mathcal{S}\left(\psi^{-1}(s)\right)\omega(s)\, ds \\ &= \int_0^1 (T\mathcal{R})(s)(T\mathcal{S})(s)\omega(s)\, ds \\ &= (T\mathcal{R}, T\mathcal{S})_{L^2([0,1],\omega(s)ds)}\end{aligned}$$

which completes the proof. □

Definition 2.1 (see [15]). *The ψ-fractional integral of order $\alpha > 0$ of a function $\mathcal{R} \in L^1(I, \psi'(t)\, dt)$ is defined by*

$$(I_a^{\alpha,\psi}\mathcal{R})(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\mathcal{R}(s)\, ds,$$

where Γ denotes the Gamma function.

Lemma 2.2. *Let $\alpha > \frac{1}{2}$. Then*

$$I_a^{\alpha,\psi} : L^2(I, \mu(\sigma)\, d\sigma) \to L^2(I, \mu(\sigma)\, d\sigma)$$

is a linear and bounded operator. Moreover,

$$\|I_a^{\alpha,\psi}\mathcal{R}\|_{L^2(I,\mu(\sigma)d\sigma)} \le C_{\alpha,\omega}\|\mathcal{R}\|_{L^2(I,\mu(\sigma)d\sigma)},$$

for all $\mathcal{R} \in L^2(I, \mu(\sigma)\, d\sigma)$, where $C_{\alpha,\omega} = \frac{1}{\Gamma(\alpha)}\sqrt{\frac{\|\omega^{-1}\|_{L^\infty([0,1])}}{(2\alpha-1)}\int_0^1 \omega(s)ds}$.

Proof. Let $\alpha > \frac{1}{2}$ and $\mathcal{R} \in L^2(I, \mu(\sigma)\, d\sigma)$. By Hölder's inequality, it holds

$$\left|\left(I_a^{\alpha,\psi}\mathcal{R}\right)(\sigma)\right|^2 \le \frac{1}{\Gamma(\alpha)^2}\left(\int_a^\sigma ((\psi(\sigma) - \psi(z))^{2\alpha-2}\psi'(z)\, dz\right)\left(\int_a^b \psi'(z)\mathcal{R}^2(z)\, dz\right), \quad \sigma \in I. \tag{3}$$

Using (i)–(ii), an elementary calculation yields

$$\int_a^\sigma ((\psi(\sigma) - \psi(z))^{2\alpha-2}\psi'(z)\, dz = \frac{\psi(\sigma)^{2\alpha-1}}{2\alpha - 1} \le \frac{1}{2\alpha - 1}. \tag{4}$$

On the other hand, since $\omega^{-1} \in C([0, 1])$, it holds

$$\int_a^b \psi'(z)\mathcal{R}^2(z)\, dz = \int_a^b \frac{1}{\omega(\psi(z))}\mathcal{R}^2(z)\mu(z)\, dz \le \|\omega^{-1}\|_{L^\infty([0,1])}\|\mathcal{R}\|^2_{L^2(I,\mu(\sigma)d\sigma)}. \tag{5}$$

Hence, combining (3)–(5), it holds

$$\begin{aligned}\|I_a^{\alpha,\psi}\mathcal{R}\|^2_{L^2(I,\mu(\sigma)d\sigma)} &= \int_a^b \left|\left(I_a^{\alpha,\psi}\mathcal{R}\right)(\sigma)\right|^2 \mu(\sigma)\, d\sigma \\ &\le \frac{\|\omega^{-1}\|_{L^\infty([0,1])}}{(2\alpha - 1)\Gamma(\alpha)^2}\left(\int_a^b \mu(\sigma)\, d\sigma\right)\|\mathcal{R}\|^2_{L^2(I,\mu(\sigma)d\sigma)} \\ &= \frac{\|\omega^{-1}\|_{L^\infty([0,1])}}{(2\alpha - 1)\Gamma(\alpha)^2}\left(\int_0^1 \omega(s)\, ds\right)\|\mathcal{R}\|^2_{L^2(I,\mu(\sigma)d\sigma)} \\ &= C^2_{\alpha,\omega}\|\mathcal{R}\|^2_{L^2(I,\mu(\sigma)d\sigma)},\end{aligned}$$

which proves the desired result. □

Definition 2.2 (see [3]). *Let*

$$F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_N(t) \end{pmatrix}, \quad \forall\, t \in I,$$

where $N \in \mathbb{N}$ *and* $F_i \in L^1(I, \psi'(t)\,dt)$*, for all* i*. The* ψ*-fractional integral of order* $\alpha > 0$ *of* F *is given by*

$$(I_a^{\alpha,\psi} F)(t) = \begin{pmatrix} (I_a^{\alpha,\psi} F_1)(t) \\ (I_a^{\alpha,\psi} F_2)(t) \\ \vdots \\ (I_a^{\alpha,\psi} F_N)(t) \end{pmatrix}.$$

Lemma 2.3 (see [15]). *For all* $\tau > 0$, $\kappa > -1$*, one has*

$$I_a^{\tau,\psi}(\psi(t))^\kappa = \frac{\Gamma(1+\kappa)}{\Gamma(\tau+\kappa+1)}(\psi(t))^{\tau+\kappa}.$$

For all $\tau, \kappa > 0$*, one has*

$$I_a^{\tau,\psi} I_a^{\kappa,\psi} \mathcal{R}(t) = I_a^{\tau+\kappa,\psi} \mathcal{R}(t).$$

Let

$$(\delta_\psi)^0 \mathcal{R}(t) = \mathcal{R}(t) \quad \text{and} \quad (\delta_\psi)^n \mathcal{R}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{R}(t),$$

where $n \in \mathbb{N}$.

