



# A Leader-Follower Single Allocation Hub Location Problem Under Fixed Markups

Dimitrije D. Čvokić<sup>a</sup>

<sup>a</sup>University of Banja Luka, Bulevar vojvode Petra Bojovića 1A, 78000, Banja Luka, Srpska Republic, Bosnia and Herzegovina

**Abstract.** This study examines a scenario in which two competitors, called a leader and a follower, sequentially create their hub and spoke networks to maximize their profits. It is assumed that a non-hub node can be allocated to at most one hub. The pricing is regulated with a fixed markup. Demand is split according to the logit model, and customers patronize their choice of route by a price. Two variants of this Stackelberg competition are addressed: deterministic and robust. In both cases, it was shown how to present the problem as a bi-level mixed-integer non-linear program. When it comes to the deterministic variant, a mixed-integer linear reformulation of the follower's model is given. For the robust variant, it is shown how to reformulate the follower's program as a mixed-integer conic-quadratic one. The benefits of these reformulations are that they allow the usage of state-of-the-art solvers in finding feasible solutions. As a solution approach for the leader, an alternating heuristic is proposed. Computational experiments are conducted on the set of CAB instances and thoroughly discussed, providing some managerial insights.

## 1. Introduction

Hub Location Problem (HLP) arises frequently in many logistic and engineering fields such as the design of airline passenger flow, maritime transport, post-delivery services, the communication network design, emergency services, chain stores in the supply chain, etc. The main task is to select some points (formally, nodes in a graph) that would serve as hubs and use them for flow redirection purposes, with respect to given objective. Hubs are applied to decrease the number of transportation links (called spokes) between the origin and destination nodes (O-D pairs) and to use transportation resources more efficiently. The HLPs can be classified according to the source that determines the number of hubs, the number of allocations of a non-hub node to hubs, whether the hubs are capacitated or uncapacitated, the hub location and allocation cost, etc. An interested reader is referred to papers [1–7] for a better and deeper overall understanding of the topic.

Several extensions of the classic hub location problems have been used quite successfully, as one can observe in [8–11]. This research is oriented towards a particular, more realistic extension of the classic HLP and an application of heuristic solution approach to it. The extension is based on pricing. Lambrecht et al. [12] identify the price as an important service attribute that affects the client's choice. In general, we can assume that the companies are more interested in profit maximization than in reducing unnecessary

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*Email address:* [dimitrije.cvokic@pmf.unibl.org](mailto:dimitrije.cvokic@pmf.unibl.org) (Dimitrije D. Čvokić)

costs. For the facility location problems, some of the recent studies are [13–15]. As Lüer-Villagra and Marianov [11] have argued, the location or route opening decisions can be very dependent on the revenues that a company can obtain. In turn, revenues depend on the price structure. In this study, it is assumed that prices are regulated, moreover that markups are fixed. This simplification is justified observing that regulation is a very common practice. It represents a legal norm intended to shape conduct that is a by-product of imperfection. Common examples of regulation include control on market entries, prices, wages, development, pollution, and standards of production for certain goods. Price regulation refers to the policy of setting the price by a government agency, legal statute, or regulatory authority. Under this policy, fixed, minimum, and maximum prices may be set. Regulatory pricing is not an unrealistic scenario. For example, IATA (International Air Transport Association) has a history of price regulation in the airline industry (as it was noted in [16]). The most simple form of regulation is a direct price setting, i.e., a scenario in which the markups are already determined (fixed). Despite some opposing arguments on price regulation, it is still considered by many scholars as something positive from the social point of view, because it usually reduces the economic profits of monopolies. For more information, please, see [17].

Following the work of Čvokić et al. [18], the demand is taken to be *perfectly inelastic*, as a natural setting for the fixed markup pricing. Moreover, the same assumption is taken by Lüer-Villagra and Marianov in [11] for non-regulated prices. When it comes to the economies of scale, a constant (flow-independent) discount between hubs is considered, which is in the literature addressed as the fundamental approach. The non-hub nodes can be allocated to at most one hub, i.e., a single allocation variant is considered. The source that determines the number of hubs is endogenous, therefore the goal is to determine the optimal number of hubs, too. Besides these basic settings, this study considers a duopoly. Usually, these kinds of duopolies are presented as bi-level programs, that are often, as problems, harder than the well-known NP-complete decision problems (for more insight into  $\Sigma_2^P$  hard location problems one can examine [13, 19, 20]). All these traits make this setting for the problem more interesting and more realistic, compared to the hub location problems in which the number of hubs is exogenously determined, the pricing is not included, and a response from the competition is not considered.

The competition between firms that use the hub and spoke topology has been studied mainly from a sequential location approach. There is a firm, the leader, who is going to enter a market as the first player. Another firm, the follower, also wants to enter the same market, but it is waiting for the leader to make its move. The paper of Sasaki and Fukushima [21] was the first one to address this issue. They have presented the *continuous* Stackelberg HLP in which the incumbent competes with several entrants for profit maximization. For every route, only one hub was allowed. Čvokić et al. [18] presented a variant with multiple allocations, with fixed markups, without considering a robust counterpart. Usual linearization techniques happened to be not very effective. The largest instances that could be solved by a heuristic approach had 10 nodes, i.e. they required unreasonable large running times. Mahmutogullari and Kara [22] addressed the competitive bi-level HLP in which *market share maximization* was the objective and hubs were determined exogenously. In their paper, the demand is divided among the competitors by elementary “the-winner-takes-it-all” rule: a competitor with lower route costs gets the whole demand. They have provided an exact solution method. De Araujo et al. have constructed a new exact algorithm [23]. Recently, the  $(r|p)$  hub-centroid problem under the price war was introduced in [24].

In this research, the focus is on the Stackelberg competition of uncapacitated single allocation hub location problem under fixed markups, where a logit rule was used to model the discrete choice and the profit maximization objective. The logit rule is a much more realistic approach than the already mentioned “the-winner-takes-it-all”. The reader is referred to paper [16] and book [25] for a better and deeper understanding of the logit model. Differently to other studies of bi-level hub location problems, a market coverage does not need to be total, i.e., not all non-hub nodes have to be allocated. This relaxation of classic hub location constraints leads to a scenario in which competitors seek for the best corner of the market, again making this problem more realistic, but also computationally harder. Two variants of this Stackelberg game are addressed: deterministic and robust.

Considering uncertainty has obtained significant attention in the industry and academia over the last years [26–29]. In the robust optimization, the decision or environmental variables are subject to change

after obtaining the optimal (of final) solution. Therefore, one has to consider that the obtained (optimal) solution should be acceptable with respect to slight changes in the variable values. The available literature dealing with uncertainty in the context of hub location problems is very cramped, especially for the bi-level models. In the case of classic one-level programs, Sim et al. [30] considered the stochastic travel time on each link and the incorporated service level requirements through chance constraints in stochastic p-hub problems. Yang [31] incorporated a seasonal variation of demand into an airfreight hub location problems and proposed a two-stage stochastic program method as the solution methodology. Contreras et al. [32] considered uncertain demand and transportation cost with a known probability distribution for multiple allocation uncapacitated hub location problems. Alumur et al. [33] described the effects of uncertainty in demand as well as the set-up cost by introducing a scenario-based stochastic model and demonstrated the sub-optimality of the deterministic solution of hub location problems in the presence of uncertainty. Up to the knowledge of the author, the robust optimization was not considered in the hub location literature for the bi-level models, especially when the pricing, non-elementary binary choice models, and non-total market coverage are considered.

This paper is organized as follows. Section 2 describes the problem and the corresponding bi-level mathematical model for the deterministic variant. In the same section, 3-index linear (re)formulations of the follower's and auxiliary problems are provided. In Section 3, the robust variant of the problem is presented, after which is shown how to solve the inner minimization problem. The analytic solution led to the conic quadratic reformulation of the robust variants of follower's and the corresponding auxiliary programs. As a solution approach for both variants, an alternating heuristic is proposed. The description of this metaheuristic is given in Section 4. The results of computational study are presented in Section 5. Finally, in Section 6, some concluding remarks on this research are given, addressing the possible future lines of work.

## 2. Deterministic leader-follower single allocation hub location problem under fixed markups

Here, notation and formulation for the deterministic variant are presented. The same notation is also used for constructing the robust counterpart. To differentiate the competitors, sometimes different pronouns will be used for them: "she" for the leader and "he" for the follower.

The basic setting is a graph  $G(N, A)$ , where  $N$  is the non-empty set of nodes and  $A$  is the set of arcs. Without loss of generality, we assume  $A = N \times N$ . A hub can only be established at the node  $k \in N$ . For each node  $k \in N$  there is a certain fixed costs  $f_k > 0$  assigned to it for establishing a hub at that particular location. A source for determining the number of hubs is endogenous. There are no hub capacity constraints, nor ones that prescribe the number of hubs. For every arc  $(i, k) \in A$  there is a fixed cost  $g_{ik} > 0$  for allocating  $i$  to  $k$ , and a (variable) transport cost per unit of flow  $c_{ik} > 0$ . The transport cost is a non-decreasing function of distance. A non-hub node  $i \in N$  can be allocated to exactly one hub  $k \in N$  (single allocation). The allocated spoke allows traffic in both directions and does not imply the flow direction. A non-hub node can not be allocated to another non-hub node. We use what is called the fundamental approach to model the economies of scale. The resulting hub backbone should be totally interconnected. To model the inter-hub discounts, we take that  $\chi, \alpha$  and  $\delta$  to be the (discount) transportation factors due to flow consolidation in the collection (origin to hub), transfer between hubs, and distribution (hub to destination), respectively. Concatenation of arcs composes a route, where hubs are located at the joints. At most two hubs are allowed to be on a single route. In other words, at most two stops are permitted. The transport cost  $c_{ij,kl}$  for a route  $i \rightarrow k \rightarrow l \rightarrow j$  is computed as  $c_{ij,kl} = \chi c_{ik} + \alpha c_{kl} + \delta c_{lj}$ . Customers are paying their respective expenses, i.e. the mill pricing is used. A demand  $w_{ij}$  for every O-D pair  $(i, j) \in A$  is assumed to be non-elastic and non-negative. The choice of route is based on the price. The logit model (MNL) is used for a discrete choice estimation, which is essentially a rule that determines how much of the flow is going to be captured considering given price differences. It has a sensitivity parameter  $\Theta$  with an already known positive value assigned. Larger  $\Theta$  means that customers are very sensitive to price differences, and they will mostly choose less expensive routes. Smaller  $\Theta$  means that the clients are less sensitive to price differences. It is assumed that the regulated mill pricing is used (the customer pays their transportation expenses), and the form of regulation is a direct price setting. Particularly, the markups  $\mu_{ij,kl} > 0$  are taken to be fixed. Here,

the markup is a difference between a service price  $p_{ij,kl}$  and the corresponding cost  $c_{ij,kl}$ . There are no budget constraints. Both competitors are trying to maximize their profits, instead of market share.

