Filomat 34:8 (2020), 2495–2510
https://doi.org/10.2298/FIL2008495C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weak and Strong Convergence Results for the Modified Noor Iteration of Three Quasi-Nonexpansive Multivalued Mappings in Hilbert Spaces

Watcharaporn Chaolamjiak[a], Suhel Ahmad Khan[b], Hasanen A. Hammad[c], Hemen Dutta[d]

[a]*School of Science, University of Phayao, Phayao 56000, Thailand*
[b]*Department of Mathematics, BITS-Pilani, Dubai Campus, P.O. Box 345055, Dubai, UAE*
[c]*Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[d]*Department of Mathematics, Gauhati University, Guwahati-781014, India*

Abstract. The paper aims to present an advanced algorithm by taking help of the Noor-iteration scheme along with the inertial technical term for three quasi-nonexpansive multivalued in Hilbert spaces. A weak convergence theorem under certain conditions has been given and added the CQ and shrinking projection methods to our algorithm to obtain certain strong convergence results. Furthermore, numerical experiments are provided by constructing an example and comparison results have also been incorporated.

1. Introduction and Some Basic Notions

Throughout, we assume $\gimel$ to be a real Hilbert space, $¥ \neq \emptyset$, a closed convex subset of $\gimel$, and the non-empty families of closed bounded, compact and proximinal bounded subsets of $\gimel$ shall be denoted by $CB(¥)$, $K(¥)$ and $P(¥)$, respectively. Further, we use $'' \rightharpoonup ''$ and $'' \to ''$ to denote weak convergence and strong convergence, respectively.

Now, if for all $\kappa \in \gimel$, there exists $\omega \in ¥$ such that

$$\|\kappa - \omega\| = \xi(\kappa, ¥) = \inf\{\|\kappa - \tau\| : \tau \in ¥\}.$$

then $¥ \subset \gimel$ is said to be proximinal.

On $CB(¥)$, the Hausdorff metric $\daleth_\xi (= \daleth)$ is defined for all $\Xi, \Theta \in CB(¥)$, as follows:

$$\daleth_\xi(\Xi, \Theta) = \max\left\{\sup_{\kappa\in\Xi} \xi(\kappa, \Theta), \sup_{\omega\in\Theta} \xi(\omega, \Xi)\right\},$$

where $\xi(\kappa, \Theta) = \inf_{\ell\in\Theta}\{\|\kappa - \ell\|\}$.

2010 *Mathematics Subject Classification*. Primary 54H25; Secondary 47H10

Keywords. Weak and strong convergence, Quasi-nonexpansive multivalued mapping, Noor-iteration, inertial technical term, CQ and shrinking projection methods

Received: 28 July 2019; Accepted: 05 July 2020

Communicated by Dragan S. Djordjević

The first author is supported by the Thailand Science Research and Innovation, Grant/Award Number: IRN62W0007; University of Phayao, Thailand.

Email addresses: c-wchp007@hotmail.com (Watcharaporn Chaolamjiak), khan.math@gmail.com (Suhel Ahmad Khan), h_elmagd89@yahoo.com (Hasanen A. Hammad), hemen_dutta08@rediffmail.com (Hemen Dutta)

The nonexpansive (n.e.) conditions of a single-valued mapping $\mathfrak{I} : ¥ \to ¥$ as well as that of a multi-valued mapping $\mathfrak{I} : ¥ \to CB(¥)$ are given respectively by means of the inequalities $\left\|\mathfrak{I}\kappa - \mathfrak{I}\omega\right\| \leq \|\kappa - \omega\|$ and $\urcorner(\mathfrak{I}\kappa, \mathfrak{I}\omega) \leq \|\kappa - \omega\|$, for all $\kappa, \omega \in ¥$. Also, an element $\wp \in ¥$ is said to be a fixed point of a single-valued mapping (multi-valued mappings) if $\wp = \mathfrak{I}\wp$ ($\wp \in \mathfrak{I}\wp$), respectively. We denote by $\Omega(\mathfrak{I})$, the set of all fixed points for the mapping $\mathfrak{I}$.

Suppose that $\Omega(\mathfrak{I}) \neq \emptyset$, $\mathfrak{I} : ¥ \to CB(¥)$ is a multi-valued mapping and I is the identity mapping. Then,
• $\mathfrak{I}$ is called quasi-nonexpansive (q.n.e.) mapping if for all $\kappa \in ¥$ and $\wp \in \Omega(\mathfrak{I})$,

$$\urcorner(\mathfrak{I}\kappa, \mathfrak{I}\wp) \leq \|\kappa - \wp\|.$$

• $I - \mathfrak{I}$ is called demiclosed at $\omega \in ¥$ if $\{\kappa_n\}_{n=1}^{\infty} \subset ¥$ such that $\kappa_n \rightharpoonup \kappa$ and $\{\kappa_n - \tau_n\} \to \omega$ for some $\tau_n \in \mathfrak{I}\kappa_n$ imply $\kappa - \omega \in \mathfrak{I}\kappa$.

Some earlier interesting and useful results related to fixed points involving n.e. single-valued mappings studied by many researchers are available in [1, 5, 11–13, 16, 24, 27–29]. An important finding by Mann [19] to approximate fixed point of a single-valued nonexpansive mapping in Hilbert spaces is the following:

$$\kappa_{n+1} = \vartheta_n\kappa_n + (1 - \vartheta_n)\mathfrak{I}\kappa_n, \ \forall n \in \mathbb{N}. \tag{1}$$

Later, Ishikawa [15] developed a generalization of the above iterative algorithm (1) by Mann as: for an arbitrary $\kappa_0 \in ¥$,

$$\begin{cases} \kappa_{n+1} = \vartheta_n\kappa_n + (1 - \vartheta_n)\mathfrak{I}\phi_n, \\ \phi_n = \sigma_n\kappa_n + (1 - \sigma_n)\mathfrak{I}\kappa_n, \ n \geq 0, \end{cases} \tag{2}$$

where $\{\vartheta_n\}$ and $\{\sigma_n\}$ are sequences in $[0, 1]$.

Ishikawa's iterative algorithm (2) was further extended by Noor [22] as: for an arbitrary $\kappa_1 \in ¥$,

$$\begin{cases} \kappa_{n+1} = \vartheta_n\kappa_n + (1 - \vartheta_n)\mathfrak{I}\phi_n, \\ \phi_n = \sigma_n\kappa_n + (1 - \sigma_n)\mathfrak{I}\omega_n, \\ y_n = \mu_n\kappa_n + (1 - \mu_n)\mathfrak{I}\kappa_n, \ n \geq 1, \end{cases}$$

where $\{\vartheta_n\}$, $\{\sigma_n\}$ and $\{\mu_n\}$ are sequences in $[0, 1]$.

Ishikawa iteration method converges weakly even in Hilbert spaces while Mann's iteration has the weak convergence theorem only (see [4]). A strong algorithm for modified Mann algorithm was given by Nakajo and Takahashi [21], which is called CQ-algorithm: for an arbitrary $\kappa_\circ \in ¥$,

$$\begin{cases} \omega_n = \vartheta_n\kappa_n + (1 - \vartheta_n)\mathfrak{I}\kappa_n, \\ C_n = \{\tau \in ¥ : \|\omega_n - \tau\| \leq \|\kappa_n - \tau\|\}, \\ Q_n = \{\tau \in ¥ : \langle\kappa_\circ - \kappa_n, \kappa_n - \tau\rangle\}, \\ \kappa_{n+1} = P_{Q_n \cap C_n}(\kappa_\circ). \end{cases}$$

They proved that if the sequence $\{\vartheta_n\}$ is bounded above by 1, then the sequence $\{\kappa_n\}$ converges strongly to $P_{Fix(\mathfrak{I})}(\kappa_0)$.

Mann's iteration method (1) was also modified by Takahashi et al. [31] by involving just one closed convex set for a family of n.e. mappings $\{\kappa_n\}$ as: given $\kappa_\circ \in \beth$,

$$\begin{cases} ¥_1 = ¥, \ \kappa_1 = P_{C_1}\kappa_\circ \\ \omega_n = \vartheta_n\kappa_n + (1 - \vartheta_n)\mathfrak{I}_n\kappa_n, \\ C_n = \{\tau \in ¥ : \|\omega_n - \tau\| \leq \|\kappa_n - \tau\|\}, \\ \kappa_{n+1} = P_{C_{n+1}}(\kappa_\circ). \end{cases}$$

They were able to establish that if $\vartheta_n \leq \ell$, for all $n \geq 1$ and for some $0 < \ell < 1$, then the sequence $\{\kappa_n\}$ converges strongly to $P_{Fix(\mathfrak{I})}(\kappa_0)$.

