Stieltjes Classes for Discrete Distributions of Logarithmic Type

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Abstract. Stieltjes classes play a significant role in the moment problem since they permit to expose explicitly an infinite family of probability distributions all having equal moments of all orders. Mostly, the Stieltjes classes have been considered for absolutely continuous distributions. In this work, they have been considered for discrete distributions. New results on their existence in the discrete case are presented.

1. Introduction

Stieltjes classes initially appeared in [11], while the name may be viewed as present-day. For good reasons, J. Stoyanov [12] suggested to use the name 'Stieltjes classes' and triggered their systematic study, which is still in progress. See, for example [6, 8, 13]. Mostly, Stieltjes classes have been considered for absolutely continuous distributions. However, they may also be used to construct sets of discrete distributions with the same sequence of moments. For the sequel, we need the following definitions, which are discrete analogues of those introduced in [12].

Definition 1.1. *Let X be a random variable possessing a discrete distribution with probability mass function $p_X = p$ given by $p(x_j) = p_j$, $j \in \mathbb{N}_0$. A sequence $h = \{h_j\}_{j \in \mathbb{N}_0}$ is a perturbation for p if*

$$M_h := \sup_j |h_j| = 1 \quad and \quad \sum_{j=0}^{\infty} x_j^k p_j h_j = 0 \quad for \ all \quad k \in \mathbb{N}_0.$$

It has to be noticed that not all probability mass functions own perturbations. Clearly, the existence of a perturbation implies the moment indeterminacy of the underlying distribution. In this connection, the next definition can be formulated.

Definition 1.2. *Given a probability mass function p and a perturbation h, the set*

$$\mathbf{S} := \{g : g = p(1 + \varepsilon h), \ \varepsilon \in [-1, 1]\}$$

is called a (discrete) Stieltjes class for p generated by h.

Examples of discrete Stieltjes classes are provided in [2] and [13, Section 11]. It has to be pointed out that, since [2] had appeared before [12] was published, the name 'Stieltjes class' had not been used in the former. However, the results of [2] can be easily restated in terms of Stieltjes classes. It is worth mentioning that C. Berg characterizes discrete moment-indeterminate distributions possessing Stieltjes classes as the ones which are not extreme points in the set of distributions with the same moment sequences. He also constructed an example of a moment-indeterminate distribution, whose probability mass function has no Stieltjes classes. See [2, Propositions 1.1 and 2.2].

In the present paper, the existence of Stieltjes classes is investigated related to certain families of discrete distributions. More precisely, given a non-negative integer-valued random variable X, this study aims to examine the presence of Stieltjes classes for the probability mass function of $Y = a^X, a > 0, a \neq 1$. We refer to the distributions of random variables of this form as *logarithmic type*. Obviously, when $a \in (0, 1)$, the distribution of Y is moment-determinate as it has a bounded support. Therefore, only the case $a > 1$ will be considered. Denote by $p_j = \mathbf{P}\{X = j\}, j \in \mathbb{N}_0$. Correspondingly, the probability generating function of X can be written as:

$$f(z) = \sum_{j=0}^{\infty} p_j z^j, \quad z \in \mathbb{C}.$$

Clearly, Y has finite moments of all orders if and only if $f(z)$ is entire. Some conditions in terms of coefficients and growth estimates of $f(z)$ for the Stieltjes classes of P_Y to exist are established. It will be proved that, under the condition

$$p_j \geqslant Ca^{-j(j-1)/2} \quad \text{for all } j \in \mathbb{N}_0 \text{ and some } C > 0,$$

the probability mass function of Y has a perturbation and, as a result, the distribution is moment-indeterminate. On the other hand, if

$$p_j = o\left(a^{-j(j-1)/2}\right) \quad \text{as } j \to \infty,$$

then no perturbation functions exist. The application of these results to the case when X has a log-concave distribution is provided.

Last but not least, it has to be acknowledged that this study is motivated by Examples 11.7 and 11.8 of [13].

2. Results and examples

To begin with, let us recall some facts and notation from the q-calculus, which will be needed for the sequel.