Definition 2.3 (see [2]). *Let* $n \in \mathbb{N}$, $n-1 < \alpha < n$ *and* $\mathcal{R}, \psi \in C^n(I)$*. The* ψ*-Caputo fractional derivative of order* α *of* $\mathcal{R}$ *is given by*

$$({}^C D_a^{\alpha,\psi} \mathcal{R})(t) = I_a^{n-\alpha,\psi} (\delta_\psi)^n \mathcal{R}(t).$$

Lemma 2.4 (see [2]). *Let* $n \in \mathbb{N}$ *and* $n-1 < \alpha < n$*.*

1. *If* $\mathcal{R} \in C(I)$*, then*
$${}^C D_a^{\alpha,\psi} I_a^{\alpha,\psi} \mathcal{R}(t) = \mathcal{R}(t).$$
2. *If* $\mathcal{R} \in C^{n-1}(I)$*, then*

$$\mathcal{R}(t) - I_a^{\alpha,\psi}\, {}^C D_a^{\alpha,\psi} \mathcal{R}(t) = \sum_{i=0}^{n-1} \frac{(\delta_\psi)^i \mathcal{R}(a)}{i!} (\psi(t))^i. \tag{6}$$

3. ψ-polynomial basis

We start this section by recalling two fundamental results from Hilbertian Analysis (see e.g. [3, 10]).

Let $(\mathcal{X}, (\cdot,\cdot)_\mathcal{X})$ be a Hilbert space with norm $\|\cdot\|_\mathcal{X}$ and $\{e_i\}_{i\geq 0}$ a Hilbertian basis. The orthogonal projection on $\operatorname{span}\{e_i\}_{i=0}^{K-1}$, $K \in \mathbb{N}$, is denoted by Π_K. For all $z \in \mathcal{X}$, one has

$$\Pi_K(z) = (z, e_0)_\mathcal{X} e_0 + \cdots + (z, e_{K-1})_\mathcal{X} e_{K-1}.$$

Lemma 3.1. *For all* $z \in \mathcal{X}$*, one has*

$$\|\Pi_K(z)\|_\mathcal{X} \leq \|z\|_\mathcal{X}$$

and

$$\|\Pi_K(z) - z\|_\mathcal{X} \to 0 \text{ as } K \to \infty.$$

Lemma 3.2. *Let $\{z_K\}_{K\in\mathbb{N}} \subset \mathcal{X}$ be the sequence given by*

$$z_K = \sum_{i=0}^{K-1} \lambda_i e_i,$$

where $\{\lambda_n\}_{n\geq 0} \subset \mathbb{R}$. Suppose that there exists $z \in \mathcal{X}$ such that

$$\|z_K - z\|_{\mathcal{X}} \to 0 \text{ as } K \to \infty.$$

Then

$$z_K = \Pi_K(z), \quad \forall\, K \in \mathbb{N}.$$

Further, for $i \in \mathbb{N}$, let S_i be a polynomial of degree i, defined by

$$S_i(s) = \sum_{n=0}^{i} a_n(i) s^n, \quad \forall\, s \in [0,1].$$

Assume that $\{S_i : i = 0, 1, \cdots\}$ is a Hilbertian basis of $L^2([0,1], \omega(s)\,ds)$. We introduce the ψ-polynomials $S_{i,\psi}$ of degree i, defined by

$$S_{i,\psi}(t) = S_i(\psi(t)), \quad \forall\, t \in I. \tag{7}$$

Lemma 3.3. *The set $\{S_{i,\psi} : i = 0, 1, \cdots\}$ is a Hilbertian basis of $L^2(I, \mu(t)\,dt)$.*

Proof. First, we claim that $\{S_{i,\psi} : i = 0, 1, \cdots\}$ is orthonormal. Indeed, for two non-negative integers i and j, by Lemma 2.1, we have

$$\begin{aligned}
(S_{i,\psi}, S_{j,\psi})_{L^2(I,\mu(t)dt)} &= (TS_{i,\psi}, TS_{j,\psi})_{L^2([0,1],\omega(s)ds)} \\
&= \int_0^1 TS_{i,\psi}(s) TS_{j,\psi}(s) \omega(s)\,ds \\
&= \int_0^1 S_{i,\psi}(\psi^{-1}(s)) S_{j,\psi}(\psi^{-1}(s)) \omega(s)\,ds \\
&= \int_0^1 S_i(s) S_j(s) \omega(s)\,ds \\
&= (S_i, S_j)_{L^2([0,1],\omega(s)ds)},
\end{aligned}$$

which proves the claim.

Next, we claim that for any $f \in L^2(I, \mu(t)\,dt)$, we have

$$f = \sum_{i=0}^{\infty} (f, S_{i,\psi})_{L^2(I,\mu(t)dt)} S_{i,\psi}. \tag{8}$$

Let $f \in L^2(I, \mu(t)\,dt)$. By Lemma 2.1, we know that $Tf \in L^2([0,1], \omega(s)\,ds)$. Using the fact that $\{S_i : i = 0, 1, 2, \cdots\}$ is a Hilbertian basis of $L^2([0,1], \omega(s)\,ds)$, it holds

$$(Tf)(s) = \sum_{i=0}^{\infty} (Tf, S_i)_{L^2([0,1],\omega(s)ds)} S_i(s), \quad \forall\, s \in [0,1].$$

Hence, for all $t \in I$, we get

$$\begin{aligned}
f(t) &= (Tf)(\psi(t)) \\
&= \sum_{i=0}^{\infty} (Tf, S_i)_{L^2([0,1],\omega(s)ds)} S_i(\psi(t)) \\
&= \sum_{i=0}^{\infty} (Tf, S_i)_{L^2([0,1],\omega(s)ds)} S_{i,\psi}(t).
\end{aligned}$$

On the other hand, by Lemma 2.1, we have

$$(Tf, S_i)_{L^2([0,1],\omega(s)ds)} = (Tf, TS_{i,\psi})_{L^2([0,1],\omega(s)ds)} = (f, S_{i,\psi})_{L^2(I,\mu(t)dt)}.$$

Then

$$f(t) = \sum_{i=0}^{\infty} (f, S_{i,\psi})_{L^2(I,\mu(t)dt)} S_{i,\psi}(t), \quad \forall\, t \in I,$$

which yields (8). Therefore, we conclude that $\{S_{i,\psi} : i = 0, 1, \cdots\}$ is a Hilbertian basis of $L^2(I, \mu(t)\, dt)$. $\square$