The notion of nonspreading mappings was given by Kohsaka and Takahashi [17, 18] in 2008 for Banach spaces, and obtained some interesting results related to fixed points as well as common fixed points for single and commutative family of nonspreading mappings, respectively. Let $\kappa, \omega \in ¥$ be any two elements. We call a mapping $\Im : ¥ \to ¥$:

♦ nonspreading [17] if

$$2\left\|\Im\kappa - \Im\omega\right\|^2 \le \left\|\kappa - \Im\omega\right\|^2 + \left\|\omega - \Im\kappa\right\|^2,$$

♦ nonspreading [14] if and only if

$$\left\|\Im\kappa - \Im\omega\right\|^2 \le \|\kappa - \omega\|^2 + 2\langle \kappa - \Im\omega, \omega - \Im\omega\rangle,$$

♦ hybrid [32] if

$$\left\|\Im\kappa - \Im\omega\right\|^2 \le \|\kappa - \omega\|^2 + \langle \kappa - \Im\kappa, \omega - \Im\omega\rangle,$$

or equivalently, a mapping $\Im : ¥ \to ℷ$ is called hybrid iff

$$3\left\|\Im\kappa - \Im\omega\right\|^2 \le \|\kappa - \omega\|^2 + \left\|\omega - \Im\kappa\right\|^2 + \left\|\kappa - \Im\omega\right\|^2.$$

The results of Kohsaka and Takahashi [17, 18] seems to motivate the works of Iemoto and Takahashi [14], Takahashi [32], Cholamjiak and Cholamjiak [6] respectively in Hilbert spaces. A multi-valued mapping $T : ¥ \to CB(¥)$ is called a hybrid if

$$3\urcorner(\Im\kappa, \Im\omega)^2 \le \|\kappa - \omega\|^2 + \xi(\omega, \Im\kappa)^2 + \xi(\kappa, \Im\omega)^2,$$

for all $\kappa, \omega \in ¥$. $\Im$ is a quasi-nonexpansive, whenever $\Im$ is hybrid and $\Omega(\Im) \neq \emptyset$. One can find more details and counter example in ([7]).

In 2001, the heavy ball method [25, 26] was applied to inertial proximal point algorithm by Alvarez and Attouch [3] as follows:

$$\begin{cases} \omega_n = \kappa_n + \wp_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (I + r_n\Upsilon)^{-1}\omega_n, \ n \ge 1. \end{cases} \tag{3}$$

by using the proximal point algorithm for maximal monotone operators. They demonstrate that the algorithm (3) converges weakly to zero of Υ, if $\{r_n\}$ is increasing and $\{\wp_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \wp_n\|\kappa_n - \kappa_{n-1}\|^2 < \infty. \tag{4}$$

Also, the hypothesis (4) holds for $\wp_n < 1/3$, $\wp_n$. The terms $\wp_n$ and $\wp_n(\kappa_n - \kappa_{n-1})$ refer to extrapolation factor and inertia, respectively. The inertial concept is beneficial for improving the efficacy of algorithms as well as for better convergence properties [2, 9, 10, 23].

The above works inspire us to introduce a more advanced iteration scheme by modifying the scheme of Noor with the inertial technical term aiming at finding common fixed points of three quasi-nonexpansive multivalued mappings. We aim to obtain first weak convergence theorems, and then strong convergence theorems by using CQ and shrinking projection methods in combination with the modified Noor iteration scheme. We shall also present a comparison between our inertial projection and the standard projection algorithms. Further, we shall discuss some numerical experiments and examine the convergence rate of our algorithm.

2. Necessary Lemmas and a Condition

This section of the paper basically recalls some necessary results as lemmas those will be required to understand the main findings of the paper.

Lemma 2.1. *[32] For each $\kappa, \omega \in \gimel$ and a real number $\mho$, we have*
(i) $\|\kappa + \omega\|^2 \leq \|\kappa\|^2 + 2\langle \omega, \kappa + \omega \rangle$,
(ii) $\|\mho\kappa + (1-\mho)\omega\|^2 = \mho\,\|\kappa\|^2 + (1-\mho)\,\|\omega\|^2 - \mho(1-\mho)\,\|\kappa - \omega\|^2$.

Lemma 2.2. *[20] For each $\kappa, \omega, \upsilon \in \gimel$ and $\eth \in \mathbb{R}$, the following set is closed and convex:*

$$\{\eta \in ¥ : \left\|\omega - \eta\right\|^2 \leq \left\|\kappa - \eta\right\|^2 + \langle \upsilon, \eta \rangle + \eth\}.$$

Lemma 2.3. *[21] Let $P_C : \gimel \to ¥$ be the metric projection. Then*

$$\|\omega - P_{¥}\kappa\|^2 + \|\kappa - P_{¥}\kappa\|^2 \leq \|\kappa - \omega\|^2$$

for all $\kappa \in \gimel$ and $\omega \in ¥$.

Lemma 2.4. *[3] Assume the sequences $\{\Lambda_n\}$, $\{\nu_n\}$ and $\{\vartheta_n\}$ in $\mathbb{R}^+$ be such that $\forall n \geq 1$, $\sum_{n=1}^{\infty} \nu_n < +\infty$,*

$$\Lambda_{n+1} \leq \Lambda_n + \vartheta_n(\Lambda_n - \Lambda_{n-1}) + \nu_n.$$

If there is a real number ϑ satisfy $0 \leq \vartheta_n \leq \vartheta < 1$. Then the assertions hold:
(i) $\Sigma_{n\geq 1}[\Lambda_n - \Lambda_{n-1}]_+ < +\infty$, *where* $[\rho]_+ = \max\{\rho, 0\}$;
(ii) $\exists \Lambda^* \in \mathbb{R}^+$ *such that* $\lim_{n\to+\infty} \Lambda_n = \Lambda^*$.

Lemma 2.5. *[30] Assume that $\complement$ is a Banach space that satisfies the Opial's condition and that $\{\kappa_n\}$ is a sequence in $\complement$. Assume $p, r \in \complement$ are such that $\lim_{n\to\infty} \left\|\kappa_n - p\right\|$ and $\lim_{n\to\infty} \left\|\kappa_n - p\right\|$ holds. If the subsequences $\{\kappa_{n_j}\}$ and $\{\kappa_{m_j}\}$ of $\{\kappa_n\}$ converge weakly to p and q, respectively, then $p = r$.*

Lemma 2.6. *[6] Let $\Im : ¥ \to K(¥)$ be a hybrid multivalued mapping. Let $\{\kappa_n\}$ be a sequence in ¥ such that $\kappa_n \rightharpoonup r$ and $\lim_{n\to\infty} \|\kappa_n - \omega_n\| = 0$ for some $\omega_n \in \Im\kappa_n$. Then $r \in \Im r$.*

Lemma 2.7. *[6] Let $\Im : ¥ \to K(¥)$ be a hybrid multivalued mapping with $\Omega(\Im) \neq \emptyset$, then $\Omega(\Im)$ is closed.*

Condition(A). Let ¥ be a subset of a real Hilbert space $\gimel$. A multivalued mapping $\Im : ¥ \to CB(¥)$ is said to satisfy Condition (A) if $\|\kappa - \ell\| = d(\kappa, \Im\ell)$ for all $\kappa \in \gimel$ and $\ell \in \Omega(\Im)$.

Lemma 2.8. *[6] Let $\Im : ¥ \to K(¥)$ be a hybrid multivalued mapping with $\Omega(\Im) \neq \emptyset$. If $\Im$ satisfies Condition (A), then $\Omega(\Im)$ is convex.*

Remark 2.9. *[6] We see that if $\Im p = \{p\}$ for all $p \in \Omega(\Im)$, then $\Im$ satisfies Condition (A). It is known that the best approximation operator $P_{\Im}$, which is defined by $P_{\Im}\kappa = \{\omega \in \Im\kappa : \|\omega - \kappa\| = \xi(\kappa, \Im\kappa)\}$, also satisfies Condition (A).*

Lemma 2.10. *[8] Let $\Im : \gimel \to CB(\gimel)$ be a quasi-nonexpansive mapping with $\Omega(\Im) \neq \emptyset$. Then $\Omega(\Im)$ is closed.*

Lemma 2.11. *[8] Let $\Im : \gimel \to CB(\gimel)$ be a quasi-nonexpansive mapping with $\Omega(\Im) \neq \emptyset$. If $\Im$ satisfies Condition (A), then $\Omega(\Im)$ is convex.*

3. Main Results and Discussion

This section is for stating and proofing the main results of this paper, i.e., the weak and strong convergence theorems.