Given $q > 0$ and $j \in \mathbb{N}_0$, the shifted q-factorial $(q; q)_j$ is defined as:

$$(q; q)_0 = 1, \qquad (q; q)_j = \prod_{s=1}^{j}(1 - q^s) \qquad \text{and} \qquad (q; q)_\infty := \prod_{s=1}^{\infty}(1 - q^s).$$

The following identity established by Euler - see, for example, [4, formula (1.23)] - will be used repeatedly:

$$\prod_{j=0}^{\infty}(1 + q^j t) = \sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{(q; q)_j} t^j, \quad 0 < q < 1, \quad t \in \mathbb{C}. \tag{1}$$

Throughout the text, the letter C - with or without indexes - denotes a positive constant whose value is of no concern. Note that the same letter may be assigned to denote constants with different numerical values.

Theorem 2.1. *Let a random variable X have probability mass function $p_X(j) = p_j$, $j \in \mathbb{N}_0$, and $Y = a^X$, $a > 1$. If Y has finite moments of all orders and*

$$p_j \geqslant Ca^{-j(j-1)/2} \quad \text{for all} \quad j \geqslant 0, \tag{2}$$

then a perturbation of p_Y exists and, therefore, the distribution of Y is moment-indeterminate.

Proof. To establish this result, it suffices to find a bounded non-zero sequence $\tilde{h} = \{\tilde{h}_j\}_{j=0}^{\infty}$ satisfying

$$\sum_{j=0}^{\infty} a^{kj} p_j \tilde{h}_j = 0 \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{3}$$

Set

$$\tilde{h}_j = \frac{(-1)^j a^{-j(j-1)/2}}{(1/a; 1/a)_j p_j}, \quad j \in \mathbb{N}_0. \tag{4}$$

Obviously, $\tilde{h} \neq 0$, and owing to (2),

$$|\tilde{h}_j| \leqslant \frac{a^{-j(j-1)/2}}{(1/a; 1/a)_j Ca^{-j(j-1)/2}} \leqslant \frac{1}{C(1/a; 1/a)_{\infty}} =: C_1.$$

The validity of (3) follows immediately from (1) because for

$$\varphi(t) := \prod_{s=0}^{\infty} \left(1 - \frac{t}{a^s}\right)$$

one has

$$0 = \varphi(a^k) = \sum_{j=0}^{\infty} a^{kj} \frac{(-1)^j a^{-j(j-1)/2}}{(1/a; 1/a)_j} = \sum_{j=0}^{\infty} a^{kj} p_j \tilde{h}_j \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Taking $h := \tilde{h}/M_{\tilde{h}}$, where $M_{\tilde{h}} = \sup_j |\tilde{h}_j|$, one obtains a perturbation h for p_Y and in this way completes the proof. □

Corollary 2.2. *Let X and Y be as in Theorem 2.1. Then, the set*

$$\mathbf{S} = \left\{ g = \{g_j\} : g_j = p_j(1 + \varepsilon h_j), \varepsilon \in [-1, 1] \right\},$$

where h is constructed by means of (4), forms a Stieltjes class for p_Y.

Example 2.3. *(Log-Poisson distribution) Let X have Poisson distribution with parameter λ. Then, the probabilities $p_j = \lambda^j e^{-\lambda}/j!$, $j \in \mathbb{N}_0$ satisfy condition (2) for every $a > 1$. Indeed, with the help of Stirling's formula, it can be observed that*

$$p_j \sim C \exp\{j \ln \lambda - (j + 1/2) \ln j + j\} \geqslant C_1 \exp\left\{-\frac{j(j-1)}{2} \ln a\right\}.$$

Hence, by Theorem 2.1, the distribution of $Y = a^X$, $a > 1$, is moment-indeterminate and a Stieltjes class for p_Y can be written in the form:

$$\mathbf{S} := \{g : g = p_Y(1 + \varepsilon h), \ \varepsilon \in [-1, 1]\},$$

where $h = \tilde{h}/M_{\tilde{h}}$ and $\tilde{h}_j = \dfrac{(-1)^j e^{\lambda} j! \, a^{-j(j-1)/2}}{\lambda^j (1/a; 1/a)_j}, \quad j \in \mathbb{N}_0.$

The result of Theorem 2.1 can be complemented by the next result. The proof below is similar to the one presented in Lemma 2.6 of [7]. For the convenience of the readers, it will be presented in detail here.