Next, given a function $f \in L^2(I, \mu(t)\, dt)$ and $K \in \mathbb{N}$, we denote by $\Pi_K(f)$ its orthogonal projection on

$$\text{span}\{S_{i,\psi} : i = 0, 1, \cdots, K-1\},$$

that is,

$$\Pi_K(f)(t) = \sum_{i=0}^{K-1} (f, S_{i,\psi})_{L^2(I,\mu(t)dt)} S_{i,\psi}(t), \quad \forall\, t \in I. \tag{9}$$

Using Lemmas 3.1 and 3.3, we get

Lemma 3.4. *Let $f \in L^2(I, \mu(t)\, dt)$. Then*

$$\lim_{K\to\infty} \|f - \Pi_K(f)\|_{L^2(I,\mu(t)dt)} = 0.$$

4. Operational matrices of integrations

Let

$$F(t) = \begin{pmatrix} F_0(t) \\ F_1(t) \\ \vdots \\ F_{K-1}(t) \end{pmatrix}, \quad \forall\, t \in I,$$

where $K \in \mathbb{N}$. We assume that $F \in L^2(I; \mathbb{R}^K, \mu(t)\, dt)$, that is, $\{F_i\}_{i=0}^{K-1} \subset L^2(I, \mu(t)\, dt)$. Let

$$(\mathbf{\Pi_K} F)(t) = \begin{pmatrix} (\Pi_K F_0)(t) \\ (\Pi_K F_1)(t) \\ \vdots \\ (\Pi_K F_{K-1})(t) \end{pmatrix}, \quad \forall\, t \in I, \tag{10}$$

where Π_K is given by (9).

For $U, V \in L^2(I; \mathbb{R}^K, \mu(t)\, dt)$, the notation $U \simeq_K V$ means that $V = \mathbf{\Pi_K} U$. For $K \gg 1$, let

$$\Phi_{K,\psi}(t) = \begin{pmatrix} S_{0,\psi}(t) \\ S_{1,\psi}(t) \\ \vdots \\ S_{K-1,\psi}(t) \end{pmatrix}, \quad \forall\, t \in I. \tag{11}$$

Using Lemma 2.2, we deduce that

Lemma 4.1. *Let $\alpha > \frac{1}{2}$. Then*

$$I_a^{\alpha,\psi} \Phi_{K,\psi} \in L^2(I; \mathbb{R}^K, \mu(t)\, dt).$$

For $\alpha > 0$, let

$$\Omega(\tau, i, \alpha) = \sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha} G(i, n, \alpha), \quad i, \tau \in \{0, 1, \cdots, K-1\},$$

where

$$\Delta_{\tau,n,\alpha} = \frac{a_n(\tau) n!}{\Gamma(\alpha + n + 1)} \tag{12}$$

and

$$G(i, n, \alpha) = \sum_{\ell=0}^{i} a_\ell(i) \int_0^1 s^{\alpha+n+\ell} \omega(s)\, ds. \tag{13}$$

Consider the matrix

$$M^\alpha_{K\times K} = (M^\alpha_{ij})_{1\le i,j\le K}, \quad M^\alpha_{i,j} = \Omega(i-1, j-1, \alpha).$$

Theorem 4.1. *Let $\alpha > \frac{1}{2}$. Then*

$$I_a^{\alpha,\psi} \Phi_{K,\psi} \simeq_K M^\alpha_{K\times K} \Phi_{K,\psi}.$$

Proof. First, observe that $M^\alpha_{K\times K}\Phi_{K,\psi} \in L^2(I; \mathbb{R}^K, \mu(t)\, dt)$. Similarly, by Lemma 4.1, since $\alpha > \frac{1}{2}$, we have $I_a^{\alpha,\psi}\Phi_{K,\psi} \in L^2(I; \mathbb{R}^K, \mu(t)\, dt)$. Hence, we have to prove that

$$M^\alpha_{K\times K}\Phi_{K,\psi} = \mathbf{\Pi_K}\left(I_a^{\alpha,\psi}\Phi_{K,\psi}\right),$$

i.e., for all $\tau \in \{0, 1, \cdots, K-1\}$,

$$\Pi_K\left(I_a^{\alpha,\psi} S_{\tau,\psi}\right) = \sum_{j=1}^{K} \Omega(\tau, j-1, \alpha) S_{j-1,\psi}. \tag{14}$$

Fixing τ, one has

$$\Pi_K\left(I_a^{\alpha,\psi} S_{\tau,\psi}\right) = \sum_{i=0}^{K-1} (I_a^{\alpha,\psi} S_{\tau,\psi}, S_{i,\psi})_{L^2(I,\mu(t)dt)} S_{i,\psi}. \tag{15}$$

On the other hand, for all $i \in \{0, 1, \cdots, K-1\}$, using Lemma 2.3, we have

$$\begin{aligned}
(I_a^{\alpha,\psi} S_{\tau,\psi}, S_{i,\psi})_{L^2(I,\mu(t)dt)} &= \sum_{n=0}^{\tau} a_n(\tau)\left(I_a^{\alpha,\psi}\psi^n, S_{i,\psi}\right)_{L^2(I,\mu(t)dt)} \\
&= \sum_{n=0}^{\tau} \frac{a_n(\tau) n!}{\Gamma(\alpha+n+1)}\left(\psi^{\alpha+n}, S_{i,\psi}\right)_{L^2(I,\mu(t)dt)} \\
&= \sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha}\left(\psi^{\alpha+n}, S_{i,\psi}\right)_{L^2(I,\mu(t)dt)}.
\end{aligned}$$

