Fractional Integral Inequalities of Hermite-Hadamard Type for Convex Functions With Respect to a Monotone Function

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Abstract. In the literature, the left-side of Hermite–Hadamard's inequality is called a midpoint type inequality. In this article, we obtain new integral inequalities of midpoint type for Riemann–Liouville fractional integrals of convex functions with respect to increasing functions. The resulting inequalities generalize some recent integral inequalities and Riemann–Liouville fractional integral inequalities established in earlier works. Finally, applications of our work are demonstrated via the known special functions.

1. Introduction

A function $g : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval $\mathcal{I}$, if the inequality

$$g(\eta x + (1 - \eta)y) \le \eta g(x) + (1 - \eta)g(y) \tag{1}$$

holds for all $x, y \in \mathcal{I}$ and $\eta \in [0, 1]$. We say that g is concave, provided $-g$ is convex.

For convex functions (1), many equalities and inequalities have been established, *e.g.*, Ostrowski type inequality [1], Opial inequality [2], Hardy type inequality [3], Olsen type inequality [4], Gagliardo-Nirenberg type inequality [5], midpoint and trapezoidal type inequalities [6, 7] and the Hermite–Hadamard type (HH-type) inequality [8] that will be used in our study, which is defined by:

$$g\left(\frac{u + v}{2}\right) \le \frac{1}{v - u} \int_u^v g(x)dx \le \frac{g(u) + g(v)}{2}, \tag{2}$$

where $g : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is assumed to be a convex function on $\mathcal{I}$ where $u, v \in \mathcal{I}$ with $u < v$.

A huge number of researchers in the field of applied and pure mathematics have devoted their efforts to modify, generalize, refine, and extend the Hermite–Hadamard inequality (2) for convex and other classes of convex functions; see for further details [8–12].

In 2013, the HH-type inequality (2) has been generalised to fractional integrals of Riemann–Liouville type by Sarikaya et al [13]. Their result is as follows, for an L^1 convex function $fg : \mathcal{I} \to \mathbb{R}$, and for any $\mu > 0$:

$$g\left(\frac{u + v}{2}\right) \le \frac{\Gamma(\mu + 1)}{2(v - u)^\mu}\left[I_{u^+}^\mu g(v) + I_{v^-}^\mu g(u)\right] \le \frac{g(u) + g(v)}{2}, \tag{3}$$

where $I_{u^+}^{\mu}$ and $I_{v^-}^{\mu}$ denote left-sided and right-sided Riemann-Liouville fractional integrals of order $\mu > 0$, respectively, defined as [14]:

$$
\begin{aligned}
I_{u^+}^{\mu}g(x) &= \frac{1}{\Gamma(\mu)} \int_u^x (x-t)^{\mu-1}g(t)dt, \quad x > u, \\
I_{v^-}^{\mu}g(x) &= \frac{1}{\Gamma(\mu)} \int_x^v (t-x)^{\mu-1}g(t)dt, \quad x < v.
\end{aligned}
\tag{4}
$$

If we take $\mu = 1$ in (3) we obtain (2), it is clear that inequality (3) is a generalization of Hermite–Hadamard inequality (2). Many important results have been derived from the Riemann–Liouville fractional operators, including in different types of fractional calculus, e.g. tempered fractional integrals [15], those of Hilfer type [16], those models of fractional calculus involving Mittag-Leffler kernels [17], fractional integrals with respect to functions [18], and many others can be found in [19–27]. But so far such inequalities have not been investigated for fractional integrals of a twice differentiable convex function with respect to a monotone function. For this reason, we recall the Riemann–Liouville fractional integrals of a function with respect to a monotone function.

Definition 1.1 ([14, 28, 29]). *Let g be an L^1 function, $\mu > 0$, and let $I \subseteq (-\infty, \infty)$ be a finite or infinite interval of real numbers $\mathbb{R}$ such that $u, v \in I$. Let ψ be an increasing and positive function on the interval I such that $\psi' \in C^1(I)$ with $\psi'(x) \neq 0$ for all $x \in I$. Then, the left and right-sided ψ-Riemann–Liouville fractional integrals of order μ of a function g with respect to ψ on I are defined by:*

$$
\begin{aligned}
I_{u^+}^{\mu;\psi}g(x) &= \frac{1}{\Gamma(\mu)} \int_u^x \psi'(t)(\psi(x) - \psi(t))^{\mu-1}g(t)dt, \\
I_{v^-}^{\mu;\psi}g(x) &= \frac{1}{\Gamma(\mu)} \int_x^v \psi'(t)(\psi(t) - \psi(x))^{\mu-1}g(t)dt.
\end{aligned}
\tag{5}
$$

It is important to note that if we set $\psi(x) = x$ in (5), then ψ-Riemann–Liouville fractional integral reduces to Riemann–Liouville fractional integral (4).

