



## On the B-Discrete Spectrum

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**Abstract.** In this paper, we introduce the B-discrete spectrum of an unbounded closed operator and we prove that a closed operator has a purely B-discrete spectrum if and only if it has a meromorphic resolvent. After that, we study the stability of the B-discrete spectrum under several type of perturbations and we establish that two closed invertible linear operators having quasisimilar totally paranormal inverses have equal spectra and B-discrete spectra.

### 1. Introduction

Let  $C(X)$  be the set of linear closed operators defined from a Banach space  $X$  to  $X$  and  $L(X)$  be the Banach algebra of bounded linear operators defined from  $X$  to  $X$ . We write  $D(T)$ ,  $N(T)$  and  $R(T)$  for the domain, nullspace and range of an operator  $T \in C(X)$ . An operator  $T \in C(X)$  is called a *Fredholm* operator [17] if both the nullity  $n(T) = \dim N(T)$  of  $T$  and the defect  $d(T) = \text{codim } R(T)$  of  $T$  are finite. The index  $i(T)$  of a Fredholm operator  $T$  is defined by  $i(T) = n(T) - d(T)$ . It is well known that if  $T$  is a Fredholm operator, then  $R(T)$  is closed.

The class of bounded linear B-Fredholm operators, which is a natural extension of the class of Fredholm operators was introduced in [1], and the class of unbounded linear closed B-Fredholm operators acting on a Banach space was studied in [4].

Recall [9] that a bounded linear operator  $T$  is called a meromorphic operator if  $\lambda = 0$  is the only possible point of accumulation of its spectrum  $\sigma(T)$  and every non-zero isolated point of  $\sigma(T)$  is a pole of the resolvent  $R_\mu(T) = (T - \mu I)^{-1}$  of  $T$ , which is defined on the resolvent set  $\rho(T)$  of  $T$ . If we also require that each non-zero eigenvalue of  $T$  has finite multiplicity, then  $T$  will be called a Riesz operator.

A first result linking bounded B-Fredholm operators to the class  $\mathfrak{M}$  of linear bounded meromorphic operators comes from the following theorem, established in [2, Theorem 2.11].

**Theorem 1.1.** *Let  $T \in L(X)$ . Then  $T$  is a meromorphic operator if and only if*

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a B-Fredholm operator}\} \subset \{0\}.$$

Recall that for  $T \in C(X)$ , its *descent*  $\delta(T)$  and its *ascent*  $a(T)$  are defined by  $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$  and  $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ . We set formally  $\inf \emptyset = \infty$ .

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A closed linear operator  $T \in C(X)$  is said to be *Drazin invertible* if  $a(T)$  and  $\delta(T)$  are both finite. In this case and if the resolvent set  $\rho(T)$  of  $T$  is nonempty, then  $a(T) = \delta(T)$ ,  $R(T^{a(T)})$  is closed and  $X = R(T^{a(T)}) \oplus N(T^{a(T)})$ .

The *Drazin spectrum* of  $T$  is defined by:  $\sigma_{\mathcal{D}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not Drazin invertible}\}$ .

The set of Browder operators is defined by  $\mathcal{B}(X) = \{T \in \Phi(X) \mid a(T) < \infty \text{ and } \delta(T) < \infty\}$  and the *Browder spectrum* of  $T$  is defined by:  $\sigma_{\mathcal{B}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}(X)\}$ .

For a closed linear operator  $T \in C(X)$ , the discrete spectrum  $\sigma_d(T)$  of  $T$  is defined as the set of all complex numbers  $\lambda$  in  $\sigma(T)$  such that  $T - \lambda I$  is a Browder operator, that is  $\sigma_d(T) = \sigma(T) \setminus \sigma_{\mathcal{B}}(T)$ , the complement of the Browder spectrum in the spectrum.

Analogously, we define here the B-discrete spectrum for closed operators as a natural extension of the discrete spectrum.

**Definition 1.2.** Let  $T \in C(X)$ . Then the B-discrete spectrum  $\sigma_{bd}(T)$  of  $T$  is defined by  $\sigma_{bd}(T) = \sigma(T) \setminus \sigma_{\mathcal{D}}(T)$ , the complement of the Drazin spectrum in the spectrum.