Next, for $n \in \{0, 1, \cdots, \tau\}$, by Lemma 2.1, we get

$$\begin{aligned}
\left(\psi^{\alpha+n}, S_{i,\psi}\right)_{L^2(I,\mu(t)dt)} &= (T\psi^{\alpha+n}, TS_{i,\psi})_{L^2([0,1],\omega(s)ds)} \\
&= \int_0^1 s^{\alpha+n} S_{i,\psi}(\psi^{-1}(s))\omega(s)\, ds \\
&= \sum_{\ell=0}^{i} a_\ell(i) \int_0^1 s^{\alpha+n+\ell}\omega(s)\, ds \\
&= G(i, n, \alpha).
\end{aligned}$$

Therefore, for all $i \in \{0, 1, \cdots, K-1\}$, it holds

$$(I_a^{\alpha,\psi} S_{\tau,\psi}, S_{i,\psi})_{L^2(I,\mu(t)dt)} = \sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha} G(i, n, \alpha). \tag{16}$$

Using (15) and (16), (14) follows. □

5. The numerical approach and its convergence

Consider the coupled system of fractional differential equations (1) subject to the initial conditions (2). It is supposed that $p_1, p_2 \in L^2(I, \mu(t)\,dt)$ and (1)–(2) admits a unique solution $(x, y) \in C^m(I) \times C^m(I)$ (see e.g. [4]). From (1), one has ${}^C D_a^{\alpha,\psi} x, {}^C D_a^{\beta,\psi} y \in L^2(I, \mu(t)\,dt)$. For $K \in \mathbb{N}$, $K \gg 1$, for all $t \in I$, we have

$$\Pi_K({}^C D_a^{\alpha,\psi} x)(t) = \sum_{i=0}^{K-1} ({}^C D_a^{\alpha,\psi} x, S_{i,\psi})_{L^2(I,\mu(t)dt)} S_{i,\psi}(t)$$

and

$$\Pi_K({}^C D_a^{\beta,\psi} y)(t) = \sum_{i=0}^{K-1} ({}^C D_a^{\beta,\psi} y, S_{i,\psi})_{L^2(I,\mu(t)dt)} S_{i,\psi}(t),$$

where Π_K is given by (9). Then, we may write

$$\Pi_K({}^C D_a^{\alpha,\psi} x)(t) = H_K^{\alpha} \Phi_{K,\psi}(t) \tag{17}$$

and

$$\Pi_K({}^C D_a^{\beta,\psi} y)(t) = H_K^{\beta} \Phi_{K,\psi}(t), \tag{18}$$

where $\Phi_{K,\psi}$ is defined by (11),

$$H_K^{\alpha} = \left(({}^C D_a^{\alpha,\psi} x, S_{0,\psi})_{L^2(I,\mu(t)dt)}, ({}^C D_a^{\alpha,\psi} x, S_{1,\psi})_{L^2(I,\mu(t)dt)}, \cdots, ({}^C D_a^{\alpha,\psi} x, S_{K-1,\psi})_{L^2(I,\mu(t)dt)} \right)$$

and

$$H_K^{\beta} = \left(({}^C D_a^{\beta,\psi} y, S_{0,\psi})_{L^2(I,\mu(t)dt)}, ({}^C D_a^{\beta,\psi} y, S_{1,\psi})_{L^2(I,\mu(t)dt)}, \cdots, ({}^C D_a^{\beta,\psi} y, S_{K-1,\psi})_{L^2(I,\mu(t)dt)} \right).$$

Using Lemma 3.1, we obtain the following result.

Lemma 5.1. *We have*

$$\lim_{K\to\infty} \max \left\{ \left\| H_K^{\alpha} \Phi_{K,\psi} - {}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)}, \left\| H_K^{\beta} \Phi_{K,\psi} - {}^C D_a^{\beta,\psi} y \right\|_{L^2(I,\mu(t)dt)} \right\} = 0.$$

Since $\alpha, \beta > \frac{1}{2}$, we deduce from Lemmas 2.2 and 5.1 the following result.

Lemma 5.2. *We have*

$$\lim_{K\to\infty} \max \left\{ \left\| H_K^{\alpha} I_a^{\alpha,\psi} \Phi_{K,\psi} - I_a^{\alpha,\psi}\, {}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)}, \left\| H_K^{\beta} I_a^{\beta,\psi} \Phi_{K,\psi} - I_a^{\beta,\psi}\, {}^C D_a^{\beta,\psi} y \right\|_{L^2(I,\mu(t)dt)} \right\} = 0.$$

Next, we shall prove the following convergence result.

Lemma 5.3. *We have*

$$\lim_{K\to\infty} \max \left\{ \left\| H_K^{\alpha} \mathbf{\Pi_K}(I_a^{\alpha,\psi} \Phi_{K,\psi}) - I_a^{\alpha,\psi}\, {}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)}, \left\| H_K^{\beta} \mathbf{\Pi_K}(I_a^{\beta,\psi} \Phi_{K,\psi}) - I_a^{\beta,\psi}\, {}^C D_a^{\beta,\psi} y \right\|_{L^2(I,\mu(t)dt)} \right\}$$
$$= 0,$$

where $\mathbf{\Pi_K}$ *is given by* (10).

Proof. First, we have

$$\left\| H_K^\alpha \mathbf{\Pi_K}(I_a^{\alpha,\psi}\Phi_{K,\psi}) - I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)} \leq \left\| H_K^\alpha \mathbf{\Pi_K}(I_a^{\alpha,\psi}\Phi_{K,\psi}) - \Pi_K(I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x) \right\|_{L^2(I,\mu(t)dt)} + \left\| I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x - \Pi_K(I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x) \right\|_{L^2(I,\mu(t)dt)}.$$

Note that

$$H_K^\alpha \mathbf{\Pi_K}(I_a^{\alpha,\psi}\Phi_{K,\psi}) = \Pi_K\left(H_K^\alpha I_a^{\alpha,\psi}\Phi_{K,\psi}\right),$$

which implies by Lemma 3.1 that

$$\begin{aligned}
&\left\| H_K^\alpha \mathbf{\Pi_K}(I_a^{\alpha,\psi}\Phi_{K,\psi}) - I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)} \\
&\leq \left\| \Pi_K\left(H_K^\alpha I_a^{\alpha,\psi}\Phi_{K,\psi} - I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x\right) \right\|_{L^2(I,\mu(t)dt)} + \left\| I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x - \Pi_K(I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x) \right\|_{L^2(I,\mu(t)dt)} \\
&\leq \left\| H_K^\alpha I_a^{\alpha,\psi}\Phi_{K,\psi} - I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)} + \left\| I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x - \Pi_K(I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x) \right\|_{L^2(I,\mu(t)dt)}.
\end{aligned}$$