Theorem 3.1. *Let $\Im, \Im_2, \Im_2 : ¥ \to CB(¥)$ be quasi-nonexpansive multivalued mappings with $\Omega(\Im_1) \cap \Omega(\Im_2) \cap \Omega(\Im_3) \neq \emptyset$ and $I - \Im_i$ is demiclosed at 0 for all $i \in \{1, 2, 3\}$. Let $\{\kappa_n\}$ be a sequence generated by*

$$\begin{cases} \kappa_0, \kappa_1 \in ¥ \text{ chosen arbitrary,} \\ \zeta_n = \kappa_n + \theta_n(\kappa_n - \kappa_{n-1}), \\ \omega_n \in \mu_n x_n + (1-\mu_n)\Im_1\zeta_n, \\ \tau_n \in \sigma_n\kappa_n + (1-\sigma_n)\Im_2\omega_n, \\ \kappa_{n+1} \in \vartheta_n\kappa_n + (1-\vartheta_n)\Im_3\tau_n, \end{cases}$$

for all $n \geq 1$, where $\{\vartheta_n\}$, $\{\sigma_n\}$ and $\{\mu_n\} \subset (0,1)$. Assume that the following conditions hold:
(i) $\sum_{n=1}^{\infty} \theta_n \|\kappa_n - \kappa_{n-1}\| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \vartheta_n < \limsup_{n\to\infty} \vartheta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \sigma_n < \limsup_{n\to\infty} \sigma_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \mu_n < \limsup_{n\to\infty} \mu_n < 1$.
If $\mathfrak{I}_1, \mathfrak{I}_2$ and $\mathfrak{I}_3$ satisfy Condition (A), then $\{\kappa_n\}$ is weakly convergent to a common fixed point of $\mathfrak{I}_1, \mathfrak{I}_2$ and $\mathfrak{I}_3$.

Proof. Let $p \in \Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3)$. Since $\mathfrak{I}_1, \mathfrak{I}_2$ and $\mathfrak{I}_3$ satisfy Condition (A), for $\hbar_n \in \mathfrak{I}_1\zeta_n$, $\wp_n \in \mathfrak{I}_2\omega_n$ and $\aleph_n \in \mathfrak{I}_3\tau_n$, we have

$$\begin{aligned}
\|\kappa_{n+1} - p\| &\leq \vartheta_n\|\kappa_n - p\| + (1-\vartheta_n)\|\aleph_n - p\| \\
&= \vartheta_n\|\kappa_n - p\| + (1-\vartheta_n)\xi(\aleph_n, \mathfrak{I}_3 p) \\
&\leq \vartheta_n\|x_n - p\| + (1-\vartheta_n)\daleth(\mathfrak{I}_3\tau_n, \mathfrak{I}_3 p) \\
&\leq \vartheta_n\|x_n - p\| + (1-\vartheta_n)\|\tau_n - p\| \\
&\leq \vartheta_n\|x_n - p\| + (1-\vartheta_n)\Big(\sigma_n\|\kappa_n - p\| + (1-\sigma_n)\|\wp_n - p\|\Big) \\
&= \vartheta_n\|x_n - p\| + (1-\vartheta_n)\Big(\sigma_n\|\kappa_n - p\| + (1-\sigma_n)\xi(\wp_n, \mathfrak{I}_2 p)\Big) \\
&\leq \vartheta_n\|x_n - p\| + (1-\vartheta_n)\Big(\sigma_n\|\kappa_n - p\| + (1-\sigma_n)\daleth(\mathfrak{I}_2\omega_n, \mathfrak{I}_2 p)\Big) \\
&\leq (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\| + (1-\vartheta_n)(1-\sigma_n)\|\omega_n - p\| \\
&\leq (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\| + (1-\vartheta_n)(1-\sigma_n)\Big(\mu_n\|\kappa_n - p\| + (1-\mu_n)\|\hbar_n - p\|\Big) \\
&= (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\| + (1-\vartheta_n)(1-\sigma_n)\Big(\mu_n\|\kappa_n - p\| + (1-\mu_n)\xi(\hbar_n, \mathfrak{I}_1 p)\Big) \\
&\leq (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\| + (1-\vartheta_n)(1-\sigma_n)\Big(\mu_n\|\kappa_n - p\| + (1-\mu_n)\daleth(\mathfrak{I}_1\zeta_n, \mathfrak{I}_1 p)\Big) \\
&\leq (\vartheta_n + (1-\vartheta_n)\sigma_n + (1-\vartheta_n)(1-\sigma_n)\mu_n)\|\kappa_n - p\| + (1-\vartheta_n)(1-\sigma_n))(1-\mu_n)\|\zeta_n - p\| \\
&\leq \|\kappa_n - p\| + \theta_n\|\kappa_n - \kappa_{n-1}\|.
\end{aligned}$$

From Lemma 2.4 and the assumption (i), we obtain $\lim_{n\to\infty}\|\kappa_n - p\|$ exists, in particular, $\{\kappa_n\}$ is bounded and also $\{\tau_n\}$, $\{\omega_n\}$ and $\{\zeta_n\}$. By using Lemma 2.1 (ii), we have

$$\begin{aligned}
\left\|\kappa_{n+1} - p\right\|^2 &= \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\left\|\aleph_n - p\right\|^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\
&= \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\xi(\aleph_n, \mathfrak{I}_3 p)^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\
&\leq \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\daleth(\mathfrak{I}_3\tau_n, \mathfrak{I}_3 p)^2 - \alpha_n(1-\alpha_n)\|\kappa_n - \aleph_n\|^2 \\
&\leq \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\left\|\tau_n - p\right\|^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2, \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\tau_n - p\right\|^2 &= \left\|\sigma_n(\kappa_n - p) + (1-\sigma_n)(\wp_n - p)\right\|^2 \\
&= \sigma_n\left\|\kappa_n - p\right\|^2 + (1-\sigma_n)\left\|\wp_n - p\right\|^2 - \sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2 \\
&= \sigma_n\left\|\kappa_n - p\right\|^2 + (1-\sigma_n)\xi(\wp_n, \mathfrak{I}_2 p)^2 - \sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2 \\
&\leq \sigma_n\left\|\kappa_n - p\right\|^2 + (1-\sigma_n)\daleth(\mathfrak{I}_2\omega_n, \mathfrak{I}_2 p)^2 - \sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2 \\
&\leq \sigma_n\left\|\kappa_n - p\right\|^2 + (1-\sigma_n)\left\|\omega_n - p\right\|^2 - \sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2. \qquad (6)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\omega_n - p\|^2 &= \mu_n\|\kappa_n - p\|^2 + (1-\mu_n)\|\hbar_n - p\|^2 - \mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2 \\ &= \mu_n\|\kappa_n - p\|^2 + (1-\mu_n)\xi(\hbar_n, \mathfrak{I}_1 p)^2 - \mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2 \\ &\leq \mu_n\|\kappa_n - p\|^2 + (1-\mu_n)\daleth(\mathfrak{I}_1\zeta_n, \mathfrak{I}_1 p)^2 - \mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2 \\ &\leq \mu_n\|\kappa_n - p\|^2 + (1-\mu_n)\|\zeta_n - p\|^2 - \mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2 \\ &\leq \mu_n\|\kappa_n - p\|^2 + 2(1-\mu_n)\theta_n\langle\kappa_n - \kappa_{n-1}, \zeta_n - p\rangle - \mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2. \end{aligned} \tag{7}$$