**Remark 2.1.** In the literature, the markup is also expressed as a percentage over the cost, i.e. as  $\frac{p_{ij,kl}-c_{ij,kl}}{c_{ij,kl}} \times 100\%$ .

**Remark 2.2.** The fixed markup pricing is equivalent to the fixed margin pricing, because the margin is defined as a difference between service price and cost over the price:  $m_{ij,kl} = \frac{p_{ij,kl}-c_{ij,kl}}{p_{ij,kl}}$ . In other words, we have that  $m_{ij,kl} = \frac{\mu_{ij,kl}}{p_{ij,kl}}$ .

2.1. Mathematical formulation of the deterministic variant

We will use the following variables to describe the choices made by the leader and the follower:

- $x_{ik} = 1$  if the leader allocates a node  $i$  to hub  $k \in N$ , and 0 otherwise;
- $u_{ij,kl}$  is the fraction of flow going from  $i \in N$  to  $j \in N$  through the leader’s hubs located at  $k, l \in N$ ;
- $y_{ik} = 1$  if the follower allocates a node  $i$  to hub  $k \in N$ , and 0 otherwise;
- $v_{ij,kl}$  is the fraction of flow going from  $i \in N$  to  $j \in N$  through the follower’s hubs located at  $k, l \in N$ .

Also, to represent the sequence of variables, we will use a more compact notation:  $x = (x_{ij})_{i,j \in N}$ ,  $y = (y_{ij})_{i,j \in N}$ . The solution set of follower’s problem, for a given leader’s solution  $x$ , is shortly denoted as  $F(x)$ . The optimal solutions are denoted with the star, as usual.

The leader-follower single allocation hub location under fixed markups can be represented as a bi-level mixed-integer non-linear mathematical program. For the leader, the following model is proposed.

$$\max \sum_{i,j,k,l \in N} \mu_{ij,kl} \omega_{ij} u_{ij,kl} - \sum_{k \in N} f_k x_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) x_{ik} - \sum_{k,l \in N} g_{kl} x_{kk} x_{ll} \tag{1}$$

subject to

$$u_{ij,kl} = \frac{x_{ik} x_{jl} e^{-\Theta(c_{ij,kl} + \mu_{ij,kl})}}{\sum_{s,t \in N} x_{is} x_{jt} e^{-\Theta(c_{ij,st} + \mu_{ij,st})} + \gamma_{ij}^*} \tag{2}$$

$$\gamma_{ij}^* = \sum_{s,t \in N} y_{is}^* y_{jt}^* e^{-\Theta(c_{ij,st} + \mu_{ij,st})} \tag{3}$$

$$\sum_{k \in N} x_{ik} \leq 1, \quad \forall i \in N \tag{4}$$

$$x_{ik} \leq x_{kk}, \quad \forall i, k \in N \tag{5}$$

$$y^* \in F(x) \tag{6}$$

$$x_{ik} \in \{0, 1\}, \quad \forall i, k \in N \tag{7}$$

The profit (1) is calculated as a sum over all net flow incomes minus the cost of network. The net income for a route  $i \rightarrow k \rightarrow l \rightarrow j$  is calculated as a product of margin and demand, where  $u_{ij,kl}$  indicates how much of the total demand is going to be captured by the leader. On the other hand, the network cost is composed of the sum of hub location costs, the sum of non-hub to hub allocation costs, and sum that represents the cost of inter-hub connections. The leader’s market share for a particular price on a route  $i \rightarrow k \rightarrow l \rightarrow j$  is given by (2). The equations (3) characterize the effect of the follower’s optimal solution on the leader’s market share. The constraint set (4) denotes that each non-hub node if allocated is connected to exactly one hub, while (5) stipulates that non-hub node can be allocated only to a hub. Constraint (6) denotes that for a given leader’s solution  $x$ , only the optimal follower’s one  $y^*$  is considered. Finally, constraints (7) state the domain of decision variables. The total number of constraint in the leader’s model is  $|N|^2 + |N| + 1$ . There are  $|N|^2$  binary variables and zero continuous.

It is worth to recall that for the bi-level problems, a solution  $(x, y)$  is called *semi-feasible* if  $x$  is feasible for the leader,  $y \in F(x)$ , but  $y$  is not optimal. For a solution to be *feasible*, it is required that  $y$  is optimal.

The follower’s problem can be formulated similarly as the leader’s.

$$\max \sum_{i,j,k,l \in N} \mu_{ij,kl} w_{ij,kl} v_{ij,kl} - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} y_{kk} y_{ll} \tag{8}$$

subject to

$$v_{ij,kl} = \frac{y_{ik} y_{jl} e^{-\Theta(c_{ij,kl} + \mu_{ij,kl})}}{\sum_{s,t \in N} y_{is} y_{jt} e^{-\Theta(c_{ij,st} + \mu_{ij,st})} + \eta_{ij}} \tag{9}$$

$$\eta_{ij} = \sum_{s,t \in N} x_{ik} x_{jl} e^{-\Theta(c_{ij,st} + \mu_{ij,st})} \tag{10}$$

$$\sum_{k \in N} y_{ik} \leq 1, \quad \forall i \in N \tag{11}$$

$$y_{ik} \leq y_{kk}, \quad \forall i, k \in N \tag{12}$$

$$y_{ik} \in \{0, 1\}, \quad \forall i, k \in N \tag{13}$$

The profit (8) is calculated as a sum over all net flow incomes minus the cost of network. Follower’s route is represented with the product  $y_{ik} y_{jl}$ . The net income for a route  $i \rightarrow k \rightarrow l \rightarrow j$  is calculated as a product of margin and demand, where  $v_{ij,kl}$  indicates how much of the total demand is going to be captured by the follower. On the other hand, the network cost is composed of the sum of hub location costs, the sum of non-hub to hub allocation costs, and sum that represents the cost of inter-hub connections. The follower’s market share for a particular price on a route  $i \rightarrow k \rightarrow l \rightarrow j$  is given by (9). The equations (10) characterize the effect of the leader’s optimal solution on the follower’s market share. The constraint set (11) denotes that each non-hub node is allocated to at most one hub, while (12) stipulates that non-hub node can be allocated only to a hub. Finally, the constraints (13) state the domain of decision variables. The total number of constraints in the follower’s model is  $|N|^2 + |N|$ . There are  $|N|^2$  binary variables and zero continuous.

Because the sum of objective functions is not a constant, an issue can arise when the follower’s problem has several optimizers. For example, the leader can be additionally harmed if the follower chooses a solution, among optimal ones, that is the most inconvenient for the leader. This is called a pessimistic scenario (or expectation). On the other hand, the follower can choose a hub and spoke topology, among optimal ones, to provide maximal profit to the leader, as a way to instrument the social welfare of the players (but not on the cost of his profit). This is called an optimistic scenario (or expectation), which we will be assumed here. In other words, we are ought to consider the following auxiliary problem.

$$\max \sum_{i,j,k,l \in N} \mu_{ij,kl} w_{ij,kl} u_{ij,kl} - \sum_{k \in N} f_k x_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) x_{ik} - \sum_{k,l \in N} g_{kl} x_{kk} x_{ll} \tag{14}$$

subject to (2)-(3), (9)-(13) and

$$\sum_{i,j,k,l \in N} \mu_{ij,kl} w_{ij,kl} v_{ij,kl} - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} y_{kk} y_{ll} \geq F^* \tag{15}$$

where  $F^*$  represents the optimal objective value of the follower’s problem. As we can see, in the auxiliary problem, there is only one additional constraint, compared to the (narrow) follower’s model.

**Remark 2.3.** Essentially, the follower’s problem can be seen as a bi-objective program, for which the preferred solutions are obtained by an a priori lexicographic method.

**Remark 2.4.** Because the network installation cost is sunk, it can be omitted in the auxiliary model.

To understand the given models and follow more easily the ideas and concepts, a small dimension example is presented, with the appropriate explanation.

**Example 2.5.** Consider a setting in which the inter-hub discount is  $\alpha = 0.8$ , the price sensitivity is  $\Theta = 3$ , and the margin is taken to be 40% of the route cost. The values of demands and costs are taken from the CAB data set, considering the first nine cities in Table 1 from the paper of O’Kelly [43]. On Fig. 1, two situations are presented:

- (a) The leader makes the first move. Her hubs are located in cities 3 and 8. The leader’s profit is estimated to be 238.07.
- (b) The follower enters the market that is already served by the leader. His hub backbone is a singleton, i.e. only the city 3 is a hub. In this situation the profits of competitors are:
  - -205.08 for the leader – she is heading into a bankrupt
  - 29.96 for the follower.

Obviously, the leader needs to choose her network very carefully, i.e., for every leader’s network she needs to consider the follower’s best response and how that will affect her profit.

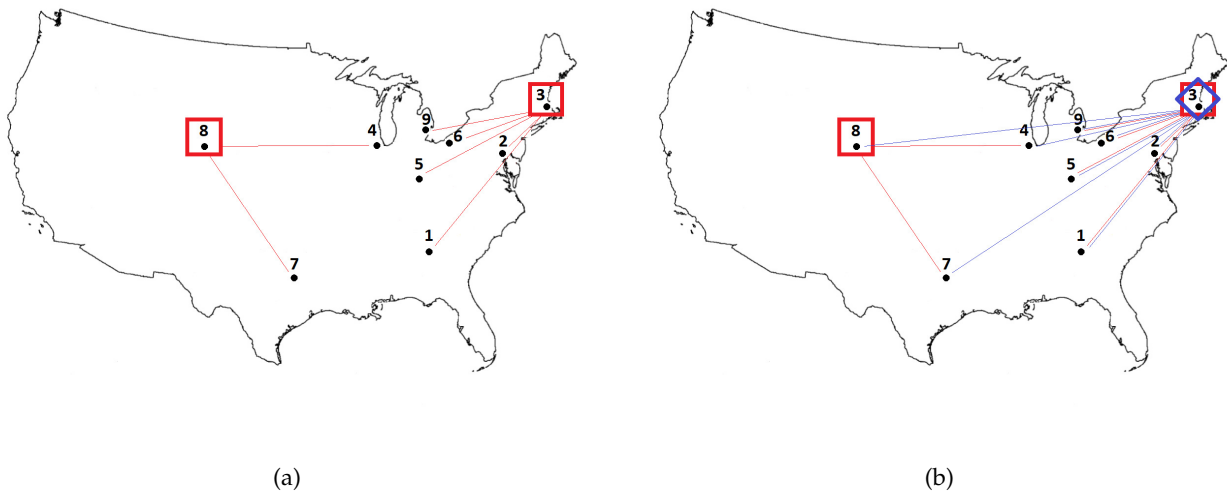


Figure 1: An example of the leader-follower scenario: (a) the leader (red) makes the move, (b) the follower (blue) reacts to the given leader’s network. Hubs are denoted by rectangles. It is assumed that the hubs in one competitor’s network are mutually interconnected. The numbers correspond to the first 9 cities from the Table 1 in [43].