Recall the standard notation $M(r; f) := \max_{|z|=r} |f(z)|$ where $f(z)$ is analytic in $\{z : |z| \leqslant r\}$. The following estimate will be needed.

Lemma 2.4. *Let $a > 1$ and $\rho_a(z) = \sum\limits_{j=0}^{\infty} a^{-j(j-1)/2} z^j$. Then,*

$$M(r; \rho_a) \leqslant C \exp \left\{ \frac{\ln^2 r}{2 \ln a} + \frac{\ln r}{2} \right\}, \quad C = C(a) \text{ and } r \geqslant r_0. \tag{5}$$

Proof. First, we estimate the sum of the following auxiliary series: $S_b(r) := \sum\limits_{j=0}^{\infty} b^{-j^2} r^j$, where $b > 1$ and $r > 0$. Clearly,

$$S_b(r) = \sum_{j=0}^{\infty} \exp\left\{ -j^2 \ln b + j \ln r \right\} = \exp\left\{ \frac{\ln^2 r}{4 \ln b} \right\} \sum_{j=0}^{\infty} \exp\left\{ -\ln b \left(j - \frac{\ln r}{2 \ln b} \right)^2 \right\}$$

$$= \exp\left\{ \frac{\ln^2 r}{4 \ln b} \right\} \sum_{j=0}^{\infty} b^{-\left(j - \ln r/(2 \ln b) \right)^2}.$$

Observe that, for every $t > 0$, there holds:

$$\sum_{j=0}^{\infty} b^{-(j-t)^2} = \sum_{j=0}^{\lfloor t \rfloor} b^{-(j-t)^2} + \sum_{j=\lfloor t \rfloor+1}^{\infty} b^{-(j-t)^2} \leqslant \sum_{j=0}^{\lfloor t \rfloor} b^{-j^2} + \sum_{j=0}^{\infty} b^{-j^2} \leqslant 2 \sum_{j=0}^{\infty} b^{-j^2} =: C(b)$$

independently of t. Consequently, for $r > 1$, we conclude that

$$S_b(r) \leqslant C(b) \exp\left\{ \frac{\ln^2 r}{4 \ln b} \right\}. \tag{6}$$

Writing $\rho_a(r) = \sum_{j=0}^{\infty} (\sqrt{a})^{-j^2} (\sqrt{a} r)^j$, one obtains with the help of (6):

$$M(r; \rho_a) = \rho_a(r) \leqslant C_1(a) \exp\left\{ \frac{\ln^2(\sqrt{a} r)}{4 \ln(\sqrt{a})} \right\} = C_2(a) \exp\left\{ \frac{\ln^2 r}{2 \ln a} + \frac{\ln r}{2} \right\}$$

as stated by (5). □

Theorem 2.5. *Let X and Y be as in Theorem 2.1. If*

$$p_j = o(a^{-j(j-1)/2}) \quad as \quad j \to +\infty,$$

then there are no perturbation functions for p_Y.

Proof. Assume that $0 \neq h = \{h_j\}_{j \geqslant 0}$ is a bounded sequence such that

$$\sum_{j=0}^{\infty} a^{kj} p_j h_j = 0 \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{7}$$

Set $c_j := p_j h_j$. Obviously, the coefficients c_j satisfy $c_j = o\left(a^{-j(j-1)/2} \right)$ as $j \to +\infty$, implying that the entire function $\phi(z) = \sum_{j=0}^{\infty} c_j z^j$ enjoys the estimate:

$$M(r; \phi) \leqslant \sum_{j=0}^{\infty} |c_j| r^j = o\left(M(r; \rho_a) \right) \quad \text{as} \quad r \to \infty. \tag{8}$$

Indeed, given $\varepsilon > 0$, there is $j_0 \in \mathbb{N}$ such that $|c_j| \leqslant \varepsilon a^{-j(j-1)/2}$ when $j > j_0$. Hence, $M(r; \phi) \leqslant \sum_{j=0}^{j_0} |c_j| r^j + \varepsilon M(r; \rho_a) =: P_{j_0}(r) + \varepsilon M(r; \rho_a)$. Meanwhile, since P_{j_0} is a polynomial and ρ_a is a transcendental entire function, it follows that $P_{j_0}(r) \leqslant \varepsilon M(r; \rho_a)$, $r \geqslant r_0$ leading to $M(r; \phi) \leqslant 2\varepsilon M(r; \rho_a)$ for r large enough. As $\varepsilon > 0$ has been selected arbitrarily, (8) follows. Plugging $r = a^k$ into (5), and using (8) one derives