Applying (6) and (7) in (5), we get

$$\begin{aligned}\|\kappa_{n+1} - p\|^2 &\leq \vartheta_n \left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)(\sigma_n\|\kappa_n - p\|^2 + (1-\sigma_n)\|\omega_n - p\|^2 \\ &\quad -\sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2) - \vartheta_n(1-\vartheta_n)\|\kappa_n - \wp_n\|^2 \\ &\leq (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\|^2 + (1-\vartheta_n)(1-\sigma_n)(\|\kappa_n - p\|^2 \\ &\quad +2(1-\mu_n)\theta_n\langle\kappa_n - \kappa_{n-1}, \zeta_n - p\rangle - \mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2) \\ &\quad -(1-\vartheta_n)\sigma_n(1-\sigma_n)\|\kappa_n - \aleph_n\|^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\ &= \|\kappa_n - p\|^2 + 2(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\theta_n\langle\kappa_n - \kappa_{n-1}, \zeta_n - p\rangle \\ &\quad -(1-\vartheta_n)(1-\sigma_n)\mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2 \\ &\quad -(1-\vartheta_n)\sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} &(1-\vartheta_n)(1-\sigma_n)\mu_n(1-\mu_n)\|\kappa_n - \hbar_n\|^2 + (1-\vartheta_n)\sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2 + \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\ \leq\ &\|\kappa_n - p\|^2 - \|\kappa_{n+1} - p\|^2 + 2(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\theta_n\langle\kappa_n - \kappa_{n-1}, \zeta_n - p\rangle \end{aligned}$$

Then by conditions (i)-(iv) and $\lim_{n\to\infty}\left\|\kappa - p\right\|$ exists, we deduce

$$\lim_{n\to\infty}\|\kappa_n - \hbar_n\| = \lim_{n\to\infty}\|\kappa_n - \wp_n\| = \lim_{n\to\infty}\|x_n - \aleph_n\| = 0. \tag{8}$$

By the assumption (i) and (8), we have

$$\begin{aligned}\|\hbar_n - \zeta_n\| &\leq \|\hbar_n - \kappa_n\| + \|\kappa_n - \zeta_n\| \\ &\leq \|\hbar_n - \kappa_n\| + \theta_n\|\kappa_n - \kappa_{n-1}\| \to 0, \end{aligned} \tag{9}$$

as $n \to \infty$. By the definition of $\{\omega_n\}$ and (8), we obtain

$$\|\omega_n - \kappa_n\| = (1-\mu_n)\|\omega_n - \kappa_n\| \to 0, \tag{10}$$

as $n \to \infty$. From (8) and (10), we have

$$\|\wp_n - \omega_n\| \leq \|\wp_n - \kappa_n\| + \|\kappa_n - \omega_n\| \to 0, \tag{11}$$

as $n \to \infty$. By the definition of $\{\tau_n\}$ and (iii), we have

$$\|\tau_n - \kappa_n\| = (1-\sigma_n)\|\wp_n - \kappa_n\| \to 0, \tag{12}$$

as $n \to \infty$. From (8) and (12), we get

$$\|\aleph_n - \tau_n\| \leq \|\aleph_n - \kappa_n\| + \|\kappa_n - \tau_n\| \to 0, \tag{13}$$

as $n \to \infty$. Since $\{\kappa_n\}$ is bounded, there exists a subsequence $\{\kappa_{n_j}\}$ of $\{\kappa_n\}$ such that $\kappa_{n_j} \rightharpoonup r$ some $r \in ¥$. From (12), we also have $\tau_{n_j} \rightharpoonup r$. Since $I - \mathfrak{I}_1$ is demiclosed at 0 and (9), we obtain $r \in \mathfrak{I}_1 r$. From (10), we know that $\omega_{n_j} \rightharpoonup r$. Since $I - \mathfrak{I}_2$ is demiclosed at 0 and (11), we also have $r \in \mathfrak{I}_2 r$. It follows from (8) and (9) that $\zeta_{n_j} \rightharpoonup r$. Again by $I - \mathfrak{I}_3$ is demiclosed at 0, we obtain $r \in \mathfrak{I}_3 r$. This implies that $r \in \Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3)$. Now we show that $\{\kappa_n\}$ converges weakly to r. We take another subsequence $\{\kappa_{m_j}\}$ of $\{\kappa_n\}$ converging weakly to some $r' \in \Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3)$. Since $\lim_{n\to\infty}\left\|\kappa_n - p\right\|$ exists, from Lemma 2.5, $r' = r$. So, we have reached the end of the proof. □

The following theorems give the strong convergence of our proposed iteration.

Theorem 3.2. *Let* $\Im, \Im_2, \Im_2 : ¥ \to CB(¥)$ *be quasi-nonexpansive multivalued mappings with* $\Omega(\Im_1) \cap \Omega(\Im_2) \cap \Omega(\Im_3) \neq \emptyset$ *and* $I - \Im_i$ *is demiclosed at 0 for all* $i \in \{1,2,3\}$. *Let* $\{\kappa_n\}$ *be a sequence generated by*

$$\begin{cases} \kappa_0, \kappa_1 \in ¥,\ ¥_1 = ¥, \\ \zeta_n = \kappa_n + \theta_n(\kappa_n - \kappa_{n-1}), \\ \omega_n \in \mu_n x_n + (1-\mu_n)\Im_1\zeta_n, \\ \tau_n \in \sigma_n\kappa_n + (1-\sigma_n)\Im_2\omega_n, \\ \mho_n \in \vartheta_n\kappa_n + (1-\vartheta_n)\Im_3\tau_n, \\ ¥_{n+1} = \{p \in ¥_n : \|\mho_n - p\|^2 \le \|\kappa_n - p\|^2 + 2\theta_n^2\|\kappa_n - \kappa_{n-1}\|^2 \\ \qquad\qquad -2\theta_n(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle\}, \\ \kappa_{n+1} = P_{¥_{n+1}}\kappa_1, \text{ for all } n \ge 1, \end{cases}$$

where $\{\vartheta_n\}, \{\sigma_n\}$ *and* $\{\mu_n\} \subset (0,1)$. *Assume that the following hold:*
(i) $\sum_{n=1}^{\infty} \theta_n\|\kappa_n - \kappa_{n-1}\| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \vartheta_n < \limsup_{n\to\infty} \vartheta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \sigma_n < \limsup_{n\to\infty} \sigma_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \mu_n < \limsup_{n\to\infty} \mu_n < 1$.
If $\Im_1, \Im_2$ *and* $\Im_3$ *satisfy Condition (A), then* $\{\kappa_n\}$ *is strongly convergent to a common fixed point of* $\Im_1, \Im_2$ *and* $\Im_3$.

Proof. We will divide the proof into the following steps:
Step 1. We show that $\{\kappa_n\}$ is well-defined. Since $\Im_1, \Im_2$ and $\Im_3$ satisfy Condition (A), from Lemmas 2.10-2.11, $\Omega(\Im_1) \cap \Omega(\Im_2) \cap \Omega(\Im_3)$ is closed and convex. Firstly, we show that C_n is closed and convex for all $n \ge 1$. By induction on n that $¥_n$ is closed and convex. For $n = 1$, $¥_1 = ¥$ is closed and convex. Assume that $¥_n$ is closed and convex for some $n \in \mathbb{N}$. From the definition $¥_{n+1}$ and Lemma 2.2, we have that $¥_{n+1}$ also closed and convex. Hence $¥_n$ is closed and convex for all $n \in \mathbb{N}$ Next, we show that $\Omega(\Im_1) \cap \Omega(\Im_2) \cap \Omega(\Im_3) \subseteq ¥_n$ for each $n \ge 1$. Since $\Im_1, \Im_2$ and $\Im_3$ satisfy Condition (A), for each $p \in \Omega(\Im_1) \cap \Omega(\Im_2) \cap \Omega(\Im_3)$, $\hbar_n \in \Im_1\zeta_n$, $\wp_n \in \Im_2\omega_n$ and $\aleph_n \in \Im_3\tau_n$, we get

$$\begin{aligned}
\left\|\mho_n - p\right\|^2 &= \left\|\vartheta_n(\kappa_n - p) + (1-\vartheta_n)(\aleph_n - p)\right\|^2 \\
&\le \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\left\|\aleph_n - p\right\|^2 \\
&= \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\xi(\aleph_n, \Im_3 p)^2 \\
&\le \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\urcorner(\Im_3\tau_n, \Im_3 p)^2 \\
&\le \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\left\|\tau_n - p\right\|^2 \\
&\le \vartheta_n\left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)(\sigma_n\|\kappa_n - p\|^2 + (1-\sigma_n)\|\wp_n - p\|^2) \\
&= (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\|^2 + (1-\vartheta_n)(1-\sigma_n)\xi(\wp_n, \Im_2 p)^2 \\
&\le (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\|^2 + (1-\vartheta_n)(1-\sigma_n)\urcorner(\Im_2\omega_n, \Im_2 p)^2 \\
&\le (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\|^2 + (1-\vartheta_n)(1-\sigma_n)\|\omega_n - p\|^2 \\
&\le (\vartheta_n + (1-\vartheta_n)\sigma_n)\|\kappa_n - p\|^2 \\
&\quad +(1-\vartheta_n)(1-\sigma_n)(\mu_n\|\kappa_n - p\|^2 + (1-\mu)\|\hbar_n - p\|^2) \\
&= (\vartheta_n + (1-\vartheta_n)\sigma_n + (1-\vartheta_n)(1-\sigma_n)\mu_n)\|\kappa_n - p\|^2 \\
&\quad +(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\xi(\hbar_n, \Im_1 p)^2 \\
&\le (\vartheta_n + (1-\vartheta_n)\sigma_n + (1-\vartheta_n)(1-\sigma_n)\mu_n)\|\kappa_n - p\|^2 \\
&\quad +(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\urcorner(\Im_1\zeta_n, \Im_1 p)^2 \\
&\le (\vartheta_n + (1-\vartheta_n)\sigma_n + (1-\vartheta_n)(1-\sigma_n)\mu_n)\|\kappa_n - p\|^2 \\
&\quad +(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\|\zeta_n - p\|^2
\end{aligned}$$