### 2.2. Linear reformulation of the follower’s model

The product  $y_{kk}y_{ll}$  (in (8) and (15)) can be substituted by a real non-negative variable  $h_{kl}$  (for all  $k, l \in N$ ), accompanied by additional constraint sets

$$h_{kl} - y_{kk} - y_{ll} + 1 \geq 0, \quad \forall k, l \in N \tag{16}$$

$$h_{kl} \geq 0, \quad \forall k, l \in N. \tag{17}$$

The net flow income can be rewritten as  $\sum_{i,j \in N} w_{ij} \sum_{k \in N} R_{ij,k}$ , where  $R_{ij,k}$  is a real non-negative variable and the following additional constraints are satisfied:

$$R_{ij,k} \leq y_{ik}M, \quad \forall i, j, k \in N \tag{18}$$

$$R_{ij,k} \leq \sum_{l \in N} \mu_{ij,kl} \frac{e^{-\Theta(c_{ij,kl} + \mu_{ij,kl})}}{e^{-\Theta(c_{ij,kl} + \mu_{ij,kl})} + \eta_{ij}} y_{jl}, \quad \forall i, j, k \in N \tag{19}$$

$$R_{ij,k} \geq 0, \quad \forall i, j, k \in N, \tag{20}$$

where  $M$  is a sufficiently large number. Note that we do not need products  $y_{is}y_{jt}$  (for  $s, t \in N$ ) in the denominator, i.e., the whole sum can be reduced to the simple exponential term. In particular, if  $y_{ik} = 0$  or  $y_{jl} = 0$ , then the whole product is 0. On the other hand, if  $y_{ik} = y_{jl} = 1$ , then because of (11) summation is not needed and  $s = k \wedge t = l$ . Moreover, the product  $y_{ik}y_{jl}$  is equal to 1, thus it is redundant, too.

The follower's problem can be written as the following mixed-integer linear program

$$\max \sum_{i,j \in N} w_{ij} \sum_{k \in N} R_{ij,k} - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} h_{kl} \tag{21}$$

subject to (10)-(13) and (16)-(20).

This reformulation of the original model has  $2|N|^3 + 2|N|^2 + |N|$  constraints,  $|N|^2$  binary variables, and  $|N|^3 + |N|^2$  continuous ones. Compared to the original follower's model, the number of constraints and continuous variables is increased. On the other hand, in both models, the number of binary variables is the same.

Additionally, we can similarly reformulate the auxiliary model, taking into account Remark 2. Firstly, the constraint (15) can be written as

$$\sum_{i,j \in N} w_{ij} \sum_{k \in N} R_{ij,k} - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} h_{kl} \geq F^* \tag{22}$$

For the objective of auxiliary problem, to reformulate the net flow income term, we can introduce a new non-negative variable  $U_{ij}$  such that

$$U_{ij} \eta_{ij} + U_{ij} \gamma_{ij}^* \leq \sum_{k,l \in N} \mu_{ij,kl} x_{ik} x_{jl} e^{-\Theta(c_{ij,kl} + \mu_{ij,kl})}, \quad \forall i, j \in N \tag{23}$$

$$U_{ij} \geq 0, \quad \forall i, j \in N. \tag{24}$$

Right hand side is a constant. The second term in the left hand side is non-linear. Straightforward substitutions lead us to the following set of new constraints:

$$U_{ij} \eta_{ij} + \sum_{s \in N} \varphi_{ij,s} \leq \sum_{k,l \in N} \mu_{ij,kl} x_{ik} x_{jl} e^{-\Theta(c_{ij,kl} + \mu_{ij,kl})}, \quad \forall i, j \in N \tag{25}$$

$$\varphi_{ij,s} \leq y_{is} M_\varphi, \quad \forall i, j, s \in N \tag{26}$$

$$\varphi_{ij,s} \leq \sum_{t \in N} e^{-\Theta(c_{ij,st} + \mu_{ij,st})} \psi_{ij,t}, \quad \forall i, j, s \in N \tag{27}$$

$$\varphi_{ij,s} \geq \sum_{t \in N} e^{-\Theta(c_{ij,st} + \mu_{ij,st})} \psi_{ij,t} - (1 - y_{is}) M_\varphi, \quad \forall i, j, s \in N \tag{28}$$

$$\psi_{ij,t} \leq y_{jt} M_\psi, \quad \forall i, j, t \in N \tag{29}$$

$$\psi_{ij,t} \leq U_{ij}, \quad \forall i, j, t \in N \tag{30}$$

$$\psi_{ij,t} \geq U_{ij} - (1 - y_{jt}) M_\psi, \quad \forall i, j, t \in N \tag{31}$$

$$U_{ij} \geq 0, \quad \forall i, j \in N \tag{32}$$

$$\varphi_{ij,s} \geq 0, \quad \forall i, j, s \in N \tag{33}$$

$$\psi_{ij,t} \geq 0, \quad \forall i, j, t \in N. \tag{34}$$

where  $M_\phi$  and  $M_\psi$  are large enough numbers. Finally, we have a mixed-integer linear model for the auxiliary program:

$$\max \sum_{i,j \in N} w_{ij} U_{ij} - \sum_{k \in N} f_k x_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) x_{ik} - \sum_{k,l \in N} g_{kl} x_{kk} x_{ll} \quad (35)$$

subject to (10)-(13), (16)-(20), (22), and (25)-(34).

The reformulation of auxiliary model has  $8|N|^3 + 3|N|^2 + |N| + 1$  constraints. The number of binary variables stayed the same, but on the other hand, there are  $3|N|^3 + 2|N|^2$  continuous ones. The increase in the number of constraints and continuous variables for this reformulation is much bigger than the increase in the reformulation of the follower's model. In other words, if there is another solution for the follower, we could expect longer running times when solving the auxiliary problem, compared to the running time needed to solve the follower's model.

**Remark 2.6.** *The product of binary variables  $x_{kk}x_{ll}$  is not the subject of reformulation, because the leader has already made her move, i.e., the follower knows what are the values of her decision variables.*

### 3. Robust leader-follower single allocation hub location problem under fixed markups

The usual parameter that is considered to be affected by some small perturbation is the demand. In particular, the demand distribution is unknown, and only some of its attributes are available. One way to approach this issue is to assume that the uncertain parameter can be expressed as a linear combination of a known mean and several independent random variables, as Chen et al. [34] did it in their study. More specifically, Wagner et al. [35] showed that such affine combination for demand uncertainty is capable of capturing systematic and non-systematic risk associated with freight demand. Consequently, the uncertain demand is assumed to be affine dependent on a known mean and some independent random variables. Shahabi and Unnikrishnan [36] presented a robust formulation for the one-level uncapacitated single and multiple allocation hub location problems where demand is uncertain, and its distribution is not entirely specified. Conducted computational experiments suggested locating more hubs when accounting for demand uncertainty using robust optimization, compared to the deterministic setting. Čvokić and Stanimirović [8] studied a robust case of one-level uncapacitated single allocation hub location problem when the price-dependent demand is introduced. Computational investigation showed that a lot of weight is put on the pricing structure, which serves to suppress the impact of uncertainty.

In particular, the demand is structurally characterized by an estimated mean  $d$  and a set of independent random variables  $\tilde{u}_t$  as

$$\tilde{d}_{ij} = d_{ij} + \sum_{t \in T} b_{ijt} \zeta_t, \quad \forall i, j \in N, \quad (36)$$

where  $T$  represents the set of uncertainty sources. Alongside with that,  $b_{ijt} \in \mathbb{R}$  represents a weight associated with the corresponding random variable, i.e. weight of the uncertainty source. Each independent variable represents one source of uncertainty causing demand perturbation from the mean value. The variables  $\zeta_t \in \mathbb{R}$  ( $t \in T$ ) are referred to as the primitive uncertainty variable defined by the following three underlying assumptions:

- the mathematical expectation is zero, i.e.  $E(\zeta_t) = 0$
- $|\zeta_t| \leq 1$
- $\zeta_t$  are all independent.



The classic worst case analysis leads to an over-conservative solution. This issue is handled by taking that the norm of a vector of uncertain random variables  $\zeta$  is bounded by some  $\Omega > 0$ , which defines a so-called ellipsoidal uncertainty set  $\{\zeta \mid \|\zeta\| \leq \Omega\}$ . As a result, we have a computationally tractable formulation with the appropriate level of conservativeness (see [34]).  $\Omega$  is called the budget of uncertainty or the risk budget. More conservative decision making requires a larger risk budget so that the solutions are more “protected” against the possible demand perturbations. The choice of  $\Omega = 0$  would yield an unprotected deterministic solutions. In studies [37] and [38] the uncertainty budget, as a theoretical input for the robust optimization framework, was found to vary between values 0 and 3. We refer an interested reader to [34] and [36], for a better understanding of the topic and terminology that is used.