It's clear that for  $T \in C(X)$ , its discrete spectrum  $\sigma_d(T)$  is a subset of its B-Discrete spectrum  $\sigma_{bd}(T)$ .

**Definition 1.3.** We will say that  $T$  has a purely B-discrete spectrum if  $\sigma(T) = \sigma_{bd}(T)$ , and that  $T$  has a purely discrete spectrum if  $\sigma(T) = \sigma_d(T)$ .

**Example 1.4.** An illustrating example of an operator with purely B-discrete spectrum, is given by the Schrödinger operator with a constant magnetic field  $B \neq 0$  in  $\mathbb{R}^2$  defined by  $S_B = (\frac{1}{i} \frac{\partial}{\partial x_1} - \frac{Bx_2}{2})^2 + (\frac{1}{i} \frac{\partial}{\partial x_2} + \frac{Bx_1}{2})^2$ . Then from [8, Example 4, p.134],  $\sigma(S_B) = \{(2k+1) | B \mid |k \in \mathbb{N}\}$ . Its B-discrete spectrum coincides exactly with the set of its Landau levels, ([8, p.136]), while its discrete spectrum is the empty set. Moreover, each eigenvalue of  $S_B$  has an infinite multiplicity. As  $S_B$  is self-adjoint, we have  $\sigma(S_B) = \sigma_{bd}(S_B)$ , (See Corollary 2.11). Thus  $L$  has a purely B-discrete spectrum, but its discrete spectrum is empty.

The discrete spectrum has important applications in the study of physical operators. However, as shown by Example 1.4, the discrete spectrum may be empty while we have discrete energy levels! Moreover, as shown by Example 2.6, the discrete spectrum does not give a clear idea on the nature of isolated points of the spectrum: are there poles of infinite rank or essential singularities of the resolvent? But an isolated point of the spectrum which is not in the B-discrete spectrum is always an essential singularity.

**Definition 1.5.** Let  $T \in C(X)$ , with a non-empty resolvent set. We will say that  $T$  has a meromorphic (resp. Riesz or compact) resolvent if there exists a scalar  $\lambda$  in the resolvent set  $\rho(T)$  of  $T$  such that  $(T - \lambda I)^{-1}$  is a bounded linear meromorphic (resp. Riesz or compact) operator.

**Remark 1.6.** It's easily seen that if  $T$  has a meromorphic (resp. Riesz or compact) resolvent, then for all scalar  $\lambda$  in the resolvent set  $\rho(T)$  of  $T$ ,  $(T - \lambda I)^{-1}$  is a bounded linear meromorphic (resp. Riesz or compact) operator.

In the second section of this paper, we characterize closed invertible operators with non-empty resolvent set having a purely B-discrete spectrum, by showing that this the case if and only if the operator considered has a meromorphic resolvent and if and only if its B-Fredholm spectrum  $\sigma_{BF}(T)$  is empty. We show also that if  $T \in C(X)$  has a nonempty resolvent set and  $\lambda$  is an isolated point of its spectrum, then  $\lambda$  is in its B-discrete spectrum if and only if  $T - \lambda I$  is a B-Fredholm operator. When  $T$  is an hereditarily normaloid operator (Definition 2.8), then the B-discrete spectrum  $\sigma_{bd}(T)$  of  $T$  is the set of all isolated points of its spectrum  $\sigma(T)$ .

In the third section, we study the stability of the B-discrete spectrum under several type of perturbations. As an example of the results obtained, we show that if  $A$  and  $T$  are two commuting closed linear operators with nonempty resolvent sets, and if for some  $\lambda \in \rho(A) \cap \rho(T)$  the operator  $(\lambda I - A)^{-1} - (\lambda I - T)^{-1}$  is of finite rank, then  $A$  has a purely B-discrete spectrum if and only if  $T$  has a purely B-discrete spectrum. Moreover, we prove two spectral mapping theorems for the B-discrete spectrum.