$$\leq \|\zeta_n - p\|^2 + 2(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\theta_n\langle \kappa_n - \kappa_{n-1}, \zeta_n - p\rangle$$
$$\leq \|\zeta_n - p\|^2 + 2\theta_n^2\|\kappa_n - \kappa_{n-1}\|^2 - 2\theta_n(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle.$$

Therefore, $p \in ¥_n$, $n \geq 1$. This implies that $\Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3) \subset ¥_n$ for each $n \geq 1$ and so $¥_n \neq \emptyset$. Hence $\{\kappa_n\}$ is well-defined.

Step 2. We show that $\kappa_n \longrightarrow a \in ¥$ as $n \to \infty$. From $\kappa_n \in P_{¥_n}\kappa_1$, $¥_{n+1} \subseteq ¥_n$ and $\kappa_{n+1} \in ¥_n$, we have

$$\|\kappa_n - \kappa_1\| \leq \|\kappa_{n+1} - \kappa_1\|, \ \forall n \geq 1, \tag{14}$$

On the other hand, since $\Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3) \subset ¥_n$, we get

$$\|\kappa_n - \kappa_1\| \leq \|a - \kappa_1\|, \ \forall n \geq 1, \tag{15}$$

for all $a \in \Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3)$. The inequalities (14) and (15) imply that the sequence $\{\kappa_n - \kappa_1\}$ is bounded and nondecreasing, hence $\lim_{n\to\infty} \|\kappa_n - \kappa_1\|$ exists.
For $m > n$, by the definition of $¥_n$, we have $\kappa_m \in P_{¥_m}\kappa_1 \in ¥_m \subseteq ¥_n$. By Lemma 2.3, we obtain that

$$\|\kappa_m - \kappa_n\|^2 \leq \|\kappa_m - \kappa_1\|^2 - \|\kappa_n - \kappa_1\|^2. \tag{16}$$

Since $\lim_{n\to\infty} \|\kappa_n - \kappa_1\|$ exists, it follows from (16) that $\lim_{n,m\to\infty} \|\kappa_m - \kappa_n\| = 0$. Hence $\{\kappa_n\}$ is a Cauchy sequence in ¥ and so that $\kappa_n \longrightarrow a \in ¥$ as $n \to \infty$.
Step 3. We show that

$$\lim_{n\to\infty} \|\hbar_n - \zeta_n\| = \lim_{n\to\infty} \|\wp_n - \omega_n\| = \lim_{n\to\infty} \|\aleph_n - \tau_n\| = 0,$$

From step 2, we have that $\lim_{n\to\infty} \|\kappa_{n+1} - \kappa_n\| = 0$. Since $\kappa_{n+1} \in ¥_n$, we have

$$\begin{aligned}
&\|\mho_n - \kappa_n\| \\
\leq\ & \|\mho_n - \kappa_{n-1}\| + \|\kappa_{n+1} - \kappa_n\| \\
\leq\ & \sqrt{\|\kappa_n - \kappa_{n+1}\|^2 + 2\theta_n^2\|\kappa_n - \kappa_{n-1}\|^2 - 2\theta_n(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\langle \kappa_n - \kappa_{n+1}, \kappa_{n-1} - \kappa_n\rangle} \\
& + \|\kappa_{n+1} - \kappa_n\|.
\end{aligned} \tag{17}$$

By the assumption (i) and (17), we obtain

$$\lim_{n\to\infty} \|\mho_n - \kappa_n\| = 0. \tag{18}$$

Since $\mathfrak{I}_1$, satisfies condition (A), by Lemma 2.1, we have

$$\begin{aligned}
\left\|\mho_n - p\right\|^2 &\leq \vartheta_n \left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\|\aleph_n - p\|^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\
&= \vartheta_n \left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\xi(\aleph_n, \mathfrak{I}_3 p)^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\
&\leq \vartheta_n \left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\urcorner(\mathfrak{I}_3\tau_n, \mathfrak{I}_3 p)^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2 \\
&\leq \vartheta_n \left\|\kappa_n - p\right\|^2 + (1-\vartheta_n)\|\tau_n - p\|^2 - \vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2.
\end{aligned} \tag{19}$$

Replacing (6) and (7) in (19), we get

$$\begin{aligned}
\|\mho_n - p\|^2 \leq\ & \left\|\kappa_n - p\right\|^2 + 2(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\theta_n\langle \kappa_n - \kappa_{n-1}, \zeta_n - p\rangle \\
& -\mu_n(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\|\kappa_n - \hbar_n\|^2 - (1-\vartheta_n)\sigma_n(1-\sigma_n)\|\kappa_n - \wp_n\|^2 \\
& -\vartheta_n(1-\vartheta_n)\|\kappa_n - \aleph_n\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned} & \mu_n(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\|\kappa_n-\hbar_n\|^2+(1-\vartheta_n)\sigma_n(1-\sigma_n)\|\kappa_n-\wp_n\|^2+\vartheta_n(1-\vartheta_n)\|\kappa_n-\aleph_n\|^2 \\ \leq\ & \left\|\kappa_n-p\right\|^2-\|\mho_n-p\|^2+2(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\theta_n\langle\kappa_n-\kappa_{n-1},\zeta_n-p\rangle. \end{aligned} \tag{20}$$

By our assumptions (i)-(iv), (18) and (20), we obtain

$$\lim_{n\to\infty}\|\kappa_n-\hbar_n\|=\lim_{n\to\infty}\|\kappa_n-\wp_n\|=\lim_{n\to\infty}\|\kappa_n-\aleph_n\|=0. \tag{21}$$

From (9)-(13), by the same proof in Theorem 3.1, we obtain

$$\lim_{n\to\infty}\|\hbar_n-\zeta_n\|=\lim_{n\to\infty}\|\omega_n-\kappa_n\|=\lim_{n\to\infty}\|\wp_n-\omega_n\|=\lim_{n\to\infty}\|\tau_n-\kappa_n\|=\lim_{n\to\infty}\|\aleph_n-\tau_n\|=0. \tag{22}$$

From Step 2, we know that $\kappa_n \to a \in ¥$. It follows from (21, 22) that $\zeta_n \to a$. Since $I-\mathfrak{I}_1$ is demiclosed at 0, we obtain $a\in\Omega(\mathfrak{I}_1)$. Similarly, we obtain that $a\in\Omega(\mathfrak{I}_2)$ and $a\in\Omega(\mathfrak{I}_3)$. This implies that $a\in\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)$.