Next, using Lemmas 3.1 and 5.2, we deduce that

$$\lim_{K\to\infty} \left\| H_K^\alpha \mathbf{\Pi_K}(I_a^{\alpha,\psi}\Phi_{K,\psi}) - I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)} = 0.$$

Following the same argument as above, we get

$$\lim_{K\to\infty} \left\| H_K^\beta \mathbf{\Pi_K}(I_a^{\beta,\psi}\Phi_{K,\psi}) - I_a^{\beta,\psi}\,{}^C D_a^{\beta,\psi} y \right\|_{L^2(I,\mu(t)dt)} = 0.$$

This ends the proof. □

Further, since $\alpha,\beta > \frac{1}{2}$, Theorem 4.1 and Lemma 5.3 yield the following result.

Lemma 5.4. *We have*

$$\lim_{K\to\infty} \max\left\{ \left\| H_K^\alpha M_{K\times K}^\alpha \Phi_{K,\psi} - I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x \right\|_{L^2(I,\mu(t)dt)}, \left\| H_K^\beta M_{K\times K}^\beta \Phi_{K,\psi} - I_a^{\beta,\psi}\,{}^C D_a^{\beta,\psi} y \right\|_{L^2(I,\mu(t)dt)} \right\} = 0.$$

Given a vector $X \in \mathbb{R}^K$, we denote by X^T its transpose. Using (6) and the initial conditions (2), we get

$$I_a^{\alpha,\psi}\,{}^C D_a^{\alpha,\psi} x(t) = x(t) - \sum_{i=0}^{m-1} \frac{x_i}{i!}(\psi(t))^i \tag{19}$$

and

$$I_a^{\beta,\psi}\,{}^C D_a^{\beta,\psi} y(t) = y(t) - \sum_{i=0}^{m-1} \frac{y_i}{i!}(\psi(t))^i. \tag{20}$$

Let

$$\Pi_K\left(\sum_{i=0}^{m-1} \frac{x_i}{i!}(\psi(t))^i\right) = Z_K^T \Phi_{K,\psi}(t), \quad \forall\, t \in I \tag{21}$$

and

$$\Pi_K\left(\sum_{i=0}^{m-1} \frac{y_i}{i!}(\psi(t))^i\right) = W_K^T \Phi_{K,\psi}(t), \quad \forall\, t \in I, \tag{22}$$

where $Z_K, W_K \in \mathbb{R}^K$ (the known vectors). Let $\{x_K\}, \{y_K\} \subset L^2(I, \mu(t)\,dt)$ be the sequences defined by

$$x_K(t) = \left(H_{K,\alpha} M^{\alpha}_{K\times K} + Z^T_K\right)\Phi_{K,\psi}(t), \quad \forall\, t \in I \tag{23}$$

and

$$y_K(t) = \left(H_{K,\beta} M^{\beta}_{K\times K} + W^T_K\right)\Phi_{K,\psi}(t), \quad \forall\, t \in I. \tag{24}$$

Theorem 5.1. *We have*

$$\lim_{K\to\infty} \max\left\{\|x_K - x\|_{L^2(I,\mu(t)dt}, \|y_K - y\|_{L^2(I,\mu(t)dt}\right\} = 0.$$

Proof. Writing

$$\|x_K - x\|_{L^2(I,\mu(t)dt)} = \left\|\left(H_{K,\alpha} M^{\alpha}_{K\times K} + Z^T_K\right)\Phi_{K,\psi} - x\right\|$$

and using (19), (21), we obtain

$$\begin{aligned} \|x_K - x\|_{L^2(I,\mu(t)dt)} &= \left\|\left(H_{K,\alpha} M^{\alpha}_{K\times K} + Z^T_K\right)\Phi_{K,\psi} - I_a^{\alpha,\psi}\, {}^{C}D_a^{\alpha,\psi} x - \sum_{i=0}^{m-1} \frac{x_i}{i!}\psi^i\right\|_{L^2(I,\mu(t)dt)} \\ &\leq \left\|H_{K,\alpha} M^{\alpha}_{K\times K}\Phi_{K,\psi} - I_a^{\alpha,\psi}\, {}^{C}D_a^{\alpha,\psi} x\right\|_{L^2(I,\mu(t)dt)} + \left\|\Pi_K\left(\sum_{i=0}^{m-1} \frac{x_i}{i!}\psi^i\right) - \sum_{i=0}^{m-1} \frac{x_i}{i!}\psi^i\right\|_{L^2(I,\mu(t)dt)}. \end{aligned}$$

Hence, using Lemmas 3.1 and 5.4, we obtain by taking $K \to \infty$,

$$\lim_{K\to\infty} \|x_K - x\|_{L^2(I,\mu(t)dt)} = 0.$$

Similarly, one has

$$\lim_{K\to\infty} \|y_K - y\|_{L^2(I,\mu(t)dt)} = 0,$$

which completes the proof. □

From the previous result, (x, y) can be obtained as limits of the sequences $\{x_K\}$ and $\{y_K\}$ defined by (23)–(24). However, before applying Theorem 5.1, we have to compute the unknown vectors $H_{K,\alpha}$ and $H_{K,\beta}$.