$$M(a^k; \phi) = o(a^{k(k+1)/2}), \quad k \to \infty. \tag{9}$$

On the other hand, by (7), $\phi(a^k) = 0$ for all $k \in \mathbb{N}_0$. Applying Jensen's Theorem [14, §3.61, formula (2), page 126] in the annulus $\{z \in \mathbb{C} : 1 \leqslant |z| \leqslant q^{-k}\}$, one obtains

$$\int_1^{a^k} \frac{n(t; \phi)}{t} dt \leqslant \ln M(a^k; \phi) + C_1,$$

where $n(t; \phi)$ is the number of zeros counting multiplicities of ϕ in the annulus $\{z \in \mathbb{C} : 1 \leqslant |z| \leqslant t\}$. Since ϕ has zeros at $1, a, \ldots, a^k$, one obtains:

$$\int_1^{a^k} \frac{n(t; \phi)}{t} dt \geqslant \frac{k(k+1)}{2} \ln a,$$

implying that

$$\ln C M(a^k; \phi) \geqslant \frac{k(k+1)}{2} \ln a, \quad k \in \mathbb{N},$$

which, however, contradicts (9). The proof is complete. $\square$

To illustrate the results of Theorems 2.1 and 2.5, consider the following example.

Example 2.6. *(Log-Heine distribution) It is said that a random variable X has the Heine distribution with parameter $\lambda > 0$ - and is written $X \sim Heine(\lambda)$ - if its probability mass function is given by:*

$$p_j = e_q(-\lambda) \frac{q^{j(j-1)/2}((1-q)\lambda)^j}{(q; q)_j}, \quad j \in \mathbb{N}_0, \ \lambda > 0, \ 0 < q < 1, \tag{10}$$

where $e_q(t)$ denotes the following q-analogue of the exponential function:

$$e_q(t) = \prod_{j=0}^{\infty} \left(1 - t(1-q)q^j\right)^{-1}, \quad 0 < q < 1, \quad t < 1/(1-q).$$

This distribution is regarded as a q-analogue of the Poisson distribution since, when $q \to 1^-$, the Poisson distribution with parameter λ is recovered. It is an important q-distribution, see [4, Section 2.3].

If $X \sim Heine(\lambda)$, the distribution of $Y = a^X, a > 0, a \neq 1$ is called log-Heine. It is not difficult to see that the Heine distribution possesses finite moments of all orders and, moreover, it is moment-determinate as its moment-generating function exists for all real numbers. The same is true for log-Heine distribution with $a \in (0, 1)$ since it has a bounded support. Now, let $X \sim Heine(\lambda)$ and $a > 1$. Taking into account that $(q; q)_{\infty} < (q; q)_j < 1$, one obtains, using (10),

$$C_1 \cdot [\lambda(1-q)]^j (qa)^{j(j-1)/2} \leqslant \frac{p_j}{a^{-j(j-1)/2}} \leqslant C_2 \cdot [\lambda(1-q)]^j (qa)^{j(j-1)/2}.$$

By virtue of Theorems 2.1 and 2.5, the last estimate implies that Stieltjes classes for the log-Heine distribution exist if and only if either $a > 1/q$ or $a = 1/q$ and $\lambda(1 - q) \geqslant 1$.

The next statement deals with discrete log-concave distributions. By definition, a discrete distribution on integers is said to be *log-concave* if its probability mass function satisfies the condition:

$$p_j^2 \geqslant p_{j-1}p_{j+1} > 0, \quad j \in \mathbb{N}. \tag{11}$$

Such distributions have been studied by many authors from different angles and have been found interesting for applications. See, for example, [1, 5], and [10].