In the fourth section, we show that if  $S, T$  are two closed invertible linear operators having quasisimilar totally paranormal inverses (Definition 2.9), then their spectra and their B-discrete spectra are equal.

## 2. B-discrete spectrum

We begin this section by characterizing operators with purely B-discrete spectrum.

**Theorem 2.1.** *Let  $T \in C(X)$  with a nonempty resolvent set. Then  $T$  has a purely B-discrete spectrum if and only if  $T$  has a meromorphic resolvent.*

*Proof.* Suppose that  $T$  has a purely B-discrete spectrum. So for all  $\lambda \in \mathbb{C}$ ,  $T - \lambda I$  is Drazin invertible operator. Since the resolvent set of  $T$  is non-empty, there exists  $\mu \in \mathbb{C}$ , such that  $T - \mu I$  is invertible. From [5, Theorem 3.6],  $(T - \mu I)^{-1} - \frac{1}{\lambda} I$  is Drazin invertible for all  $\lambda \neq 0$ , and  $\frac{1}{\lambda}$  is a pole of the resolvent of  $(T - \mu I)^{-1}$ . So  $(T - \mu I)^{-1}$  is a meromorphic operator. Hence  $T$  has a meromorphic resolvent.

Conversely, if  $T$  has a meromorphic resolvent, we can assume without loss of generality that  $T$  is invertible and that  $T^{-1}$  is a meromorphic operator. If  $\lambda \notin \sigma(T)$ , then  $T - \lambda I$  is invertible. If  $\lambda \in \sigma(T)$ , then  $\lambda \neq 0$ . Since  $T^{-1}$  is a meromorphic operator, then from [5, Theorem 3.6],  $\frac{1}{\lambda}$  is a pole of  $T^{-1}$  and again from [5, Theorem 3.6],  $\lambda$  is a pole of  $T$ . Therefore  $T - \lambda I$  is Drazin invertible for all  $\lambda \in \mathbb{C}$  and  $T$  has a purely B-discrete spectrum.  $\square$

**Remark 2.2.** Analogous result of Theorem 2.1 for the discrete spectrum can be deduced from [14, Theorem 2], that is  $T$  has a purely discrete spectrum if and only if  $T$  has a Riesz resolvent.

**Corollary 2.3.** *Let  $T \in C(X)$  with a nonempty resolvent set. Then  $T$  has a purely B-discrete spectrum if and only if  $\sigma_{BF}(T) = \emptyset$ .*

*Proof.* Suppose that  $T$  has a purely B-discrete spectrum. Then for all  $\lambda \in \mathbb{C}$ ,  $T - \lambda I$  is Drazin invertible. From [4, Theorem 2.9], it follows that  $T - \lambda I$  is a B-Fredholm operator. Hence  $\sigma_{BF}(T) = \emptyset$ .

Conversely assume that  $\sigma_{BF}(T) = \emptyset$ , and let  $\mu$  be in the resolvent set of  $T$ . Then from [5, Theorem 3.6],  $\sigma_{BF}((T - \mu I)^{-1}) \subset \{0\}$  and from Theorem 1.1,  $(T - \mu I)^{-1}$  is a meromorphic operator. Hence  $T$  has a meromorphic resolvent. From Theorem 2.1, it follows that  $T$  has purely B-discrete spectrum.  $\square$

**Example 2.4.** *Let  $A$  be the shift operators on the Hilbert space  $l^2(\mathbb{N})$ .*

$$\begin{aligned} A : D(A) \subset l^2(\mathbb{N}) &\longrightarrow l^2(\mathbb{N}) \\ x = (x_n)_{n \geq 0} &\longmapsto Ax = (0, x_1, 2x_2, 3x_3, \dots), \end{aligned}$$

where  $D(A) = \{x = (x_n)_{n \geq 0} \in l^2(\mathbb{N}) : \sum_{n \geq 0} n^2 |x_n|^2 < \infty\}$

It is easy to verify that the complex number  $i$  ( $i^2 = -1$ ) is in  $\rho(A)$ . Moreover one can check easily that  $\sigma(A) = \mathbb{N}$  and that  $\sigma_{BF}(A) = \emptyset$ . Hence  $A$  has a purely B-discrete spectrum.