First, by Lemma 3.2 and Theorem 5.1, it holds

Lemma 5.5. *For all $K \in \mathbb{N}$, we have*

$$(x_K, y_K) = (\Pi_K(x), \Pi_K(y)).$$

Further, for $i = 1, 2$, let $Q_{K,i} \in \mathbb{R}^K$ be the (known) vectors satisfying

$$\Pi_K(p_i)(t) = Q^T_{K,i}\Phi_{K,\psi}(t), \quad \forall\, t \in I. \tag{25}$$

Using (1), we get

$$\begin{cases} \Pi_K\left({}^{C}D_a^{\alpha,\psi} x\right) = A_{11}\Pi_K(x) + A_{12}\Pi_K(y) + \Pi_K(p_1), \\ \Pi_K\left({}^{C}D_a^{\beta,\psi} y\right) = A_{21}\Pi_K(x) + A_{22}\Pi_K(y) + \Pi_K(p_2). \end{cases}$$

By (17)–(25) and Lemma 5.5, it holds

$$\begin{cases} H_{K,\alpha}\left(I_{K\times K} - A_{11}M^{\alpha}_{K\times K}\right) + H_{K,\beta}\left(-A_{12}M^{\beta}_{K\times K}\right) &= A_{11}Z^T_K + A_{12}W^T_K + Q^T_{K,1}, \\ H_{K,\alpha}\left(-A_{21}M^{\alpha}_{K\times K}\right) + H_{K,\beta}\left(I_{K\times K} - A_{22}M^{\beta}_{K\times K}\right) &= A_{21}Z^T_K + A_{22}W^T_K + Q^T_{K,2}, \end{cases}$$

which can be written as

$$\begin{cases} H_{K,\alpha}\mathcal{A}_{11} + H_{K,\beta}\mathcal{A}_{12} &= U_1 \\ H_{K,\alpha}\mathcal{A}_{21} + H_{K,\beta}\mathcal{A}_{22} &= U_2, \end{cases} \tag{26}$$

where

$$\begin{aligned}
\mathcal{A}_{11} &= I_{K\times K} - A_{11}M^{\alpha}_{K\times K},\\
\mathcal{A}_{12} &= -A_{12}M^{\beta}_{K\times K},\\
\mathcal{A}_{21} &= -A_{21}M^{\alpha}_{K\times K},\\
\mathcal{A}_{22} &= I_{K\times K} - A_{22}M^{\beta}_{K\times K},\\
U_1 &= A_{11}Z_K^T + A_{12}W_K^T + Q_{K,1}^T,\\
U_2 &= A_{21}Z_K^T + A_{22}W_K^T + Q_{K,2}^T
\end{aligned}$$

and $I_{K\times K}$ is the unit matrix. One can shows that (26) is equivalent to

$$\left(H_{K,\alpha}, H_{K,\beta}\right)\begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{21}\\ \mathcal{A}_{12} & \mathcal{A}_{22}\end{pmatrix} = (U_1, U_2)\,. \tag{27}$$

Note that it is assumed that

$$\begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{21}\\ \mathcal{A}_{12} & \mathcal{A}_{22}\end{pmatrix}$$

is invertible. Otherwise, one may increase, iteratively, the number K by one, until the matrix becomes invertible. Next, after solving (27), (x, y) can be approximated by (23)–(24), for $K \gg 1$.

6. Numerical experiments

6.1. The model example

As a model example, consider the coupled system of fractional differential equations

$$\begin{cases} {}^{C}D_0^{\alpha,\psi}x(t) = \dfrac{2}{\Gamma(3-\alpha)}x(t) + \dfrac{2}{\Gamma(3-\alpha)}y(t) + \dfrac{2}{\Gamma(3-\alpha)}\left(\psi(t)\right)^{2-\alpha}\left(1 - (\psi(t))^{\alpha} - (\psi(t))^{\alpha+1}\right) \\ {}^{C}D_0^{\beta,\psi}y(t) = \dfrac{6}{\Gamma(4-\beta)}x(t) + \dfrac{6}{\Gamma(4-\beta)}y(t) + \dfrac{6}{\Gamma(4-\beta)}\left(\psi(t)\right)^{2} - \left((\psi(t))^{1-\beta} - 1 - (\psi(t))\right) \end{cases} ;\ t \in I \tag{28}$$

subject to the initial conditions

$$x(0) = y(0) = 0, \tag{29}$$

where $(\alpha, \beta) = (0.9, 0.6)$ and $I = [0, 1]$. one can show easily that

$$(x^*(t), y^*(t)) = (\psi(t)^2, \psi(t)^3), \quad \forall\, t \in I, \tag{30}$$

is the exact solution to (28)–(29).

Our aim in this section is to illustrate the effectiveness of the approach presented in Section 5 by computing the numerical solution (x, y) to (28)–(29) using different types of polynomials.

The absolute error at any point $t \in I$ will be denoted by

$$E(t) = |x^*(t) - x(t)| + |y^*(t) - y(t)|.$$

6.2. ψ-shifted polynomials

We will use two types of polynomials.

6.2.1. ψ-shifted polynomials of Legendre-type

The shifted Legendre polynomials are given by (see e.g. [18])

$$L_j(s) = (1+2j)^{\frac{1}{2}} \sum_{n=0}^{j} (-1)^{j-n} \frac{(n+j)!}{(j-n)!(n!)^2} s^n, \quad \forall\, s \in [0,1].$$

Moreover, the set $\{L_i\}_{i\geq 0}$ is a Hilbertian basis of $L^2([0,1])$ (see e.g. [16]). The ψ-shifted polynomials of Legendre-type $S_{i,\psi}$, are given by (see [3])

$$S_{i,\psi}(t) = L_i(\psi(t)),$$

for all $t \in I$, i.e., $S_{i,\psi}$ is defined by (7) with

$$a_n(i) = (-1)^{i-n}(1+2i)^{\frac{1}{2}} \frac{(i+n)!}{(i-n)!(n!)^2}, \tag{31}$$

for all $n = 0, 1, \cdots, i$. Hence, using (31), (12)–(13) with $\omega \equiv 1$, for $r \in \{\alpha, \beta\}$ and $j, \tau \in \{0, 1, \cdots, K-1\}$, we get

$$\Delta_{\tau,n,r} = (1+2\tau)^{\frac{1}{2}} (-1)^{\tau-n} \frac{(n+\tau)!}{n!(\tau-n)!\Gamma(r+n+1)}$$

and

$$G(j,n,r) = (1+2j)^{\frac{1}{2}} \sum_{\ell=0}^{i} (-1)^{j-\ell} \frac{(\ell+j)!}{(j-\ell)!(\ell!)^2(r+n+\ell+1)}.$$