### 3.1. Mathematical formulation of the robust variant

The robust leader-follower single allocation hub location problem under fixed markups can be represented as a bi-level mixed-integer non-linear mathematical program. The robust variant for the leader’s problem can be neatly represented as a max-min model

$$\max \left( \min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ijkl} \tilde{w}_{ij} u_{ij,kl} \right) - \sum_{k \in N} f_k x_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) x_{ik} - \sum_{k,l \in N} g_{kl} x_{kk} x_{ll} \tag{37}$$

where

$$\tilde{w}_{ij} = w_{ij} + \sum_{t \in T} b_{ijt} \zeta_t, \quad \forall i, j \in N \tag{38}$$

and constraints (2)-(7) must be satisfied. In the same manner, the robust variant for the follower can be represented as

$$\max \left( \min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ijkl} \tilde{w}_{ij,kl} v_{ij,kl} \right) - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} y_{kk} y_{ll} \tag{39}$$

where constraints (9)-(13) are satisfied.

Additionally, as in the deterministic case, we need to define the auxiliary problem for the optimistic scenario

$$\max \left( \min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ijkl} \tilde{w}_{ij} u_{ij,kl} \right) - \sum_{k \in N} f_k x_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) x_{ik} - \sum_{k,l \in N} g_{kl} x_{kk} x_{ll} \tag{40}$$

subject to constraints (2)-(3), (9)-(13) and an additional one

$$\left( \min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ijkl} \tilde{w}_{ij} v_{ij,kl} \right) - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} y_{kk} y_{ll} \geq \tilde{F}^* \tag{41}$$

where  $\tilde{F}^*$  is the optimal objective value for the follower’s problem.

In all these robust models, the number of constraints is the same as in the deterministic counterpart.

3.2. Solution of the inner minimization problem

Considering that the inner minimization problem is defined in terms of the variables  $\zeta_t$ , the sum of net flow incomes (39) for the follower can be rewritten in the following manner

$$\min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ij,kl} \left( w_{ij} + \sum_{t \in T} b_{ijt} \zeta_t \right) v_{ij,kl} \tag{42}$$

$$= \sum_{i,j,k,l \in N} \mu_{ij,kl} w_{ij} v_{ij,kl} + \left( \min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ij,kl} \left( \sum_{t \in T} b_{ijt} \zeta_{ijt} \right) v_{ij,kl} \right) \tag{43}$$

To reduce our program to a computationally tractable formulation we focus our attention to the new problem

$$\min_{\|\zeta\| \leq \Omega} \sum_{i,j,k,l \in N} \mu_{ij,kl} \left( \sum_{t \in T} b_{ijt} \zeta_{ijt} \right) v_{ij,kl} \tag{44}$$

In order to solve it, a Lagrangian relaxation scheme can be used to determine the expression for the primitive uncertainty variables  $\zeta_t$ . The associated Lagrangian function is

$$L(\zeta, \lambda) = \sum_{i,j,k,l \in N} \sum_{t \in T} \mu_{ij,kl} b_{ijt} \zeta_{ijt} v_{ij,kl} + \lambda (\|\zeta\| - \Omega). \tag{45}$$

for a Lagrangian multiplier  $\lambda \geq 0$ , that corresponds to  $\|\zeta\| \leq \Omega$ . The First Order Condition for the associated unconstrained optimization problem is given as

$$\frac{\partial L(\zeta, \lambda)}{\partial \zeta_t} = \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} + \lambda \frac{\zeta_t}{\|\zeta\|} = 0, \quad \forall t \in T \tag{46}$$

$$\lambda (\Omega - \|\zeta\|) = 0. \tag{47}$$

Assuming  $\lambda > 0$ , from the equation (46) we have that the optimal value for  $\zeta_t$  is

$$\zeta_t = -\frac{\|\zeta\|}{\lambda} \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl}, \quad \forall t \in T. \tag{48}$$

Observe that for  $\lambda = 0$  the equation (46) does not have any solution in terms of  $\zeta$ . Therefore, taking into the account the equation (47), we can conclude that  $\|\zeta\| = \Omega$ . Now, bearing in mind the last equation, we may write  $\|\zeta\|$  as

$$\|\zeta\| = \frac{\Omega}{\lambda} \sqrt{\sum_{t \in T} \left( \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \right)^2} \tag{49}$$

Again, from equation (47) the optimal value for  $\|\zeta\|$  is equal to  $\Omega$  and therefore the optimal value for  $\lambda$  is given as

$$\lambda = \sqrt{\sum_{t \in T} \left( \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \right)^2} \tag{50}$$

Returning back  $\lambda$  into (48), the optimal value for  $\zeta_t$  can be written as

$$\zeta_t = -\Omega \frac{\sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl}}{\sqrt{\sum_{t \in T} \left( \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \right)^2}} \tag{51}$$

The optimal value of our inner optimization problem (44) is equal to

$$-\Omega \sqrt{\sum_{t \in T} \left( \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \right)^2}. \tag{52}$$

Incorporating this solution into (39), the robust total net income equation can be converted to

$$\sum_{i,j,k,l \in N} \mu_{ij,kl} w_{ij} v_{ij,kl} - \Omega \sqrt{\sum_{t \in T} \left( \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \right)^2} \tag{53}$$

For the sake of clarity, we introduce  $\rho$  to present the last equation as

$$\sum_{i,j,k,l \in N} \mu_{ij,kl} w_{ij} v_{ij,kl} - \Omega \rho \tag{54}$$

where

$$\rho = \sqrt{\sum_{t \in T} \left( \sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \right)^2} \tag{55}$$

**Remark 3.1.** *The same procedure would lead to a closed-form expression for the solution of leader’s inner minimization problem.*

In the next section, we present how to reformulate our objective, i.e., to obtain a possible tractable conic quadratic formulation.

### 3.3. Conic quadratic reformulation of the robust follower’s model

We can further exploit the properties of the robust follower’s model, introducing auxiliary positive variables  $\pi_t$  to reformulate (53) using linear and conic quadratic inequalities (constraints). This can be done as follows

$$\sum_{t \in T} \pi_t^2 \leq \rho^2 \tag{56}$$

$$\sum_{i,j,k,l \in N} \mu_{ij,kl} b_{ijt} v_{ij,kl} \leq \pi_t, \quad \forall t \in T \tag{57}$$

$$\rho \geq 0 \tag{58}$$

$$\pi_t \geq 0, \quad \forall t \in T \tag{59}$$

Now we have all the knowledge to formulate the robust variant of the follower’s problem as a mixed-integer conic quadratic program

$$\max \sum_{i,j \in N} w_{ij} \sum_{k \in N} R_{ij,k} - \Omega \rho - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} h_{kl} \tag{60}$$

subject to (10)-(13), (16)-(20) and

$$\sum_{t \in T} \pi_t^2 \leq \rho^2 \quad (61)$$

$$\sum_{i,j,k \in N} b_{ijt} R_{ijk} \leq \pi_t, \quad \forall t \in T \quad (62)$$

$$\rho \geq 0 \quad (63)$$

$$\pi_t \geq 0, \quad \forall t \in T \quad (64)$$

This conic-quadratic reformulation of the originally non-linear and non-quadratic model has one conic-quadratic constraint,  $2|N|^3 + 2|N|^2 + |N| + |T|$  linear ones, and  $|N|^2$  binary and  $|N|^3 + |N|^2 + |T| + 1$  continuous variables. Obviously, there is an increase in the number of constraints and continuous variables, compared to the deterministic counterpart. Interestingly, the number of binary variables is still unchanged.

The reformulation of auxiliary program for the robust variant is done in the same fashion. In particular, we obtain the following model.

$$\max \sum_{i,j,k,l \in N} w_{ij} U_{ij} - \Omega \rho - \sum_{k \in N} f_k x_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) x_{ik} - \sum_{k,l \in N} g_{kl} x_{kk} x_{ll} \quad (65)$$

subject to constraints (10)-(13), (16)-(20), (25)-(34), (61)-(63), and an additional ones

$$\sum_{i,j \in N} w_{ij} \sum_{k \in N} R_{ijk} - \Omega \rho - \sum_{k \in N} f_k y_{kk} - \sum_{i,k \in N} (g_{ik} + g_{ki}) y_{ik} - \sum_{k,l \in N} g_{kl} h_{kl} \geq \tilde{F}^* \quad (66)$$

$$\sum_{t \in T} \phi_t^2 \leq \varrho^2 \quad (67)$$

$$\sum_{i,j,k,l \in N} b_{ijt} U_{ij} \leq \phi_t, \quad \forall t \in T \quad (68)$$

$$\varrho \geq 0 \quad (69)$$

$$\phi_t \geq 0, \quad \forall t \in T. \quad (70)$$

where  $\tilde{F}^*$  is the optimal objective value for the follower's problem.

This conic-quadratic reformulation of the originally non-linear and non-quadratic model has two conic-quadratic constraints and  $8|N|^3 + 3|N|^2 + |N| + 2|T| + 1$  linear ones. The number of binary variables has stayed the same, but the continuous ones count up to  $2|N|^3 + 2|N|^2 + 2|T| + 2$ . The same observation for the robust follower's problem holds for the robust auxiliary one.

#### 4. Solution approach

Metaheuristics are widely used to solve hard combinatorial optimization problems. Here, the solution approach is based on an alternating heuristic (AH), firstly presented in [39]. Some of the recent applications can be found in [13, 14, 18]. The main idea is built on Hotelling's observation that at equilibrium, on a line, the duopoly facilities tend to cluster together at some central point of the market [40]. While d'Aspermont et al. [41] have shown that this observation does not hold in the case of variable prices, it does hold in the case of fixed and equal prices (see [39]). Besides this, designing a sophisticated heuristic approach for both variants of the problem (deterministic and robust) is not an easy task. The AH is chosen because of its simplicity, robustness, and the previously mentioned observation for the fixed prices.

Essentially, one can consider the AH similar to the coordinate descent (ascend). The same underlying idea is presented in several widely popular heuristics: Blahut-Arimoto algorithm in coding theory, the

expectation-maximization algorithm in statistics, the concave-convex procedure in global optimization, k-means in machine learning, etc.

The instance is represented as a pair of finite associative arrays  $(x, y^*)$ , according to the description in subsection 2.1. The size of array is  $|N|$ . In the algorithm,  $x_i$  represents the leader's network, during the  $i$ -th iteration. Accordingly,  $y_i$  represents the corresponding follower's network.

The algorithm starts with a monopolistic solution. In other words, the alternating heuristic is a natural approach to estimate the evolution of market competition for this setting. In the literature, the AH can start from the arbitrary feasible solution, but here, intentionally, the monopolistic solution is chosen as the starting one. One reason is the already mentioned evolution of market competition, and the other is to have the ability to compare the final solution found by the AH with the monopolistic one.