**Corollary 2.5.** *Let  $T \in C(X)$ , with a nonempty resolvent set. If  $T$  has a purely discrete spectrum, then  $T$  has a purely B-discrete spectrum and  $\sigma_{bd}(T) = \sigma_d(T)$ .*

*Proof.* It follows from [14, Theorem 2], that if  $T$  has purely discrete spectrum, then  $T$  has a Riesz resolvent. So  $T$  has a meromorphic resolvent. Thus  $T$  has a purely B-discrete spectrum. In this case, we have  $\sigma(T) = \sigma_d(T)$ , so  $\sigma_B(T) = \emptyset$ . Hence  $\sigma_D(T) = \emptyset$  and  $\sigma(T) = \sigma_{bd}(T)$ .  $\square$

As shown by the example 1.4, the converse of the previous corollary is not true in general.

The following example shows that even the spectrum of an operator is discrete, its B-discrete spectrum (and also its discrete spectrum) could be empty.

**Example 2.6.** Let  $Q$  be defined for each  $x = (\xi_i) \in \ell^1$  by

$$Q(\xi_1, \xi_2, \xi_3, \dots, \xi_k, \dots) = (0, \alpha_1 \xi_1, \alpha_2 \xi_2, \dots, \alpha_{k-1} \xi_{k-1}, \dots),$$

where  $(\alpha_i)$  is a sequence of complex numbers such that  $0 < |\alpha_i| \leq 1$  and  $\sum_{i=1}^{\infty} |\alpha_i| < \infty$ . We observe that

$$\overline{R(Q^n)} \neq R(Q^n), \quad n = 1, 2, \dots$$

Indeed, for a given  $n \in \mathbb{N}$  let  $x_k^{(n)} = (1, \dots, 1, 0, 0, \dots)$  (with  $n+k$  times 1). Then the limit  $y^{(n)} = \lim_{k \rightarrow \infty} Q^n x_k^{(n)}$  exists and lies in  $\overline{R(Q^n)}$ . However, there is no element  $x^{(n)} \in \ell^1$  satisfying the equation  $Q^n x^{(n)} = y^{(n)}$  as the algebraic solution to this equation is  $(1, 1, 1, \dots) \notin \ell^1$ .

It is easy to see that  $\sigma(Q) = \sigma_D(Q) = \{0\}$  and so  $\sigma_d(T) = \sigma_{bd}(T) = \emptyset$ .

In the next theorem, we give a necessary and sufficient condition under which an isolated point in the spectrum of a closed operator is in its B-discrete spectrum.

**Theorem 2.7.** *Let  $T \in C(X)$ , with a nonempty resolvent set and let  $\lambda$  be an isolated point of its spectrum. Then  $\lambda$  is in its B-discrete spectrum if and only if  $T - \lambda I$  is a B-Fredholm operator.*

*Proof.* If  $\lambda \in \sigma_{bd}(T)$ , then  $\lambda$  is a pole of its resolvent. From [15, Theorem 1.2], there exists an integer  $p$  such that  $R((T - \lambda I)^p)$  is closed and  $X = N((T - \lambda I)^p) \oplus R((T - \lambda I)^p)$ . From [4, Theorem 2.4], it follows that  $T - \lambda I$  is a B-Fredholm operator.

Conversely assume that  $\lambda$  is isolated in  $\sigma(T)$  and that  $T - \lambda I$  is a B-Fredholm operator. We can assume without loss of generality that  $T$  is invertible, thus  $\lambda \neq 0$  and from [5, Theorem 3.6],  $T^{-1} - \frac{1}{\lambda}I$  is a B-Fredholm operator. From [2, Theorem 2.3], it follows that  $\frac{1}{\lambda}$  is a pole of  $T^{-1}$  and from [5, Theorem 3.6] it follows that  $\lambda$  is a pole of  $T$ . So  $\lambda$  is in the B-discrete spectrum of  $T$ .  $\square$