6.2.2. ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$

The shifted Jacobi polynomials with parameters $(0, -\frac{1}{2})$ are given by (see e.g. [27])

$$J_i^{(0,-\frac{1}{2})}(s) = \left(2i + \frac{1}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{i} (-1)^{i-n} \frac{\Gamma(\frac{1}{2}+i+n)}{\Gamma(\frac{1}{2}+n)(i-n)!n!} s^n, \quad \forall\, s \in [0,1].$$

The set $\left\{J_i^{(0,-\frac{1}{2})}\right\}_{i\geq 0}$ is a Hilbertian basis of $L^2([0,1], \omega(s)\, ds)$, where

$$\omega(s) = \frac{1}{\sqrt{s}}, \quad \forall\, s \in (0,1]. \tag{32}$$

We define the ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$ by

$$S_{i,\psi}(t) = J_i^{(0,-\frac{1}{2})}(\psi(t)) \quad \forall\, t \in I,$$

i.e., $S_{i,\psi}$ is given by (7) with

$$a_n(i) = \left(2i + \frac{1}{2}\right)^{\frac{1}{2}} \frac{(-1)^{i-n}\Gamma(\frac{1}{2}+i+n)}{\Gamma(\frac{1}{2}+n)(i-n)!n!}, \tag{33}$$

for all $n = 0, 1, \cdots, i$. Further, let $r \in \{\alpha, \beta\}$ and $i, \tau \in \{0, 1, \cdots, K-1\}$. For $\ell \in \{0, 1, \cdots, i\}$, we have

$$\int_0^1 s^{r+n+\ell}\omega(s)\, ds = \int_0^1 s^{r+n+\ell-\frac{1}{2}}\, ds = \frac{1}{r+n+\ell+\frac{1}{2}}. \tag{34}$$

Hence, using (33), (12)–(13) with ω given by (32), and (34), we get

$$\Delta_{\tau,n,r} = \left(2\tau + \frac{1}{2}\right)^{\frac{1}{2}} (-1)^{\tau-n} \frac{\Gamma(\tau+n+\frac{1}{2})}{\Gamma(n+\frac{1}{2})(\tau-n)!\Gamma(r+n+1)}$$

and

$$G(i,n,r) = \left(2i + \frac{1}{2}\right)^{\frac{1}{2}} \sum_{\ell=0}^{i} (-1)^{i-\ell} \frac{\Gamma(i+\ell+\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})(i-\ell)!\ell!(r+n+\ell+\frac{1}{2})}.$$

6.3. Numerical computations via ψ-shifted Legendre polynomials

In this subsection, the numerical approach presented in Section 5 will be applied for solving (28)–(29) using ψ-shifted polynomials of Legendre-type.

6.3.1. The case $\psi(t) = \frac{t(t+1)}{2}$

We consider (28)–(29) with

$$\psi(t) = \frac{t(t+1)}{2}, \quad t \in I, \tag{35}$$

where $I = [0, 1]$.

In Table 1, for $K = 10$, we give the numerical solution (x, y) to (28)–(29), as well as $E(T)$ at different points in I.

t	$(x^*(t), y^*(t))$	$(x(t), y(t))$	$E(t)$
0.0	(0.0,0.0)	(-7.4×10^{-6}, -1.3×10^{-6})	8.7×10^{-6}
0.1	(0.003025,0.000166)	(0.003027,0.000166)	2.0×10^{-6}
0.2	(0.014400,0.001728)	(0.014397,0.001727)	4.0×10^{-6}
0.3	(0.038025,0.007414)	(0.038025,0.007414)	0
0.4	(0.078400,0.021952)	(0.078401,0.021952)	1.0×10^{-6}
0.5	(0.140625,0.052734)	(0.140624,0.052734)	1.0×10^{-6}
0.6	(0.230400,0.110592)	(0.230399,0.110592)	1.0×10^{-6}
0.7	(0.354025,0.210645)	(0.354027,0.210645)	2.0×10^{-6}
0.8	(0.518400,0.373248)	(0.518398,0.373248)	2.0×10^{-6}
0.9	(0.731025,0.625026)	(0.731027,0.625026)	2.0×10^{-6}
1.0	(1.0,1.0)	(1.000010,0.999998)	1.2×10^{-5}

Table 1: Exact and numerical solutions to (28)–(29) for $K = 10$ using ψ-shifted polynomials of Legendre-type with $\psi(t) = \frac{t(t+1)}{2}$

6.3.2. The case $\psi(t) = \tan\left(\frac{\pi}{4}t\right)$

We consider (28)–(29) with

$$\psi(t) = \tan\left(\frac{\pi}{4}t\right), \quad t \in I. \tag{36}$$

In Table 2, for $K = 10$, we give the numerical solution (x, y) to (28)–(29), as well as $E(T)$ at different points in I.

6.4. Numerical computations via ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$

In this subsection, the numerical approach presented in Section 5 is applied for solving (28)–(29) using ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$.