**Remark 4.1.** *The monopolistic solution can be obtained by solving the follower's problem under the assumption that the incumbent does not serve the market.*

The follower's problem  $F(x_i)$  (in  $i$ -th iteration) is solved *exactly*, utilizing the proposed reformulations, i.e., a commercial solver can be used like CPLEX or Gurobi.

Every iteration corresponds to one callback of the exact algorithm for the follower's problem. In the same iteration, the roles are switched, i.e., for the obtained solution of follower, the leader takes his role. This is portrayed at line 8. Note that the leader's profit is computed after the follower's one.

The termination condition is a cycle detection. It is obvious that this procedure would converge because the number of all semi-feasible solutions is finite. Moreover, this way the AH could capture the Nash equilibrium if it exists. Because the follower's and auxiliary problems can be exactly solved (utilizing the proposed reformulations), the AH returns a feasible solution, not just semi-feasible.

**Remark 4.2.** *Cycles can be very long, therefore in the literature, it is not unusual to include the upper bound on the number of iterations in the termination criterion.*

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**Algorithm 1:** Alternating heuristic
 

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**Data:** instance parameter values

**Result:** a feasible solution  $(x, y^*)$

1  $i \leftarrow 1$

2  $x_i \leftarrow$  the monopolistic solution

3 **while**  $x_i \notin \{x_j \mid j \in \{1, \dots, i-1\}\}$  **do**

4      $y_i^* \leftarrow F(x_i)$  solved exactly (e.g. by a solver)

5      $z_i^{F^*} \leftarrow$  the follower's profit

6      $z_i^L \leftarrow$  the leader's profit

7      $i \leftarrow i + 1$

8      $x_i \leftarrow y_{i-1}^*$

9  $k \leftarrow \arg \max_{j \in \{1, \dots, i\}} z_j^L$

10  $x \leftarrow x_k$

11  $z^{F^*} \leftarrow z_k^{F^*}$

12  $y^* \leftarrow$  the solution of corresponding auxiliary problem for  $x$  and  $z^{F^*}$

---

Since the follower's problem itself is hard to solve, the alternating heuristic could require a significant amount of computational resources, especially in the case of robust variant. It is important to note that some authors use heuristics to solve the follower's problem (line 4), disregarding the existence of exact method. The feasible solution is found only at the end of search process. On top of that, solving the auxiliary problem can be very time consuming itself, because the logit model obviously affects the chance for finding a different optimal solution for the follower, which is quite substantial in the case of non-linear conic-quadratic programs. A preliminary investigation showed that the runtime of solving the auxiliary model can sometimes be longer than the whole while-loop in the proposed AH. Thus, it was natural to design the approach so that the auxiliary model is solved at the last step of Algorithm 1.

**Remark 4.3.** In the following two cases it is not needed to solve the auxiliary problem:

- the follower does not enter the market (entry deterrence)
- the leader does not enter the market (non-profitable setting).

**Remark 4.4.** If the best solution for the leader (line 10)) is the Nash equilibrium, solving the auxiliary problem can give a feasible solution that is not a Nash equilibrium.

## 5. Computational experiments

Computational experiments were conducted using the instances generated from the CAB data set, firstly presented by O’Kelly [43]. Instance size was taken to be 20 and 25 for the deterministic variants, and 20 for the robust ones. In total, 960 instances were examined. All mathematical programs were implemented in Python 3.6 (as a part of Anaconda suite). The Gurobi 8.0 was used as a solver, installed on Windows 10 OS. The tests were performed on an Intel(R) Core(TM) i7-7700 CPU @ 3.60GHz processor with 24.0 GiB of DDR4 RAM.

The fixed cost of opening a hub at node  $f_k = 100$  was set to be the same for all nodes, as usual for the CAB instances. The spoke allocation cost  $g_{ik}$  for every O-D pair  $(i, j)$  is computed similarly as in [42] and [11]:

$g_{ik} = f \frac{c_{ik}}{\beta_{ik}} \frac{c_{jk}}{\max_{i,k \in N} \frac{c_{ik}}{\beta_{ik}}}$ . The discount factors  $\chi$  and  $\delta$  were set to 1, and  $\alpha$  took values from the set  $\{0.2, 0.4, 0.6, 0.8\}$ .

The markup  $\mu_{ij,kl}$  is taken as the cost percentage  $\mu_{ij,kl} = p \cdot c_{ij,kl}$ , where  $p \in \{0.05, 0.1, 0.2, 0.4\}$ , that is percentages were: 5%, 10%, 20%, 40%. The price sensitivity factor  $\Theta$  took values from a set  $\{3, 6, 9, 12, 15\}$ . These are the rounded and equally distributed values of  $\Theta$  taken from [11].

When it comes to the robust variant, the risk budget value  $\Omega$  was set to 1.5, as in [36]. Two levels of uncertainty sources were considered. Five instances of the  $b_{ijt}$  parameter were randomly generated, where  $b_{ijt} \in [0, 0.2w_{ij}]$  for the lower uncertainty level, and  $b_{ijt} \in [0, 0.5w_{ij}]$  for the higher one. Preliminary experiments showed that for our bi-level model there is no sense to examine much higher levels of uncertainty sources (e.g.  $b_{ijt} \in [0, 0.8w_{ij}]$ ).

Among the usual elements of analysis in location theory, here we address the net profit margin (NPM), which is the ratio of net profits to revenues for a company or business segment. Usually expressed as a percentage, the measurement reveals the amount of profit that a company can take from its total income. In other words, the net profit margin is calculated as follows

$$\text{NPM} = \frac{\text{objective value}}{\text{revenue}} 100\% \quad (71)$$

Calculating the net profit margin of a business is a routine part of financial analysis, particularly a vertical analysis.

Four solution types were observed: the Nash equilibrium, the entry deterrence, the non-profitable setting, and the best of examined. The last type will be denoted in such fashion because both competitors have non-negative profit values and the solution type is not one of the previous ones.

All figures are created using Matplotlib Python’s package by Droettboom et al. [44]. Although the line plots are not often used in the experimentation of multiple instances, they are given in order to express pattern observations.

### 5.1. Results for the deterministic variant of LFS AHLuFM

From the Table 1 we can see that the dominant solution type is the Nash equilibrium. In general, larger instances and larger markups give more space to the competitors.

The running time data for both batches of instances (without step 6):

- $|N| = 25$ : min = 411.35s, max = 17653.35s, mean = 3296.78s,  $\sigma = 3151.66$
- $|N| = 20$ : min = 21.67s, max = 296.50s, mean = 108.72s,  $\sigma = 64.68$

Table 1: Solution type percentages for the deterministic variant.

$ N $	Nash equilibrium	Entry deterrence	Non-profitable setting	The Best of examined
25	73.75%	0.0%	0.0%	26.25%
20	58.75%	1.25%	0.0%	40.0%
<b>both</b>	<b>66.25%</b>	<b>0.625%</b>	<b>0.0%</b>	<b>33.125%</b>

where  $\sigma$  represents the standard deviation. As it can be seen, the alternating heuristic can be very time-consuming. Obviously, in the case of Nash equilibrium, the best solution was found at the end of execution time.

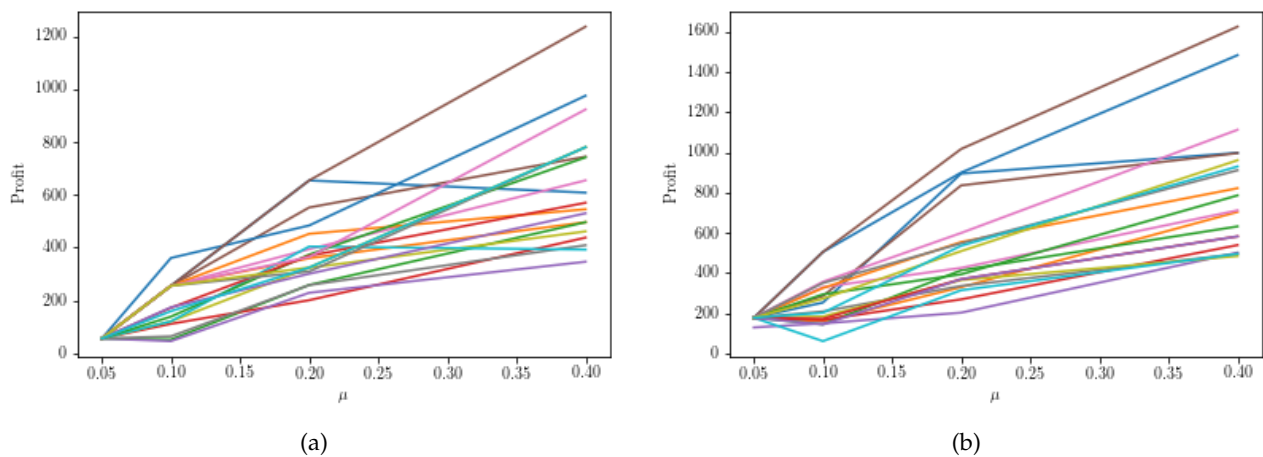


Figure 2: The effect of markup on profit: (a)  $|N| = 20$ , (b)  $|N| = 25$ . Different line colors represent different combinations of instance parameters  $\alpha$  and  $\Theta$ .

From the results of computational experiments, the effect of instance parameter  $\alpha$  is not very clear. There is no strict regularity, but it could be said that a leader should expect a higher profit if  $\alpha$  is big enough. No regularity was observed when it comes to the effect of  $\alpha$  on NPM. Interestingly, it was found that the effect of discount factor  $\alpha$  on NPM is venial in case of a monopolistic solution.

Similarly, no solid pattern was observed when it comes to the effect of price sensitivity parameter  $\Theta$  on the objective value, but differently to the discount factor  $\alpha$ , the leader could expect a higher profit and a larger NPM for *smaller* values of  $\Theta$ .

The effect of markup was the most discernible. A larger markup in most cases leads to a higher profit (Fig. 2) and bigger NPM (Fig. 3).

Basically, in almost all cases a combined effect of higher markup, larger inter-hub discount factor  $\alpha$ , and a smaller value of sensitivity parameter  $\Theta$  lead to a higher profit. The same can be said for the NPM.

The results obtained by alternating heuristic suggests that there could be a pattern in profit comparisons. In the case of Nash equilibrium, profits are equal, obviously. On the other hand, if the solution type is the best of examined, it is observed that the follower usually can gain higher profit than the leader, i.e., usually, there is no first move advantage. The opposite situation was observed only in five cases and for all of them  $|N| = 20$ . For NPM, we could say that the same pattern holds, although in one case when  $|N| = 25$  this was not true.