Step 4. We show that $a=P_{\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)}\kappa_1$. Since $a\in\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)$. Form (15), we have

$$\|a-\kappa\|\leq\|\omega-\kappa_1\|,\ \forall\omega\in\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3).$$

By the definition of the projection operator, we can conclude that $a=P_{\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)}\kappa_1$. This completes the proof. □

Theorem 3.3. *Let $\mathfrak{I},\mathfrak{I}_2,\mathfrak{I}_2: ¥\to CB(¥)$ be quasi-nonexpansive multivalued mappings with $\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)\neq\emptyset$ and $I-\mathfrak{I}_i$ is demiclosed at 0 for all $i\in\{1,2,3\}$. Let $\{\kappa_n\}$ be a sequence generated by*

$$\begin{cases} \kappa_0,\kappa_1\in ¥,\ ¥_1=¥,\\ \zeta_n=\kappa_n+\theta_n(\kappa_n-\kappa_{n-1}),\\ \omega_n\in\mu_n x_n+(1-\mu_n)\mathfrak{I}_1\zeta_n,\\ \tau_n\in\sigma_n\kappa_n+(1-\sigma_n)\mathfrak{I}_2\omega_n,\\ \mho_n\in\vartheta_n\kappa_n+(1-\vartheta_n)\mathfrak{I}_3\tau_n\\ ¥_n=\{p\in ¥_n:\|\mho_n-p\|^2\leq\|\kappa_n-p\|^2+2\theta_n^2\|\kappa_n-\kappa_{n-1}\|^2\\ \qquad\qquad -2\theta_n(1-\vartheta_n)(1-\sigma_n)(1-\mu_n)\langle\kappa_n-p,\kappa_{n-1}-\kappa_n\rangle\},\\ Q_n=\{\omega\in ¥:\langle\kappa_1-\kappa_n,\kappa_n-p\rangle\geq0\},\\ \kappa_{n+1}=P_{¥_n\cap Q_n}\kappa_1,\ \text{for all } n\geq1, \end{cases}$$

where $\{\vartheta_n\}$, $\{\sigma_n\}$ and $\{\mu_n\}\subset(0,1)$. Assume that the following hold:
(i) $\sum_{n=1}^{\infty}\theta_n\|\kappa_n-\kappa_{n-1}\|<\infty$;
(ii) $0<\liminf_{n\to\infty}\vartheta_n<\limsup_{n\to\infty}\vartheta_n<1$;
(iii) $0<\liminf_{n\to\infty}\sigma_n<\limsup_{n\to\infty}\sigma_n<1$;
(iv) $0<\liminf_{n\to\infty}\mu_n<\limsup_{n\to\infty}\mu_n<1$.
If $\mathfrak{I}_1,\mathfrak{I}_2$ and $\mathfrak{I}_3$ satisfy Condition (A), then $\{\kappa_n\}$ is strongly convergent to a common fixed point of $\mathfrak{I}_1,\mathfrak{I}_2$ and $\mathfrak{I}_3$.

Proof. By the same method of Theorem 3.2 step by step, we can conclude the proof by replacing $¥_{n+1}$ by $¥_n$, expect in Step 1. Showing that $\{\kappa_n\}$ is well-defined and $F(T_1)\cap F(T_2)\cap F(T_3)\subseteq C_n$ for each $n\geq1$. Next we show that $\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)\subseteq Q_n$ for all $n\in\mathbb{N}$. Also by mathematical induction. For $n=1$, we have $\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)\subseteq ¥=Q_1$. Assume that $\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)\subseteq Q_n$ for all $n\in\mathbb{N}$. Since κ_{n+1} is the projection of κ_1 onto $¥_n\subseteq Q_n$, we have

$$\langle\kappa_1-\kappa_{n+1},\kappa_{n+1}-\omega\rangle\geq0\ ,\forall\omega\in ¥_n\cap Q_n.$$

Thus $\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)\subseteq Q_{n+1}$. So $\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)\subseteq ¥_n\cap Q_n$. This implies that $\{\kappa_n\}$ is well-defined. We next show that $\kappa_n\to\varkappa\in ¥$ as $n\to\infty$. From the definition of Q_n, we get $\kappa_n=P_{Q_n}\kappa_1$. Since $\kappa_{n+1}\in Q_n$, we have

$$\|\kappa_n-\kappa_1\|\leq\|\kappa_{n+1}-\kappa_1\|,\ \forall n\geq\mathbb{N}. \tag{23}$$

On the other hand, we obtain

$$\|\kappa_n - \kappa_1\| \leq \|\varkappa - \kappa_1\|, \ \forall \varkappa \in \Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3). \tag{24}$$

The inequalities (23) and (24) imply that the sequence $\{\kappa_n - \kappa_1\}$ is bounded and non-decreasing, hence $\lim_{n\to\infty} \|\kappa_n - \kappa_1\|$ exists. For $m > n$, by definition of Q_n, we have $\kappa = P_{Q_m}\kappa_1 \in Q_m \subseteq Q_n$. By Lemma 2.3, we obtain that

$$\|\kappa_m - \kappa_n\|^2 \leq \|\kappa_m - \kappa_1\|^2 - \|\kappa_n - \kappa_1\|^2. \tag{25}$$

Since $\lim_{n\to\infty} \|\kappa_n - \kappa_1\|$ exists, it follows from (25) that $\lim_{n,m\to\infty} \|\kappa_m - \kappa_n\| = 0$. Hence $\{\kappa_n\}$ is a Cauchy sequence in ¥ and so $\kappa_n \to \varkappa \in$ ¥ as $n \to \infty$. In particular, we have $\lim_{n\to\infty} \|\kappa_{n+1} - \kappa_n\| = 0$. By the same proof of Step 3-4 in Theorem 3.2, we obtain $\varkappa = P_{\Omega(\mathfrak{I}_1)\cap\Omega(\mathfrak{I}_2)\cap\Omega(\mathfrak{I}_3)}\kappa_1$. □

Remark 3.4. *Let $\Omega(\mathfrak{I}) \neq \emptyset$, then a hybrid multivalued mapping $\mathfrak{I} : \beth \to K(\beth)$ is quasi-nonexpansive, by Lemma 2.10-2.11, $\Omega(\mathfrak{I})$ is closed and convex, also by Lemma 2.7, $I - \mathfrak{I}$ is demiclosed at 0, where I is the identity mapping. Now, we can finish Theorems 3.1-3.3 by another methods as follows:*

(a) If we take $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_2 :$ ¥ $\to K($¥$)$ be hybrid multivalued mappings.

(b) If we put $\mathfrak{I}_1 p = \{p\}$, $\mathfrak{I}_2 q = \{q\}$ and $\mathfrak{I}_3 r = \{r\}$ for all $p \in \Omega(\mathfrak{I}_1)$, $q \in \Omega(\mathfrak{I}_2)$ and $r \in \Omega(\mathfrak{I}_3)$ and $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_2 :$ ¥ $\to K($¥$)$ be hybrid multivalued mappings.

(c) Since $P_{\mathfrak{I}}$ satisfies Condition (A), by taking $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_2 :$ ¥ $\to P($¥$)$ be hybrid multivalued mappings, hence $P_{\mathfrak{I}_1}, P_{\mathfrak{I}_2}$ and $P_{\mathfrak{I}_3}$ are too. In this case, we can write

$$\begin{aligned} \tau_n &\to \aleph_n \in P_{\mathfrak{I}_3}\tau_n \subseteq \mathfrak{I}_3\tau_n, \\ \omega_n &\to \wp_n \in P_{\mathfrak{I}_2}\omega_n \subseteq \mathfrak{I}_2\omega_n, \\ \zeta_n &\to \hbar_n \in P_{\mathfrak{I}_1}\zeta_n \subseteq \mathfrak{I}_1\zeta_n. \end{aligned}$$

Remark 3.5. *It may also be remarked that the condition (i) can be implemented easily in numerical computation since the valued of $\|\kappa_n - \kappa_{n-1}\|$ is known before choosing θ_n. In fact, θ_n can be so chosen that it satisfies $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\varpi_n}{\|\kappa_n - \kappa_{n-1}\|}, \theta\right\} & if\ \kappa_n \neq \kappa_{n-1}, \\ \theta & otherwise, \end{cases}$$

where $\{\varpi_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} \varpi_n < \infty$.

4. Numerical Results and Comparisons

This section is for numerical experiments supporting our main findings and comparison between our proposed inertial projection method and the standard projection method.