**Definition 2.8.** *A bounded linear operator  $T$  acting on a Banach space is said to be normaloid if  $\rho(T) = \|T\|$ , where  $\rho(T)$  is the spectral radius of  $T$  or equivalently  $\|T^n\| = \|T\|^n$  for every  $n \in \mathbb{N}$ . We say that  $T$  is hereditarily normaloid if the restriction of  $T$  to any invariant subspace under  $T$  is again normaloid.*

**Definition 2.9.** *A bounded operator  $T$  on a Banach space is said to be paranormal if  $\|Tx\|^2 \leq \|T^2x\|\|x\|$  for all  $x \in X$ .  $T$  is said to be totally paranormal if  $T - \lambda I$  is paranormal for all  $\lambda \in \mathbb{C}$ .*

It is easy to see that every paranormal operator  $T$  is normaloid. Since the restriction of a paranormal operator  $T$  to any closed invariant subspace under  $T$  is also paranormal, then a paranormal operator is hereditarily normaloid.

A good example of totally paranormal operators is given by the class of hyponormal operators. Recall that a bounded linear operator  $T$  acting on Hilbert space  $H$  is said to be hyponormal if  $\|T^*x\| \leq \|Tx\|$ , for all  $x \in H$ , where  $T^*$  is the adjoint of  $T$ . From [12], we know that  $\text{Hyponormal} \subsetneq \text{Paranormal} \subsetneq \text{hereditarily normaloid} \subsetneq \text{normaloid}$  and the inclusions are all proper.

In the case of a closed invertible operator with a normaloid inverse, we can determine precisely its B-discrete spectrum.

**Theorem 2.10.** *Let  $T \in C(X)$  be a closed invertible operator having an hereditarily normaloid inverse. Then the B-discrete spectrum  $\sigma_{bd}(T)$  of  $T$  is the set of all isolated points of the spectrum  $\sigma(T)$  of  $T$ .*

*Proof.* If  $\lambda \in \sigma_{bd}(T)$ , then  $\lambda$  is a pole of its resolvent and so  $\lambda$  is an isolated point of its spectrum. Conversely assume that  $\lambda$  is an isolated point of the spectrum of  $T$ . As  $T^{-1}$  is hereditarily normaloid, then from [11, Lemma 2.1]  $\frac{1}{\lambda}$  is a pole of  $T^{-1}$  and from From [5, Theorem 3.6],  $\lambda$  is a pole of  $T$ . So  $\lambda$  is in  $\sigma_{bd}(T)$ .  $\square$

**Corollary 2.11.** *Let  $T$  be a normal operator acting on a Hilbert space  $H$ . Then the B-discrete spectrum  $\sigma_{bd}(T)$  of  $T$  is the set of all isolated points of the spectrum  $\sigma(T)$  of  $T$ .*

*Proof.* Since the resolvent set  $\rho(T)$  of  $T$  is nonempty, we can assume without loss of generality that  $T$  is invertible. Since  $T$  is normal, then from [19, Theorem 5.42],  $T^{-1}$  is a bounded normal operator. Since a normal operator is normaloid, then from Theorem 2.10,  $\sigma_{bd}(T)$  is the set of all isolated points of the spectrum  $\sigma(T)$  of  $T$ .  $\square$

Corollary 2.11 applies in particular to the case of self-adjoint operators and in particular to self-adjoint Schrödinger operators [19].

### 3. Stability of the B-discrete spectrum

In this Section, we study the stability of the B-discrete spectrum of a closed linear operator acting on a Banach space  $X$ , under the effect of several type of perturbations

**Definition 3.1.** [16] Let  $X$  be a Banach space,  $A : D(A) \subset X \rightarrow X$  and  $T : D(T) \subset X \rightarrow X$  two linear operators. We say that  $A$  commutes with  $T$  and we denote  $AT = TA$ , if

- (i)  $D(A) \subset D(T)$ .
- (ii)  $Tx \in D(A)$  whenever  $x \in D(A)$ .
- (iii)  $AT = TA$  on  $\{x \in D(A), Ax \in D(T)\}$ .