Log-concave sequences are also of importance in the theory of entire functions. Assume that the probability generating function $f(z) = \sum_{j=0}^{\infty} p_j z^j$ is entire. Then, for each $r > 0$, $\sum_{j=0}^{\infty} p_j r^j < \infty$, whence $\{p_j r^j\} \to 0$. Thence, the sequence has, for every $r > 0$, a maximal term which is not necessarily unique. Now, fix $k \in \mathbb{N}_0$ and look for $r > 0$ such that $p_k r^k$ is a maximal term of the sequence $\{p_j r^j\}_{j=0}^{\infty}$. In general, for an arbitrary sequence $\{p_j r^j\}_{j=0}^{\infty}$ such r may not exist, for example, if $p_k = 0$. Nevertheless, condition (11) guarantees that each term $p_k r^k$ is a maximal term for some $r = r_k$, see, for example, [9, Part IV, Ch. 1, problem 43]. This property is crucial for the proof of part (*ii*) in the next theorem.

Theorem 2.7. *Let X be a random variable whose probability generating function $f(z)$ is entire and $Y = a^X, a > 1$.*
   (*i*) *If*

$$0 < \limsup_{r \to \infty} \frac{\ln f(r)}{\ln^2 r} =: \beta < \infty \tag{12}$$

and $a < \exp\{1/(2\beta)\}$, then the probability mass function of Y has no Stieltjes classes.
   (*ii*) *Additionally, assume that X has a log-concave distribution. If*

$$0 < \lim_{r \to \infty} \frac{\ln f(r)}{\ln^2 r} =: \beta < \infty \tag{13}$$

and $a > \exp\{1/(2\beta)\}$, then the probability mass function of Y has Stieltjes classes, and, thus, its distribution is moment-indeterminate.

Proof. (*i*) Since the coefficients of f are positive, one has $M(r; f) = f(r)$ and the Cauchy estimates for the coefficients of f imply that $p_j \leqslant f(r)r^{-j}$. By virtue of (12), if $\varepsilon > 0$, then

$$\ln p_j \leqslant \ln f(r) - j \ln r \leqslant (\beta + \varepsilon) \ln^2 r - j \ln r, \quad r \geqslant r_0 = r_0(\varepsilon).$$

Taking the minimum with respect to r, one obtains $\ln p_j \leqslant -\frac{j^2}{4(\beta+\varepsilon)}$ for $j \geqslant j_0(\varepsilon)$. Hence,

$$\limsup_{j \to \infty} \frac{\ln p_j}{j^2} \leqslant -\frac{1}{4\beta}.$$

The latter inequality implies that if $a < \exp\{1/(2\beta)\}$, then the upper estimate for p_j gives $p_j = o\left(a^{-j(j-1)/2}\right)$, $j \to \infty$, and Theorem 2.5 is applicable.

   (*ii*) It is commonly known that, in general, it is impossible to obtain lower estimates for the coefficients of an entire function from its growth estimate. Below, under the additional condition that $\{p_j\}$ is log-concave, such estimates will be derived using the result by V. Boicuk and A. Eremenko about the Dirichlet series ([3, Theorem 3]). A version of their proof is presented showing that if the conditions of (*ii*) are satisfied, then $\ln p_j \geqslant -j^2/(4\beta) + \alpha_j$, where $\alpha_j = o(j^2)$ as $j \to \infty$.

Assume that there exists a subsequence $\{p_k\}$, $k \in K \subset \mathbb{N}_0$ such that

$$\ln p_k \leqslant -\frac{k^2}{4\gamma}, \quad 0 < \gamma < \beta, \ k \in K.$$

It follows from (11) that each term $p_k r^k$ is a maximal term for some $r = r_k$. That is, $\mu(r_k) := \max_j p_j r_k^j = p_k r_k^k$.

Then, since $\ln M(r_k) / \ln \mu(r_k) \to 1$ as $k \to \infty$ (we refer to [9, Part IV, Ch. 1, problem 54], for every $\delta > 1$ and k large enough, there holds: $\ln M(r_k; f) \leqslant \delta \ln \mu(r_k)$. Select $\delta > 1$ in such a way that $\delta \gamma < \beta$. Therefore, for $k \in K$, one derives

$$\ln M(r_k; f) \leqslant \delta \left(\ln p_k + k \ln r_k \right) \leqslant \delta \left(-\frac{k^2}{4\gamma} + k \ln r_k \right) \leqslant \delta \max_t \left(-\frac{t^2}{4\gamma} + t \ln r_k \right) = \delta \gamma \ln^2 r_k, \quad k \geqslant k_0.$$