In this study, the setting is much more complex than in Hotelling's problem. Therefore, it is natural to investigate the (dis)similarity of hub backbones. For strings (ordered arrays), one of the usual distance measures is the edit (Levenshtein) distance — the minimum-weight series of edit operations that transform

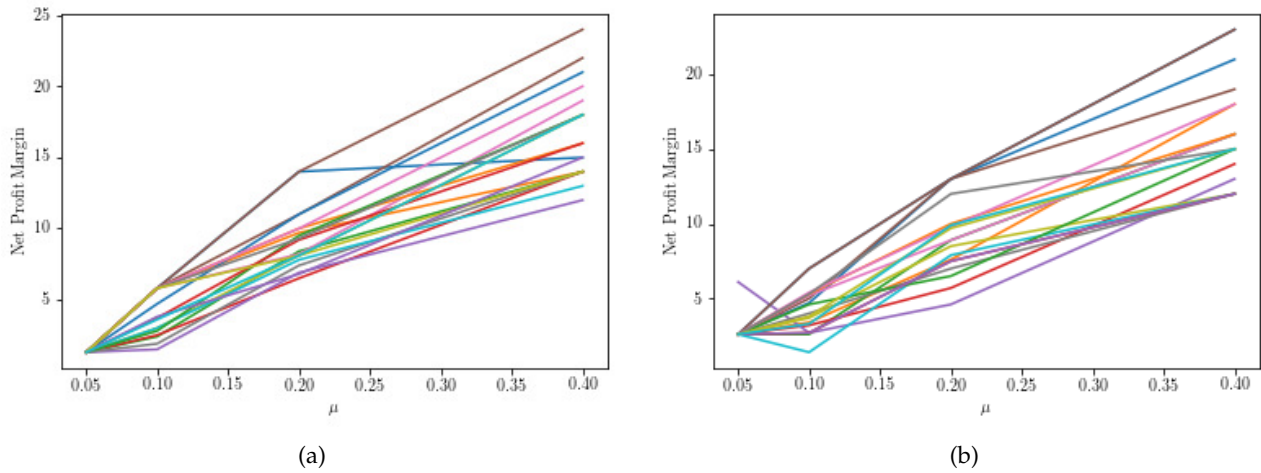


Figure 3: The effect of markup on net profit margin: (a)  $|N| = 20$ , (b)  $|N| = 25$ . Different line colors represent different combinations of instance parameters  $\alpha$  and  $\Theta$ .

one string into another. The edit operations are insertion, deletion, and substitution. Following this logic for strings, we can create a similar measure. Because sets are unordered, the minimal number of operations to transform a set  $A$  into a set  $B$  is  $\max(|A \setminus B|, |B \setminus A|)$ :  $\min(|A \setminus B|, |B \setminus A|)$  substitutions and  $\max(|A \setminus B|, |B \setminus A|) - \min(|A \setminus B|, |B \setminus A|)$  insertions xor deletions, depending on which set difference is larger.

Here, the computational experiments suggest that with the increase of instance size there is a tendency to set edit distance to become smaller. Table 2 presents the min, max, and average size of set edit distances between hub backbones of the leader and follower. On average, 0.35 edit operations were needed for conducted computational experiments.

Table 2: Min, max and average set edit distances between the leader’s and follower’s hub backbones.

$ N $	Minimal set edit distance	Maximal set edit distance	Average set edit distance
25	0	3	0.2625
20	0	4	0.4375
<b>both</b>	<b>0</b>	<b>4</b>	<b>0.35</b>

The similarity of hub backbones can be measured using the Jaccard similarity coefficient. In this case, the solutions in which the follower does not enter the market should be ignored to avoid the zero division error. The results for the Jaccard similarity index are presented in Table 3. On average, the hub backbones of competitors tend to share a lot of hubs. Interestingly, the Jaccard similarity index is higher for 25-size instances.

Table 3: Min, max and average values of Jaccard similarity indexes for the hub backbones of leader and follower.

$ N $	Minimal Jaccard similarity coefficient	Maximal Jaccard similarity coefficient	Average Jaccard similarity coefficient
25	0	1	0.905
20	0	1	0.825
<b>both</b>	<b>0</b>	<b>1</b>	<b>0.866</b>



These experimental results could look counter-intuitive at first glance, but the reason for this may lay in the fact that the smaller instances leave less “maneuvering” space for the follower. Also, the solutions of the auxiliary model were not different from the corresponding solutions of the follower’s model.

In Figure 4 we can see how typical hub and spoke topologies look like on the CAB dataset of largest US airports, as the network solutions for the competitors. It is obvious that the leader’s and follower’s networks share a lot of infrastructures. Their set edit distance is 1 and the Jaccard similarity coefficient is  $2/3$ . Also, the allocations of cities (nodes) 6, 15, 20, and 21 may catch the eye of a reader. They are all allocated to the hub 12, although these cities are in the close vicinity of hub 4. Unusual at first sight, one should keep in mind that the objective is not the minimization of transportation costs, nor the transportation time for the customers, but the maximization of the company’s own profit.

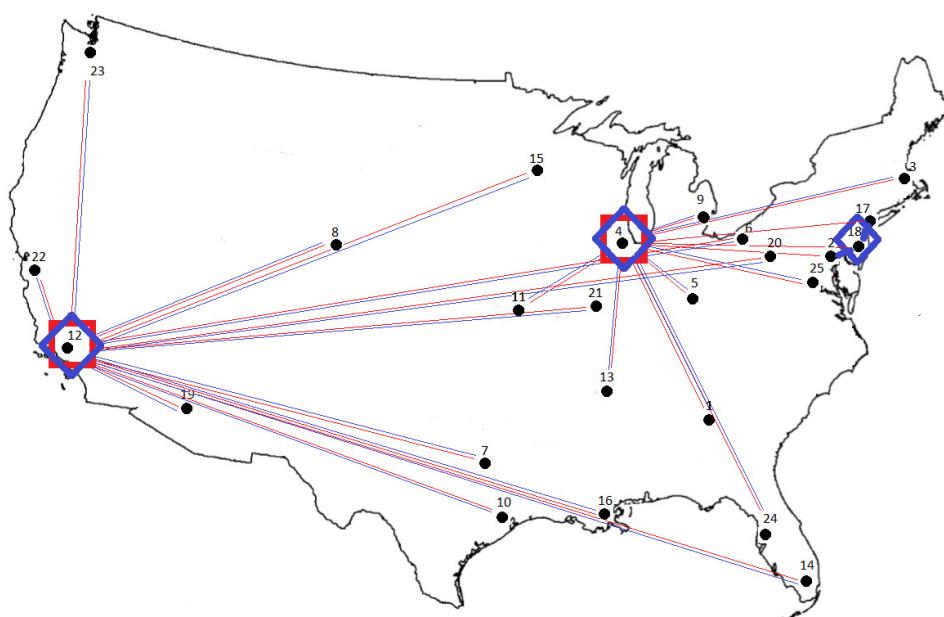


Figure 4: The leader’s (red) and follower’s (blue) networks, obtained by alternating heuristics as the best solution. The instance parameter values are:  $\alpha = 0.6$ ,  $\mu_{ij,kl} = 0.1 \cdot c_{ij,kl}$ ,  $\Theta = 9$ . The numbers correspond to the cities from the Table 1 in [43]. It is assumed that the hubs are mutually interconnected for both competitors, separately.

### 5.2. Results for the robust variant of LFS AHLuFM

In this case, the entry deterrence solution type was more often observed, compared to the deterministic case, especially for the low markups. Percentages of solution types for the robust variant are given in Table 4.

The running time data for both batches of instances (lower and higher level), excluding step 6:

- the lower level of uncertainty sources: min = 41.01s, max = 796.36s, mean = 263.97s,  $\sigma = 213.15$
- the higher level of uncertainty sources: min = 12.63s, max = 1207.59s, mean = 184.33s,  $\sigma = 272.53$

where  $\sigma$  represents the standard deviation. As expected, the computational time of alternating heuristic is longer for robust problems. Interestingly, the higher level of uncertainty sources lead to shorter runtime

Table 4: Solution type percentages for the robust variant.

Level of uncertainty sources	Nash equilibrium	Entry deterrence	Non-profitable setting	The Best of examined
lower	49.0%	40.0%	0.0%	26.0%
higher	15.25%	46.5%	25.75%	12.5%
<b>both</b>	<b>32.125%</b>	<b>43.25%</b>	<b>12.875%</b>	<b>19.25%</b>

in average. The reason for this is because there are a lot of entry deterrence solutions, that pull down the mean value.

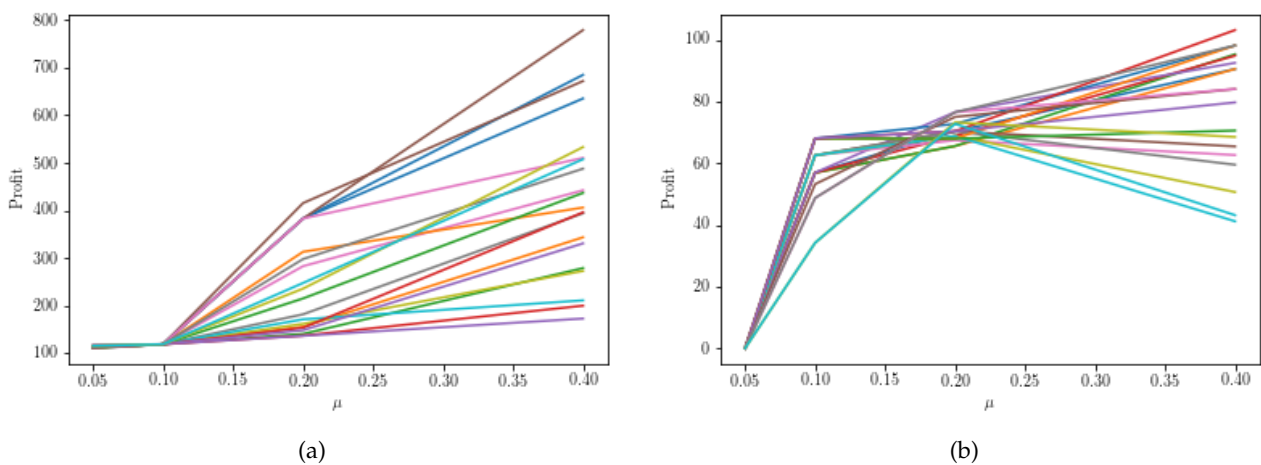


Figure 5: The effect of markup on profit: (a) *lower* level of uncertainty sources, (b) *higher* level of uncertainty sources. Different line colors represent various combinations of instance parameters  $\alpha$  and  $\Theta$ .