Example 4.1. *Let ¥ $= \beth = \mathbb{R}^3$, ¥$_1 = \{\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3 : \|\kappa\|_1 \leq 2\}$, ¥$_2 = \{\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3 : |\kappa_1| + |\kappa_2| + |\kappa_3| \leq 2\}$ and ¥$_3 = \{\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3 : \max\{|\kappa_1|, |\kappa_2|, |\kappa_3|\} \leq 2\}$. Let $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3 : \mathbb{R}^3 \to CB(\mathbb{R}^3)$ be defined by*

$$\mathfrak{I}_1\kappa = \begin{cases} \{(0,0,0)\} & if\ \kappa \in ¥_1, \\ \{\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 : \|\omega\|_2 \leq \frac{1}{\|\kappa\|_2}\} & otherwise, \end{cases}$$

$$\mathfrak{I}_2\kappa = \begin{cases} \{(0,0,0)\} & if\ \kappa \in ¥_2, \\ \{\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 : |\omega_1| + |\omega_2| + |\omega_3| \leq \frac{1}{|\kappa_1| + |\kappa_2| + |\kappa_3|}\} & otherwise \end{cases}$$

and

$$\mathfrak{I}_3\kappa = \begin{cases} \{(0,0,0)\} & if\ \kappa \in \yen_3, \\ \{\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 : \max\{|\omega_1|, |\omega_2|, |\omega_3|\} \leq \frac{1}{\max\{|\kappa_1|,|\kappa_2|,|\kappa_3|\}}\} & otherwise. \end{cases}$$

We see that $\mathfrak{I}_1$, $\mathfrak{I}_2$ *and* $\mathfrak{I}_3$ *are quasi-nonexpansive and* $\Omega(\mathfrak{I}_1) \cap \Omega(\mathfrak{I}_2) \cap \Omega(\mathfrak{I}_3) = \{(0,0,0)\}$. *Let* $\vartheta_n = \sigma_n = \mu_n = \frac{n}{2n+1}$ *and*

$$\theta_n = \begin{cases} \min\left\{\frac{1}{n^2\|\kappa_n - \kappa_{n-1}\|}, 0.5\right\} & if\,\kappa_n \neq \kappa_{n-1}, \\ 0.5 & otherwise. \end{cases}$$

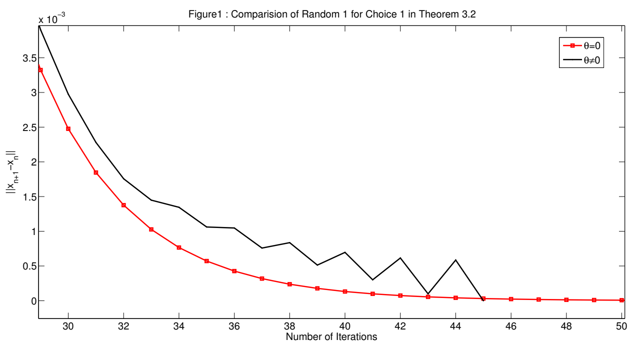
Next, we give a numerical comparison between our inertial forward-backward method (defined in Theorem 3.2) and a standard forward-backward method (i.e. $\theta_n = 0$). The stooping criterion is defined by $\|\kappa_{n+1} - \kappa_n\| < 10^{-9}$.

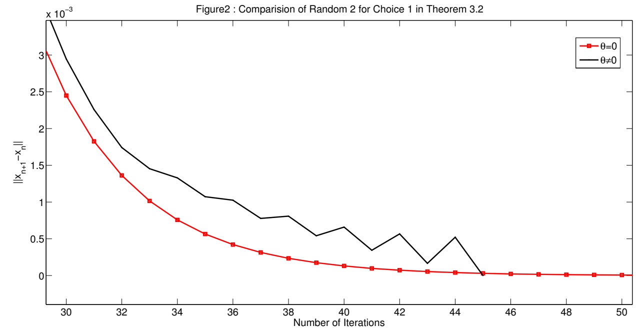
The different choices of κ_0 and κ_1 are given as follows:
Choice 1: $\kappa_0 = (-5, 1, 3)$ and $\kappa_1 = (70, -5, -1)$;
Choice 2: $\kappa_0 = (0, 0.21, -3.15)$ and $\kappa_1 = (0.22, -5.10, 3.21)$.

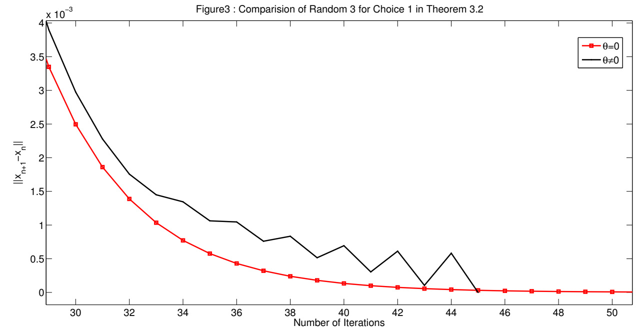
Table 1: Comparing the methods in Theorem 3.2 for $\theta_n \neq 0$ and $\theta_n = 0$ in Example 4.1

	Random	No. of Iter.		cpu (Time).	
	$\omega_n, \tau_n, \mho_n$	$\theta_n \neq 0$	$\theta_n = 0$	$\theta_n \neq 0$	$\theta_n = 0$
Choice 1	1	45	81	0.986159	1.234107
$\kappa_0 = (-5, 1, 3)$	2	45	81	1.563857	4.783340
$\kappa_1 = (70, -5, -1)$	3	45	81	0.786085	2.385316
Choice 2	1	44	72	0.016749	0.024109
$\kappa_0 = (0, 0.21, -3.15)$	2	44	72	0.015076	0.024412
$\kappa_1 = (0.22, -5.10, 3.21)$	3	44	72	0.016659	0.072746

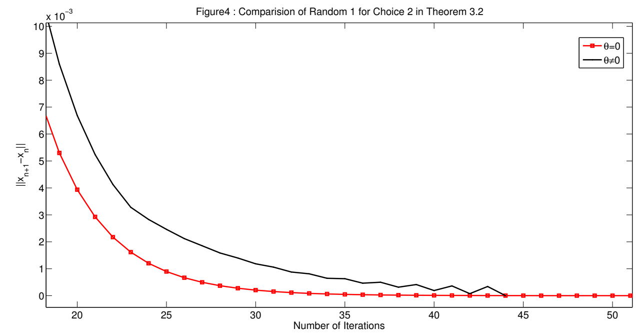
Figures 1-3: The error plotting E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each randomization of choice 1 in Table 1 is shown in the following figures, respectively.

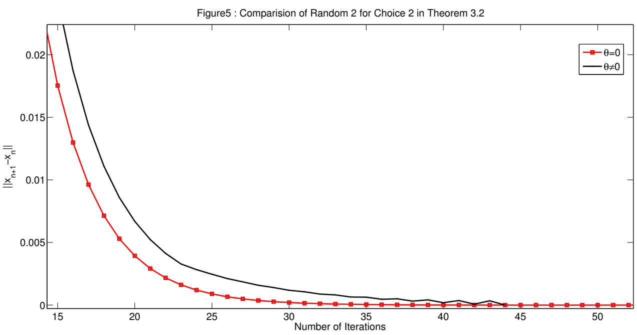


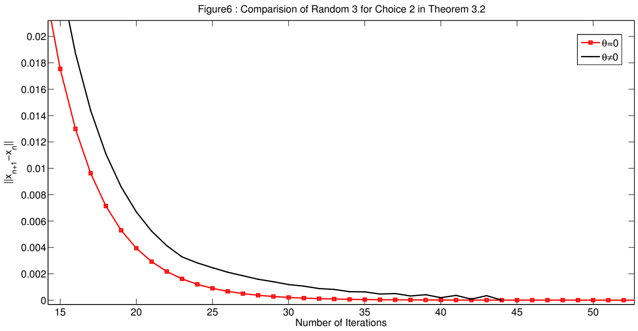




Figures 4-6: The error plotting of E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each randomization of choice 2 in Table 1 is shown in the following figures, respectively.







Similarly, we compare (Table 2) a numerical test between our inertial forward-backward method defined in Theorem 3.3 and a standard forward-backward method (i.e. $\theta_n = 0$). The valued $\|\kappa_{n+1} - \kappa_n\| < 10^{-9}$ is used for the stooping criterion.

The different choices of κ_0 and κ_1 are given as follows:
Choice 1: $\kappa_0 = (10, -13, 7)$ and $\kappa_1 = (-8, -50, -3)$;
Choice 2: $\kappa_0 = (0.12, -5.78, 1.20)$ and $\kappa_1 = (-8.20, -5.11, -0.91)$.

Table 2: Comparison the methods in Theorem 3.3 of $\theta_n \neq 0$ and $\theta_n = 0$ in Example 4.1

	Random	No. of Iter.		cpu (Time).	
	$\omega_n, \tau_n, \mho_n$	$\theta_n \neq 0$	$\theta_n = 0$	$\theta_n \neq 0$	$\theta_n = 0$
Choice 1	1	43	80	0.444689	0.704305
$\kappa_0 = (10, -13, 7)$	2	41	80	0.306636	0.362682
$\kappa_1 = (-8, -50, -3)$	3	41	80	0.448037	0.647477
Choice 2	1	37	74	0.015117	0.032576
$\kappa_0 = (0.12, -5.78, 1.20)$	2	35	74	0.013289	0.028377
$\kappa_1 = (-8.20, -5.11, -0.91)$	3	37	74	0.017730	0.024601

Figures 7-9: The error plotting E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each randomization of choice 1 in Table 2 is shown in the following figures, respectively.