Hence,

$$\limsup_{k \to \infty} \frac{\ln M(r_k; f)}{\ln^2 r_k} \leqslant \delta \gamma < \beta,$$

contrary to (13). The contradiction shows that

$$\liminf_{j \to \infty} \frac{\ln p_j}{j^2} \geqslant -\frac{1}{4\beta},$$

whence

$$p_j \geqslant \exp\{-j^2/(4\beta) + o(j^2)\}, \quad j \to \infty. \tag{14}$$

Now, if $a > \exp\{1/(2\beta)\}$, one has:

$$p_j a^{j(j-1)/2} \geqslant \exp\left\{ -\frac{j^2}{4\beta} + \frac{j(j-1)}{2} \ln a + o(j^2) \right\} = \exp\left\{ \frac{j^2}{2} \left(\ln a - \frac{1}{2\beta} \right) + o(j^2) \right\} \to +\infty \quad \text{as} \quad j \to +\infty$$

because $\ln a > 1/(2\beta)$. Since all $p_j > 0$, it follows that $p_j \geqslant C a^{-j(j-1)/2}$ for some $C > 0$ and all $j \in \mathbb{N}_0$. By Theorem 2.1, the Stieltjes classes for the probability mass function of $Y = a^X$ exist. $\square$

Remark 2.8. *It has to be pointed out that for $a = \exp\{1/(2\beta)\}$ the existence of Stieltjes classes has to be investigated in more detail, as the following examples demonstrate. If $p_j = C \exp\{-j(j-1)/(4\beta)\}$, then Stieltjes classes exist by Theorem 2.1, while for $p_j = C \exp\{-j(j+1)/(4\beta)\}$, there are no Stieltjes classes due to Theorem 2.5.*

Another example is presented by the log-Heine distribution, see (10). If $X \sim \text{Heine}(\lambda)$, then $\beta = 1/[2\log(1/q)]$ and $\exp\{1/(2\beta)\} = 1/q$. Example 2.6 shows that the log-Heine distribution $(1/q)^X$ has Stieltjes classes if and only if $\lambda \geqslant 1/(1-q)$.

Notice that in all these cases the distributions of X are log-concave.

3. Acknowledgements

References

[1] M. An, Log-Concave Probability Distributions: Theory and Statistical Testing, *SSRN Electronic Journal*, 1–29 (1996)
[2] Ch. Berg, On some indeterminate moment problems for measures on a geometric progression, *J. Comput. Appl. Math.* **99** 67–75 (1998)
[3] V. S. Boicuk, A. E. Eremenko, The growth of entire functions that are representable by Dirichlet series (Russian), *Izv. Vysš. Učebn. Zaved. Matematika* **5**(156), 93–95 (1975)
[4] Ch. A. Charalambides, *Discrete q-Distributions*, Wiley, Hoboken, New Jersey, 2016.
[5] S. Dharmadhikari and K. Joag-Dev, *Unimodality, Convexity, and Applications*, Academic Press, New York, 1988.

[6] G. D. Lin, Recent developments on the moment problem, *Journal of Statistical Distributions and Applications*, **4:5**, DOI: 10.1186/s40488-017-0059-2 (2017)

[7] S. Ostrovska, M. Turan, q-Stieltjes classes for some families of q-densities, *Statistics & Probability Letters* **146**, 118–123 (2019)

[8] A. G. Pakes, Structure of Stieltjes classes of moment-equivalent probability laws. *J. Math. Anal. Appl.* **326**(2), 1268–1290 (2007)

[9] G. Polya, G. Szegő, *Problems and Theorems in Analysis II*, Springer-Verlag, (1998)

[10] A. Saumard and J. A. Wellner, Log-Concavity and Strong Log-Concavity: a review, arXiv:1404.5886v1, 23 Apr 2014.

[11] T. J. Stieltjes. Recherches sur les fractions continues. *Annales de la Faculté des Sciences de Toulouse* **8**, J76-J122 (1894)

[12] J. Stoyanov. Stieltjes classes for moment-indeterminate probability distributions. *J. Appl. Probab.* **41A**, 281–294 (2004)

[13] J. Stoyanov, *Counterexamples in Probability*, 3nd edn. Dover Publications, New York, (2013)

[14] E.C. Titchmarsh, *The theory of functions*, Oxford University Press, New York, (1986)