Like in the deterministic case, the effect of markup was the most discernible. In most cases, a larger markup leads to a higher profit (Fig. 5). A bigger markup leaves more space for the follower, too. But in the robust optimization competitors need to immunize possible turmoils of demand. This usually requires more developed and more costly infrastructure, but the generosity of the market has to be taken into account, too. When it comes to the NPM, the effect is even more discernible (Fig. 6). In a way, this justifies taking into account the NPM. For several instances, a very interesting and counterintuitive situation was observed — the objective value for the leader was zero, although the follower’s objective value was positive. Basically, this observation says that there are instances for which it is not profitable to enter the market as the first competitor.

Comparing the leader’s hub locations for the robust and deterministic variants is done by computing the already mentioned set edit distances and Jaccard similarity indexes. For both measures, an average value was computed, based on all corresponding five instance sets of  $b_{ijt}$  values. The results for set edit distances are presented in Table 5 and findings for the Jaccard similarity indexes are given in Table 6. The entries in tables suggest that the hub backbones of competitors in robust solutions should be less similar, compared to their deterministic counterparts, when the level of uncertainty sources is becoming higher. Particularly, the set edit distances are getting larger, naturally corresponds to the smaller values of Jaccard similarity index.

Computational experiments conducted by Shahabi and Unnikrishnan [36] on the same data set showed that for the classic hub location problem (with the cost minimization objective), the robust variant implies that more hubs should be established. The similar observation holds for the monopolistic solutions in our

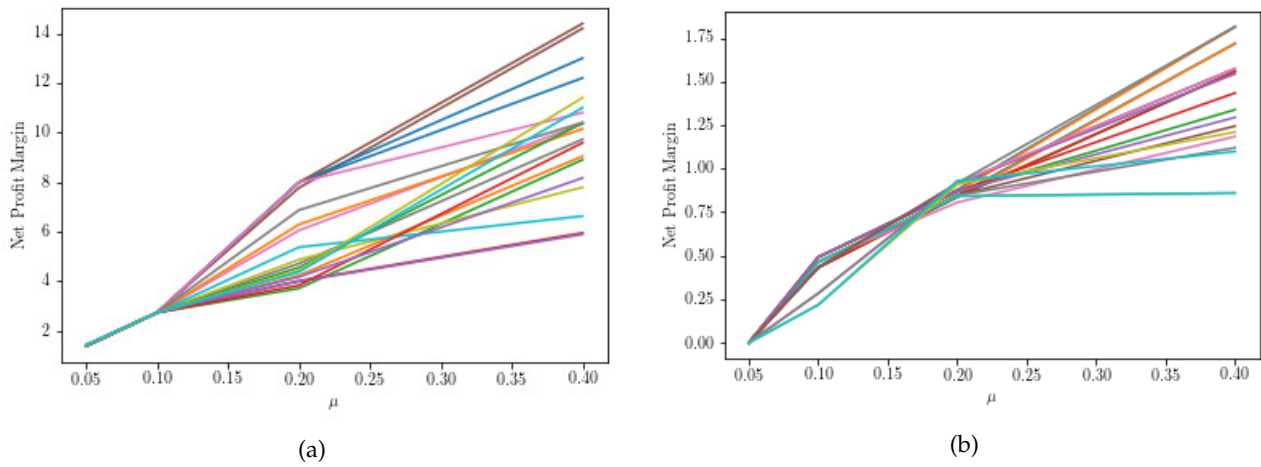


Figure 6: The effect of markup on net profit margin: (a) *the lower* level of uncertainty sources, (b) *the higher* level of uncertainty sources. Different line colors represent various combinations of instance parameters  $\alpha$  and  $\Theta$ .

Table 5: Min, max and average set edit distances between sorted lists of leader’s hubs in robust and deterministic variant.

$ N $	Minimal average set edit distance	Maximal average set edit distance	Average of average set edit distances
lower	0	3	0.72
higher	0.2	3	1.5475
<b>both</b>	<b>0</b>	<b>3</b>	<b>1.3375</b>

setting. On the other hand, because of competition, a different pattern is observed (although solutions are obtained using a heuristic approach). The following lines present the descriptive statistics for the average differences between the hub backbone size of robust solution and its deterministic counterpart.

- *the lower* level of uncertainty sources: min = -2.6, max = 0.6, mean = 0.35, median = 0.0,  $\sigma = 0.56$
- *the higher* level of uncertainty sources: min = -3.0, max = 0.6, mean = -1.125, median = -1.0,  $\sigma = 0.89$

In other words, it is reasonable to expect a smaller number of hubs in the leader’s robust solution network, compared to her deterministic hub backbone. This suggests that the effect of discrete choice rule and competitive setting should not be neglected.

Solutions of the auxiliary model were not different from the corresponding solutions of the follower’s model.

In Figure 7 we can see how typical hub and spoke topologies look like the network solutions for the competitors in the robust case. The data set corresponds to the first 20 CAB cities. As can be seen, the hub locations in the left and right sub-figures differ and the corresponding solution types are also different. On the left sub-figure, the networks of the leader and follower are the same. Contrary to this situation, on the right sub-figure, we can see that only the leader’s network is depicted. In other words, it is not profitable for the follower to enter the market, and the leader’s network corresponds to the monopolistic solution. Moreover, we can see that market coverage is not total. Particularly, the nodes (cities) 2, 5, 10, and 13 are not part of the leader’s hub and spoke topology.

Table 6: Min, max and average values of *average* Jaccard similarity indexes determined for the leader’s hub backbones in robust and deterministic variants.

$ N $	Minimal average Jaccard similarity coefficient	Maximal average Jaccard similarity coefficient	Average of average Jaccard similarity coefficient
lower	0	1	0.629
higher	0	0.8	0.184
<b>both</b>	<b>0</b>	<b>1</b>	<b>0.432</b>

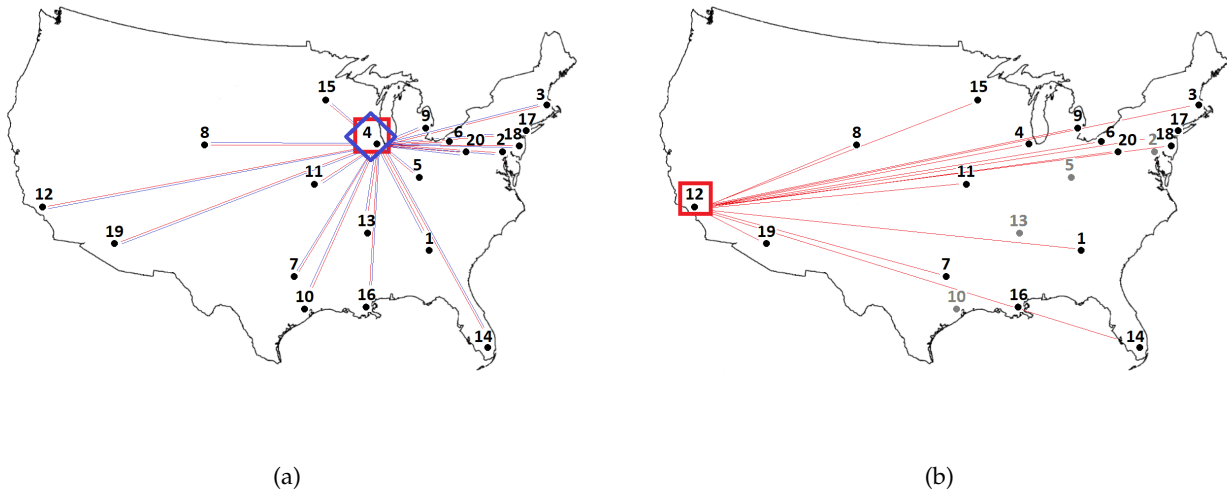


Figure 7: The leader’s (red) and follower’s (blue) networks, obtained as a solution of the robust variant, by the alternating heuristic. Hubs are denoted by rectangles. Sub-figure (a) depicts the solution when the level of uncertainty sources are lower, and sub-figure (b) when they are higher. The numbers correspond to the first 20 cities (i.e., not all 25 ones) from Table 1 in [43]. The instance parameter values are  $\alpha = 0.6$ ,  $\mu_{ij,kl} = 0.1 \cdot c_{ij,kl}$ , and  $\Theta = 9$ .

### 6. Conclusion

In this study, we have formally examined the scenario in which two competitors, called the leader and the follower, sequentially create their hub and spoke networks, for which only a single allocation is allowed. The number of hubs is not determined in advance, differently to the  $(r|p)$ -hub centroid problem. Moreover, the market does not need to be totally covered, so competitors are also seeking to find the best “corner” of the market, each for itself. It is assumed that the customers patronize a chosen route mainly according to its price. The market pricing is regulated, i.e., every route has its fixed markup. The demand is taken to be non-elastic, and the MNL is the discrete choice model. Two variants of this problem were addressed: deterministic and robust.

It was shown how to present this Stackelberg competition as the corresponding bi-level mixed-integer non-linear program. Furthermore, it can be seen how the follower’s model can be reformulated as a mixed-integer linear program in the deterministic case, and as a mixed-integer conic quadratic program in the case of robust variant. The reformulation of the robust follower’s problem is based on the analytic solution of the corresponding inner minimization problem. The direct benefit from these reformulations is that they allow the usage of state-of-the-art solvers to obtain *feasible* solutions, instead of semi-feasible ones. The same was shown for the corresponding auxiliary problems.

Computational investigation for deterministic variant showed that the majority of best-found solutions composed a Nash equilibrium, especially in the case of larger instances and higher markups. When it comes to the profit, it was observed that the higher markup, bigger inter-hub discount  $\alpha$ , and a smaller

value of sensitivity parameter  $\Theta$  lead to higher objective value. There is a strong tendency for competitors to share hub locations.