It is easily seen that if  $A$  and  $T$  are two commuting closed linear operators on a Banach space  $X$  and if  $\lambda \in \rho(A) \cap \rho(T)$ , then  $(\lambda I - A)^{-1}(\lambda I - T)^{-1} = (\lambda I - T)^{-1}(\lambda I - A)^{-1}$ .

**Theorem 3.2.** Let  $A$  and  $T$  be two closed linear operators with a nonempty resolvent sets. If  $AT = TA$  and  $(\lambda I - A)^{-1} - (\lambda I - T)^{-1}$  is a nilpotent operator for some  $\lambda \in \rho(A) \cap \rho(T)$ , then  $\sigma_{bd}(A) = \sigma_{bd}(T)$ .

*Proof.* Without loss of generality, we can assume that  $\lambda = 0$ . Let  $\mu \in \mathbb{C} \setminus \{0\}$ . Since  $T^{-1} - A^{-1}$  is a nilpotent operator commuting with  $A^{-1}$ , then  $\sigma(A^{-1}) = \sigma(A^{-1} + (T^{-1} - A^{-1})) = \sigma(T^{-1})$ . Moreover and from [3, Theorem 3.2] we know that  $\sigma_D(A^{-1}) = \sigma_D(A^{-1} + (T^{-1} - A^{-1})) = \sigma_D(T^{-1})$ . Thus  $\sigma(A) = \sigma(T)$  and  $\sigma_D(A) = \sigma_D(T)$ . Consequently we have  $\sigma_{bd}(A) = \sigma_{bd}(T)$ .  $\square$

**Corollary 3.3.** Let  $A \in C(X)$  with a nonempty resolvent set and let  $Q \in L(X)$  be a nilpotent operator satisfying  $AQ = QA$ . Then  $\sigma_{bd}(A + Q) = \sigma_{bd}(A)$ .

**Proof.** Since  $AQ = QA$  and  $Q$  is a nilpotent operator, then  $(\mu I - A)^{-1}Q$  is also a nilpotent operator for all  $\mu \in \rho(A)$ . Then  $I - (\mu I - A)^{-1}Q$  is an invertible operator and so  $\mu I - A - Q$  is also invertible. Moreover  $(\mu I - A - Q)^{-1} = (\mu I - A)^{-1}(I - (\mu I - A)^{-1}Q)^{-1} = (\mu I - A)^{-1} \sum_{k=0}^n ((\mu I - A)^{-1}Q)^k = (\mu I - A)^{-1} + (\mu I - A)^{-1}Q \sum_{k=1}^{n-1} ((\mu I - A)^{-1}Q)^{k-1}$  where  $n$  is the nilpotent-index of  $(\mu I - A)^{-1}Q$ . Hence,  $(\mu I - A - Q)^{-1} - (\mu I - A)^{-1}$  is nilpotent. From Theorem 3.2, we deduce that  $\sigma_{bd}(A + Q) = \sigma_{bd}(A)$ .

**Theorem 3.4.** Let  $A$  and  $T$  be two closed linear operators on a Banach space  $X$ . If  $AT = TA$  and for some  $\lambda \in \rho(A) \cap \rho(T)$  the operator  $(\lambda I - A)^{-1} - (\lambda I - T)^{-1}$  has a power of finite rank, then  $A$  has a purely B-discrete spectrum if and only if  $T$  has a purely B-discrete spectrum.

*Proof.* Without loss of generality we can assume that  $\lambda = 0$ . Let  $\mu \in \mathbb{C} \setminus \{0\}$ . From [5, Theorem 3.6], we see that  $\mu I - A$  is a B-Fredholm operator if and only if  $\mu^{-1}I - A^{-1}$  is a B-Fredholm operator. Since  $A^{-1} - T^{-1}$  has a power of finite rank, then from [18, Theorem 2.11],  $\mu^{-1}I - A^{-1}$  is a B-Fredholm operator if and only if  $\mu^{-1}I - T^{-1}$  is a B-Fredholm operator. From [5, Theorem 3.6] this is true if and only if  $\mu I - T$  is a B-Fredholm operator. This shows that  $\sigma_{BF}(A) = \sigma_{BF}(T)$ . Therefore  $A$  has a purely B-discrete spectrum if and only if  $T$  has a purely B-discrete spectrum.  $\square$

We conclude this section with two spectral mapping theorems for the B-discrete spectrum.