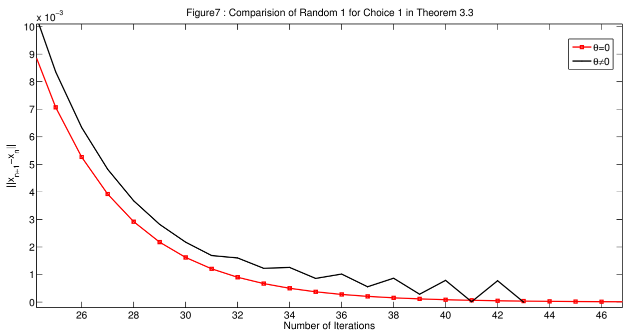


Figure8 : Comparision of Random 2 for Choice 1 in Theorem 3.3

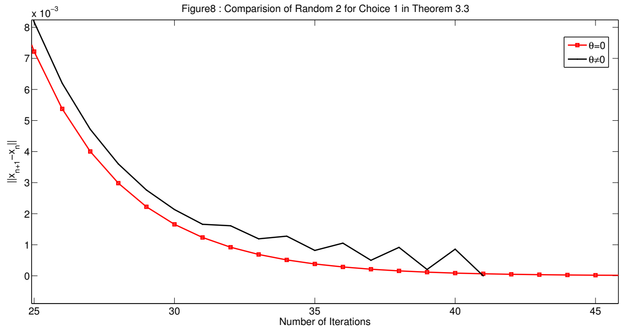
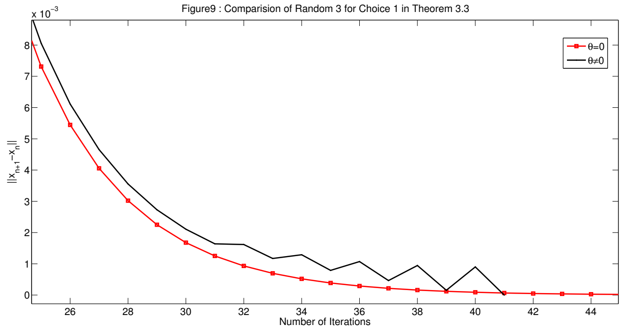


Figure9 : Comparision of Random 3 for Choice 1 in Theorem 3.3



Figures 10-12: The error plotting of E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each randomization of choice 2 in Table 2 is shown in the following figures, respectively.

Figure10 : Comparision of Random 1 for Choice 2 in Theorem 3.3

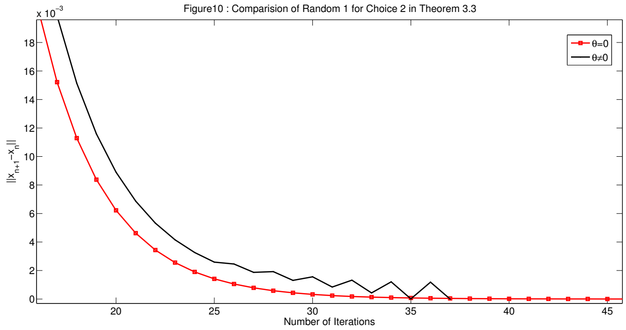
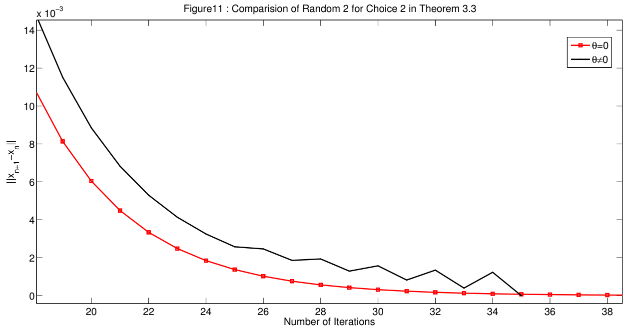
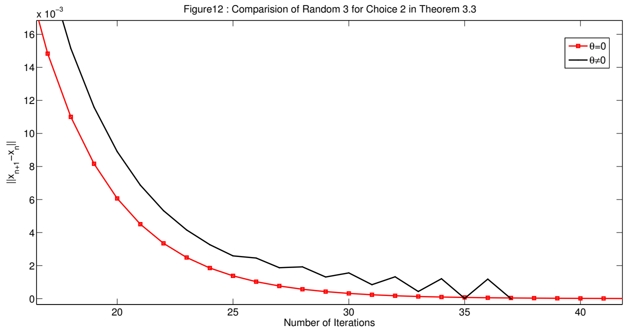


Figure11 : Comparision of Random 2 for Choice 2 in Theorem 3.3





Remark 4.2. *It is observed for Figures 1-12 that our inertial forward-backward method with the inertial technique term has a better convergence speed and requires small number of iterations than the standard forward-backward method.*

References

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, Fixed point theory for Lipschitzian-type mappings with applications, Topological Fixed Point Theory and Its Applications, 6, Springer, New York, (2009).
[2] F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces, SIAM. J. Optim., 14(3) (2004) 773-782.
[3] F. Alvarez, H. Attouch, An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Anal., 9 (2001) 3-11.
[4] H. H. Bauschke, E. Matou ková, S. Reich, Projection and proximal point methods: convergence results and counterexamples, Nonlinear Anal., 56(5) (2004) 715-738.
[5] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA, 54 (1965) 1041-1044.
[6] P. Cholamjiak and W. Cholamjiak, Fixed point theorems for hybrid multivalued mappings in Hilbert spaces, J. Fixed Point Theory Appl., 18(3) (2016) 673-688.
[7] W. Cholamjiak, N. Chutibutr, S. Weerakham, Weak and strong convergence theorems for the modified Ishikawa iterative for two hybrid multivalued mappings in Hilbert spaces, Commun. Korean Math. Soc., 33(3) (2018) 767-786.
[8] W. Cholamjiak, N. Pholasa and S. Suantai, A modified inertial shrinking projection method for solving inclusion problems and quasi-nonexpansive multivalued mappings, Comp. Appl. Math., 37 (2018) 5750-5774.
[9] Y. Dang, J. Sun, H. Xu, Inertial accelerated algorithms for solving a split feasibility problem, J. Ind. Manag. Optim., 13(3) (2017) 1383-1394.
[10] Q. L. Dong, H. B. Yuan, Y. J. Cho and Th. M. Rassias, Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings, Optim. Lett., 12 (2018) 87-102.
[11] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, 28, Cambridge University Press, Cambridge, (1990).
[12] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive mappings, Monographs and Textbooks in Pure and Applied Mathematics, 83, Marcel Dekker, Inc., New York, (1984).
[13] D. Gohde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr., 30 (1965) 251-258.
[14] S. Iemoto and W. Takahashi, Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Anal. 71(12) (2009) 2082-2089.
[15] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc., 44 (1974) 147-150.
[16] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly, 72 (1965) 1004-1006.
[17] F. Kohsaka and W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces, SIAM J. Optim., 19(2) (2008) 824-835.
[18] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. (Basel), 91(2) (2008) 166-177.
[19] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953) 506-510.
[20] C. Martinez-Yanes, H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal., 64(11) (2006) 2400-2411.
[21] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl., 279(2) (2003) 372-379.
[22] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., 251 (2000) 217-229.
[23] Y. Nesterov, A method for solving the convex programming problem with convergence rate $O(1/k^2)$, Dokl. Akad. Nauk SSSR, 269 (1983) 543-547.
[24] H. Piri, S. Rahrovi and P. Kumam, Generalization of Khan fixed point theorem, J. Math. Computer Sci., 17 (2017) 76-83.
[25] B. T. Polyak, Introduction to Optimization, Optimization Software, New York, (1987).

[26] B. T. Polyak, Some methods of speeding up the convergence of iterative methods, Zh. Vychisl. Mat. Mat. Fiz., 4 (1964) 1-17.
[27] S. Reich, Research Problems: The fixed point property for non-expansive mappings, Amer. Math. Monthly, 83(4) (1976), 266-268.
[28] S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, J. Math. Anal. Appl., 62(1) (1978) 104-113.
[29] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 67(2) (1979) 274-276.
[30] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl., 311(2) (2005) 506-517.
[31] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 341(1) (2008) 276-286.
[32] W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, J. Nonlinear Convex Anal., 11(1) (2010) 79-88.