In the case of robust variant, there was a lot of entry deterrence type solutions. One reason behind this could be the instance size, not big enough markup, and height of the level of uncertainty sources. The alternating heuristic has provided a few quite interesting solutions in which it is not profitable for the leader to enter the market. Also, differently to the conclusions for the non-competitive robust hub location problems, with the cost minimization objective, the size of hub backbone in the robust solution is usually smaller than the size of the deterministic counterpart. On the other hand, the similarity between the leader's hub locations in both variants is smaller than the similarity of the leader's and follower's hub locations in the deterministic case. This finding suggests that robust solutions are qualitatively different than the deterministic ones. The effect of markup on the net profit margin was more perceptible, compared to the effect on the objective value. In a way, this observation justified the introduction of net profit margin into the analysis of computational investigation.

One obvious line of research is the construction of exact solution approach. This way, we could obtain better insight and incidentally we would be able to estimate the quality of different heuristics. Also, promising areas of research are the investigation of polynomial and approximation hierarchies, as well as the consideration of some other ways to address the uncertainty, like fuzzy sets.

## References

- [1] S. Hsieh, S. Kao, A Survey of Hub Location Problems, *Journal of Interconnection Networks* (2019) doi:10.1142/S021926591940005X
- [2] I. Contreras: Hub Location Problems. In: Laporte, G., Nickel, S., Saldanha da Gama, F. (eds) *Location Science*. pp. 311–344. Springer, Cham, (2015)
- [3] R. Farahani, M. Hekmatfar, A. Arabani, E. Nikbaksh, Hub location problems: a review of models, classification, solution techniques, and applications, *Computers & Industrial Engineering* 64:4 (2013) 1096–1109 doi:10.1016/j.cie.2013.01.012
- [4] J. Campbell, M. O'Kelly, Twenty-five years of hub location research, *Transportation Science* 46:2 (2012) 153–169 doi:10.1287/trsc.1120.0410
- [5] Kara, B.Y., Taner, M.R.: Hub location problems: the location of interacting facilities. In: Eiselt, H.A., Marianov, V. (eds) *Foundations of Location Analysis*, 1st edn., pp. 273–288. Springer (2011) doi:10.1007/978-1-4419-7572-0\_12
- [6] M. Hekmatfar, M. Pishvae, Hub Location Problem, In: Zanjirani, R., Hekmatfar, M. (eds) *Facility Location*, pp. 243–270. Physica-Verlag HD, Heidelberg (2009) doi:10.1007/978-3-7908-2151-2\_11
- [7] S. Alumur, B. Kara, Network hub location problems: the state of the art, *European journal of Operational Research* 190(1) (2008) 1–21
- [8] D.D. Čvokić, Z. Stanimirovic, A single allocation hub location and pricing problem, *Comp. Appl. Math.* 39 (2020) doi:10.1007/s40314-019-1025-z
- [9] H. Eiselt, V. Marianov, A conditional p-hub location problem with attraction functions, *Computers & Operations Research* 36:12 (2009) 3128–3135 doi: 10.1016/j.cor.2008.11.014
- [10] S. Elhedhli, H. Wu, A Lagrangian heuristic for hub-and-spoke system design with capacity selection and congestion, *Informatics Journal on Computing* 22 (2010) 282–296
- [11] A. Luer-Villagra, V. Marianov, A competitive hub location and pricing problem, *European Journal of Operational Research* 231:3 (2013) 734–744
- [12] A. Lambrecht, K. Seim, N. Vilcassim, A. Cheema, Y. Chen, G. Crawford, K. Hosanagar, I. Raghuram, O. Koenigsberg, R. Lee, E. Miravete, O. Sahin, Price discrimination in service industries, *Mark Lett* 23 (2012) 423–438 doi: 10.1007/s11002-012-9187-0
- [13] A. Panin, M. Pashchenko, A. Plyasunov, Bilevel competitive facility location and pricing problems. *Automation and Remote Control*, 75:4 (2014) 715–727 doi:10.1134/S0005117914040110
- [14] Y. Kochetov, I. Sokolova, S. Amirgaliyeva, Z. Amirgaliyeva, Alternating heuristic and exact method for the leader-follower facility location and design problem, 2015 Twelve International Conference on Electronics Computer and Computation (ICECCO), IEEE, (2015) 1–3 doi:10.1109/ICECCO.2015.7416870
- [15] A.A. Kononov, A.A. Panin, V.A. Plyasunov, A bilevel competitive location and pricing model with non-uniform split of the demand, *J. Appl. Ind. Math.* 13, 519 (2019)
- [16] J. Grammig, R. Hujer, M. Scheidler, Discrete Choice Modelling in Airline Network Management, *Journal of Applied Econometrics* 20 (2005) 467–486
- [17] J. Hertog, Review of Economic Theories of Regulation. Discussion Paper, (2010) 10–18, Tjalling C. Koopmans Research Institute, Utrecht University School of Economics, Utrecht University, Utrecht
- [18] D. Čvokić, Y. Kochetov, A. Plyasunov, A leader-follower hub location problem under fixed markups. Yu. Kochetov et al. (Eds.) 9th International Conference on Discrete Optimization and Operations Research, DOOR 2016. LNCS 9869 (2016) 350–363 doi:10.1007/978-3-319-44914-2\_28
- [19] A. Melnikov, Computational complexity of the discrete competitive facility location problem. *Journal of Applied and Industrial Mathematics* 8:4 (2014) 557–567

- [20] S. Iellamo, E. Alekseeva, L. Chen, M. Coupechoux, Yu. Kochetov, Competitive location in cognitive radio networks, *4OR* 13:1 (2015) 81–110.
- [21] M. Sasaki, M. Fukushima, Stackelberg hub location problem, *Journal of the Operations Research Society of Japan* 44:4 (2001) 390–405.
- [22] I. Mahmutogullari, B. Kara, Hub location under competition, *European Journal of Operational Research* 250:1 (2016) 214–225.
- [23] A.C.A. de Araujo, M.C. Roboredo, A.A. Pessoa, V. Pereira, Exact Methods for the discrete multiple allocation (*rlp*) hub-centroid problem, *Computers & Operations Research* 116 (2020) doi:10.1016/j.cor.2019.104870
- [24] D.D. Čvokić, Y.A. Kochetov, A.V. Plyasunov, The competitive hub location under the price war, In: Khachay M., Kochetov Y., Pardalos P. (eds) *Mathematical Optimization Theory and Operations Research. MOTOR 2019. Lecture Notes in Computer Science*, vol 11548, pp. 133–146. Springer, Cham (2019)
- [25] K. Train, *Discrete Choice Methods with Simulation*, (2nd Edition), Cambridge University Press, Cambridge, 2012.
- [26] N.V. Sahinidis, Optimization under uncertainty: state-of-the-art and opportunities, *Computers & Chemical Engineering*, 28:6–7 (2004) 971–983 doi:10.1016/j.compchemeng.2003.09.017
- [27] X. Li, Operations Management of Logistics and Supply Chain: Issues and Directions, *Discrete Dynamics in Nature and Society*, 2014 (2014) doi:10.1155/2014/701938
- [28] K. Govindan, M. Fattahi, E. Keyvanshokoohc, Supply chain network design under uncertainty: A comprehensive review and future research directions, *European Journal of Operational Research*, 263:1 (2017) 108–141 doi:10.1016/j.ejor.2017.04.009
- [29] D. Drakulić, A. Takači, M. Marić, Fuzzy Covering Location Problems with Different Aggregation Operators, *Filomat* 31:2 (2017) 513–522 doi:10.2298/FIL1702513D
- [30] T. Sim, T. Lowe, B. Thomas The stochastic p-hub center problem with service-level constraints. *Computer & Operations Research* 36:12 (2009) 3166–3177
- [31] T. Yang, Stochastic air freight hub location and flight routes planning. *Appl. Math. Model.* 33:12 (2009) 4424–4430
- [32] I. Contreras, J. Cordeau, G. Laporte, Stochastic uncapacitated hub location. *European Journal of Operational Research* 212 (2011) 518–528
- [33] S. Alumur, S. Nickel, F. Saldanha-de-Gama Hub location under uncertainty. *Transport. Res. B* 46:4 (2012) 529–543
- [34] X. Chen, M. Sim, P. Sun A robust optimization perspective on stochastic programming. *Operat. Res.* 55:6 (2007) 1058–1071
- [35] M. Wagner, J. Bhadury, S. Peng Risk management in uncapacitated facility location models with random demands. *Computers & Operations Research*, 36:14 (2009) 1002–1011
- [36] M. Shahabi, A. Unnikrishnan, Robust hub network design problem, *Transportation Research Part E* 70 (2014) 356–373 doi:10.1016/j.tre.2014.08.003
- [37] S.J. Jeong, C.G. Lee, J.H. Bookbinder, The European freight railway system as a hub-and-spoke network, *Transport. Res. A* 41:6 (2007) 523–536
- [38] N. Gülpınar, D. Pachamanova, E. Canakoglu, Robust strategies for facility location under uncertainty. *Eur. J. Operat. Res.* 225:1 (2013) 21–35
- [39] J. Bhadury, H.A. Eiselt, J.H. Jaramillo, An Alternating Heuristic for Medianoid and Centroid Problems in the Plane, *Comput. Oper. Res.* 30 (2003) 553–565
- [40] H. Hotelling, Stability in Competition, *The Economic Journal* 39:153 (1929) 41–57 doi:10.2307/2224214
- [41] J. d’Aspremont, J.J. Gabszewicz, J.F. Thisse, On Hotelling’s stability in competition, *Econometrica* 49 (1979) 1145–50.
- [42] H. Calik, S.A. Alumur, B.Y. Kara, O.E. Karasan A tabu-search based heuristic for the hub covering problem over incomplete hub networks, *Computers & Operations Research*, 36:12 (2009) 3088–3096
- [43] M.E. O’Kelly, A quadratic integer program for the location of interacting hub facilities, *EJOR*, 32 (1987) 393–404.
- [44] M. Droettboom, T. Caswell, J. Hunter, E. Firing, J. Nielsen, N. Varoquaux, B. Root, E. de Andrade, P. Elson, D. Dale, J. Lee, A. Lee, J. Seppänen, R. May, D. Stansby, D. McDougall, A. Straw, P. Hobson, T. Yu, E. Ma, Ch. Gohlke, S. Silvester, Ch. Moad, A. Vincent, J. Schulz, P. Würtz, F. Ariza, Cimarron, T. Hirsch, N. Kniazev, (2017) *matplotlib/matplotlib: REL: v2.1.0* doi: 10.5281/zenodo.1004650