**Theorem 3.5.** Let  $A \in C(X)$  be a densely defined closed operator such that  $\rho(A) \neq \emptyset$ . Let  $P(\lambda)$  be a polynomial with complex coefficients. Then  $\sigma_{BF}(P(A)) = P(\sigma_{BF}(A))$ , and  $A$  has a purely B-discrete spectrum if and only if  $P(A)$  has a purely B-discrete spectrum.

*Proof.*  $P(A)$  is well defined and is a closed operator. From [7, Theorem 3.2] we have  $\sigma_{BF}(P(A)) = P(\sigma_{BF}(A))$ . Thus  $\sigma_{BF}(A) = \emptyset$  if and only if  $\sigma_{BF}(P(A)) = \emptyset$ .  $\square$

For an unbounded closed operator  $A$  with non-empty resolvent set, and a complex-valued functions  $f$  holomorphic on an open set containing  $\sigma(A) \cup \{\infty\}$ ,  $f(A)$  may be defined by the operational calculus introduced by Taylor in [17].

**Theorem 3.6.** *Let  $A \in C(X)$  be a closed operator with non-empty resolvent set and let  $f$  be complex-valued function holomorphic on an open set containing  $\sigma(A) \cup \{\infty\}$ . If  $f$  is an univalent function, then  $\sigma_{bd}(f(A)) = f(\sigma_{bd}(A))$ .*

*Proof.* Let  $\widetilde{\sigma}(A) = \sigma(A) \cup \{\infty\}$  and  $\widetilde{\sigma}_D(A) = \sigma_D(A) \cup \{\infty\}$  be the extended spectrum and the extended Drazin spectrum of  $A$ . From [13, Theorem 7], we know that  $f(\widetilde{\sigma}(A)) = \sigma(f(A))$  and from [6, Theorem 4.1]  $f(\widetilde{\sigma}_D(A)) = \sigma_D(f(A))$ . Therefore we have  $\sigma_{bd}(f(A)) = \sigma(f(A)) \setminus \sigma_D(f(A)) = f(\widetilde{\sigma}(A)) \setminus f(\widetilde{\sigma}_D(A)) = f(\widetilde{\sigma}(A) \setminus \widetilde{\sigma}_D(A)) = f(\sigma(A) \setminus \sigma_D(A))$ , because  $f$  is univalent. Hence  $\sigma_{bd}(f(A)) = f(\sigma_{bd}(A))$ .  $\square$

#### 4. B-discrete spectrum and quasi-similarity

A bounded linear operator  $A : X \rightarrow Y$  from the Banach spaces  $X$  to the Banach space  $Y$  is said to be quasi-invertible if it is injective and has dense range. Two bounded linear operators  $T \in L(X)$  and  $S \in L(Y)$  are quasisimilar if there exists quasi-invertible operators  $A : X \rightarrow Y$  and  $B : Y \rightarrow X$  such that  $AT = SA$  and  $BS = TB$ .

As mentioned in [10, p.89], the same proof of [10, Theorem 1] proved for hyponormal operators holds also for totally paranormal operators. Thus we formulate the following result without proof and we refer the reader to [10].

**Theorem 4.1.** *If two bounded linear operators  $T \in L(X)$  and  $S \in L(Y)$  are totally paranormal and quasisimilar, then they have the same spectrum. This is in particular the case of two quasisimilar hyponormal operators.*

Using this result, we prove now the equality of the B-discrete spectrum of two quasi-similar totally paranormal operators.