6.4.1. The case $\psi(t) = \frac{t(t+1)}{2}$

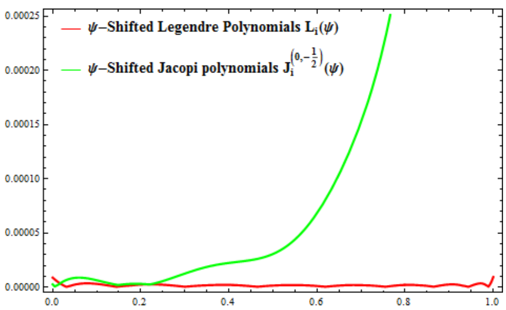
We consider (28)–(29) with ψ given by (35). In Table 3, for $K = 10$, we give the numerical solution (x, y) to (28)–(29), as well as $E(T)$ at different points in I.

t	$(x^*(t), y^*(t))$	$(x(t), y(t))$	$E(t)$
0.0	(0.0 , 0.0)	(-7.4×10^{-6},-1.3×10^{-6})	8.7×10^{-6}
0.1	(0.006193,0.000487)	(0.006194,0.000487)	1.0×10^{-6}
0.2	(0.025085,0.003973)	(0.025083,0.003972)	3.0×10^{-6}
0.3	(0.057637,0.013837)	(0.057639,0.013837)	2.0×10^{-6}
0.4	(0.105573,0.034302)	(0.105573,0.034303)	1.0×10^{-7}
0.5	(0.171573,0.071067)	(0.171571,0.071067)	2.0×10^{-6}
0.6	(0.259616,0.132281)	(0.259616,0.132281)	0
0.7	(0.375525,0.230122)	(0.375526,0.230122)	1.0×10^{-6}
0.8	(0.527864,0.383516)	(0.527862,0.383516)	2.0×10^{-6}
0.9	(0.729454,0.623012)	(0.729456,0.623012)	2.0×10^{-6}
1.0	(1.0,1.0)	(1.000010,0.999998)	1.2×10^{-5}

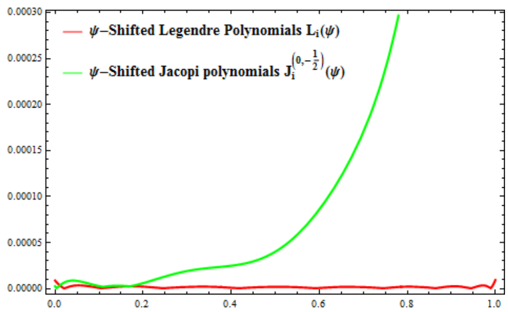
Table 2: Exact and numerical solutions to (28))–(29) for $K = 10$ using ψ-shifted polynomials of Legendre-type with $\psi(t) = \tan\left(\frac{\pi}{4}t\right)$

t	$(x^*(t), y^*(t))$	$(x(t), y(t))$	$E(t)$
0.0	(0.0,0.0)	(-1.6×10^{-6},-9.6×10^{-6})	1.12×10^{-5}
0.1	(0.003025,0.000166)	(0.003029,0.000168)	6.00×10^{-6}
0.2	(0.014400,0.001728)	(0.014399,0.001730)	3.00×10^{-6}
0.3	(0.038025,0.007414)	(0.038032,0.007420)	1.30×10^{-5}
0.4	(0.078400,0.021952)	(0.078410,0.021964)	2.20×10^{-5}
0.5	(0.140625,0.052734)	(0.140634,0.052756)	3.10×10^{-5}
0.6	(0.230400,0.110592)	(0.230425,0.110635)	6.80×10^{-5}
0.7	(0.354025,0.210645)	(0.354082,0.210741)	1.53×10^{-4}
0.8	(0.518400,0.373248)	(0.518513,0.373466)	3.31×10^{-4}
0.9	(0.731025,0.625026)	(0.731319,0.625557)	8.25×10^{-4}
1.0	(1.0,1.0)	(1.00077,1.00138)	2.15×10^{-3}

Table 3: Exact and numerical solutions to (28)–(29) for $K = 10$ using ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$ and $\psi(t) = \frac{t(t+1)}{2}$



(a) $E(t)$ for (28)–(29) with $\psi(t) = \frac{t(t+1)}{2}$



(b) $E(t)$ for (28)–(29) with $\psi(t) = \tan\left(\frac{\pi}{4}t\right)$

Figure 1: Comparison of absolute error $E(t)$ using ψ-shifted polynomials of Legendre-type and ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$ with $(\alpha, \beta, K) = (0.9, 0.6, 10)$ for the considered choices of ψ.

6.4.2. *The case* $\psi(t) = \tan\left(\frac{\pi}{4}t\right)$

We consider (28)–(29) with ψ given by (36). In Table 4, for $K = 10$, we give the numerical solution (x, y) to (28)–(29), as well as $E(T)$ at different points in I.

t	$(x^*(t), y^*(t))$	$(x(t), y(t))$	$E(t)$
0.0	(0.0 , 0.0)	(-1.59×10^{-6},-9.6×10^{-7})	2.55×10^{-6}
0.1	(0.006193,0.000487)	(0.006194,0.000489)	3.00×10^{-6}
0.2	(0.025085,0.003973)	(0.025088,0.003976)	6.00×10^{-6}
0.3	(0.057637,0.013837)	(0.057648,0.013846)	2.00×10^{-5}
0.4	(0.105573,0.034302)	(0.105582,0.034318)	2.50×10^{-5}
0.5	(0.171573,0.071067)	(0.171585,0.071095)	4.00×10^{-5}
0.6	(0.259616,0.132281)	(0.259649,0.132334)	8.60×10^{-5}
0.7	(0.375525,0.230122)	(0.375587,0.230230)	1.70×10^{-4}
0.8	(0.527864,0.383516)	(0.527983,0.383744)	3.47×10^{-4}
0.9	(0.729454,0.623012)	(0.729746,0.623540)	8.20×10^{-4}
1.0	(1.0,1.0)	(1.000770,1.001380)	2.15×10^{-3}

Table 4: Exact and numerical solutions to (28)–(29) for $K = 10$ using ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$ and $\psi(t) = \tan\left(\frac{\pi}{4}t\right)$

Figure (1a) shows a comparison of the obtained absolute error $E(t)$ for (28)–(29) using ψ-shifted polynomials of Legendre-type and ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$ in the case $\psi(t) = \frac{t(t+1)}{2}$. Figure (1b) shows a comparison of the obtained absolute error $E(t)$ for (28)–(29) using ψ-shifted polynomials of Legendre-type and ψ-shifted polynomials of Jacobi-type with parameters $(0, -\frac{1}{2})$ in the case $\psi(t) = \tan\left(4^{-1}\pi t\right)$.

The numerical tests presented in this section confirm the efficiency of the method.

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