**Theorem 4.2.** *Let  $S$  and  $T$  be two totally paranormal and quasisimilar bounded linear operators acting on a Banach space  $X$ , then  $\sigma_{bd}(S) = \sigma_{bd}(T)$ . This is in particular the case of two quasisimilar hyponormal operators.*

*Proof.* Since  $S$  and  $T$  are quasisimilar, there exists quasi-invertible operators  $A : X \rightarrow Y$  and  $B : Y \rightarrow X$  such that  $AT = SA$  and  $BS = TB$ . In this case the operators  $\tilde{S}_{[n]} : \overline{R(S^n)} \rightarrow \overline{R(S^n)}$  and  $\tilde{T}_{[n]} : \overline{R(T^n)} \rightarrow \overline{R(T^n)}$  defined as the restrictions of  $S$  and  $T$  respectively to the closure of the ranges  $R(S^n)$  and  $R(T^n)$ , are also quasisimilar. Indeed, if we consider  $P = A/\overline{R(T^n)} : \overline{R(T^n)} \rightarrow \overline{R(S^n)}$  and  $Q = B/\overline{R(S^n)} : \overline{R(S^n)} \rightarrow \overline{R(T^n)}$ , then it is easily seen that  $\overline{P(R(T^n))} = \overline{R(S^n)}$ ,  $\overline{Q(R(S^n))} = \overline{R(T^n)}$ ,  $P$  and  $Q$  are both injective,  $P\tilde{T}_{[n]} = \tilde{S}_{[n]}P$  and  $Q\tilde{S}_{[n]} = \tilde{T}_{[n]}Q$ .

Let  $\alpha \in \sigma_{bd}(T)$  be arbitrary. Then  $T - \alpha I$  is Drazin invertible and  $a(T - \alpha I) = \delta(T - \alpha I) = n < \infty$ . Since we are dealing with totally paranormal operators, we may assume without loss of generality that  $\alpha = 0$ . Therefore  $\overline{R(T^n)}$  is closed, and  $\tilde{T}_{[n]} : R(T^n) \rightarrow R(T^n)$  is invertible. On the other hand,  $\tilde{T}_{[n]} : R(T^n) \rightarrow R(T^n)$  and  $\tilde{S}_{[n]} : \overline{R(S^n)} \rightarrow \overline{R(S^n)}$  are totally paranormal quasisimilar operators. Since  $\tilde{T}_{[n]}$  is invertible, then from Theorem 4.1,  $\tilde{S}_{[n]}$  is invertible. So  $(\tilde{S}_{[n]})^n$  is invertible and  $R((\tilde{S}_{[n]})^n) = \overline{R(S^n)}$ . As  $R((\tilde{S}_{[n]})^n) \subset R(S^n)$ , then  $\overline{R(S^n)} = R(S^n)$  and  $R(S^n)$  is closed. Therefore  $0 \notin \sigma_D(S)$ . As we know from Theorem 4.1 that  $\sigma(T) = \sigma(S)$ , then  $0 \in \sigma_{bd}(S)$ . Similarly, we have  $\sigma_{bd}(S) \subset \sigma_{bd}(T)$  and so  $\sigma_{bd}(S) = \sigma_{bd}(T)$ .  $\square$

**Theorem 4.3.** *Let  $S, T$  be two closed invertible linear operators having quasisimilar totally paranormal inverses, then  $\sigma(S) = \sigma(T)$  and  $\sigma_{bd}(S) = \sigma_{bd}(T)$ .*

*Proof.* Let  $U = S^{-1}$ , and  $V = T^{-1}$ , then  $U$  and  $V$  are totally paranormal and quasi-similar. From Theorem 4.1 and Theorem 4.2, we have  $\sigma(U) = \sigma(V)$  and  $\sigma_{bd}(U) = \sigma_{bd}(V)$ . Thus  $\sigma(S) = \sigma(T)$  and if  $\lambda \in \sigma_{bd}(T)$ , then  $\frac{1}{\lambda} \in \sigma_{bd}(U)$ . Thus  $\frac{1}{\lambda} \in \sigma_{bd}(V)$  and so  $\lambda \in \sigma_{bd}(S)$ . This implies that  $\sigma_{bd}(T) \subset \sigma_{bd}(S)$ . Similarly we have  $\sigma_{bd}(S) \subset \sigma_{bd}(T)$  and then  $\sigma_{bd}(T) = \sigma_{bd}(S)$ .  $\square$

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