As we said, in this study we investigate several inequalities of midpoint type for Riemann–Liouville fractional integrals of twice differentiable convex functions with respect to increasing functions.

2. Main Results

Throughout this article, we assume that u, v and $\frac{u+v}{2}$ belong to the image of ψ. Now, we state our main lemma:

Lemma 2.1. *Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I such that $u, v \in I$ with $0 \leq u < v$. If ψ is an increasing and positive function on I and its derivative $\psi'(x)$ is continuous on I. Then, for any $\mu \in (0, 1)$, we have*

$$
\sigma_{\mu,\psi}(g; u, v) = \frac{2^{\mu-1}}{(v-u)^{\mu}} \left[\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)} \psi'(t)(v - \psi(t))^{\mu+1}(g'' \circ \psi)(t)dt \right.
$$
$$
\left. + \int_{\psi^{-1}(u)}^{\psi^{-1}\left(\frac{u+v}{2}\right)} \psi'(t)(\psi(t) - u)^{\mu+1}(g'' \circ \psi)(t)dt \right], \quad (6)
$$

where

$$
\sigma_{\mu,\psi}(g; u, v) = \frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^{\mu}} \left[I_{\psi^{-1}\left(\frac{u+v}{2}\right)^+}^{\mu;\psi}(g \circ \psi)\left(\psi^{-1}(v)\right) + I_{\psi^{-1}\left(\frac{u+v}{2}\right)^-}^{\mu;\psi}(g \circ \psi)\left(\psi^{-1}(u)\right) \right] - (\mu+1)g\left(\frac{u+v}{2}\right).
$$

Proof. From Definition 1.1, we have

$$\hbar_1 := \frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^\mu} I^{\mu:\psi}_{\psi^{-1}\left(\frac{u+v}{2}\right)^+}(g\circ\psi)\left(\psi^{-1}(v)\right) = \frac{\mu(\mu+1)2^{\mu-1}}{(v-u)^\mu}\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)}\psi'(t)(v-\psi(t))^{\mu-1}(g\circ\psi)(t)dt$$

$$= -\frac{(\mu+1)2^{\mu-1}}{(v-u)^\mu}\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)}(g\circ\psi)(t)d(v-\psi(t))^\mu.$$

Integrating by parts twice, we have

$$\hbar_1 = \frac{\mu+1}{2}g\left(\frac{u+v}{2}\right) + \frac{(\mu+1)2^{\mu-1}}{(v-u)^\mu}\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)}\psi'(t)(v-\psi(t))^\mu(g'\circ\psi)(t)dt$$

$$= \frac{\mu+1}{2}g\left(\frac{u+v}{2}\right) + \frac{1}{2}g'\left(\frac{u+v}{2}\right) + \frac{2^{\mu-1}}{(v-u)^\mu}\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)}\psi'(t)(v-\psi(t))^{\mu+1}(g''\circ\psi)(t)dt \qquad (7)$$

Analogously

$$\hbar_2 := \frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^\mu} I^{\mu:\psi}_{\psi^{-1}\left(\frac{u+v}{2}\right)^-}(g\circ\psi)\left(\psi^{-1}(u)\right)$$

$$= \frac{\mu+1}{2}g\left(\frac{u+v}{2}\right) - \frac{1}{2}g'\left(\frac{u+v}{2}\right) + \frac{2^{\mu-1}}{(v-u)^\mu}\int_{\psi^{-1}(u)}^{\psi^{-1}\left(\frac{u+v}{2}\right)}\psi'(t)(\psi(t)-u)^{\mu+1}(g''\circ\psi)(t)dt. \qquad (8)$$

It follows from (7) and (8) that

$$\hbar_1 + \hbar_2 - (\mu+1)g\left(\frac{u+v}{2}\right) = \frac{2^{\mu-1}}{(v-u)^\mu}\left[\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)}\psi'(t)(v-\psi(t))^{\mu+1}(g''\circ\psi)(t)dt\right.$$

$$\left. + \int_{\psi^{-1}(u)}^{\psi^{-1}\left(\frac{u+v}{2}\right)}\psi'(t)(\psi(t)-u)^{\mu+1}(g''\circ\psi)(t)dt\right].$$

This completes the proof of Lemma 2.1. □

Corollary 2.2. *With similar assumptions of Lemma 2.1 if*

1. $\psi(x) = x$, *we have*

$$\frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^\mu}\left[I^\mu_{\left(\frac{u+v}{2}\right)^+}g(v) + I^\mu_{\left(\frac{u+v}{2}\right)^-}g(u)\right] - (\mu+1)g\left(\frac{u+v}{2}\right)$$

$$= \frac{(v-u)^2}{8}\left[\int_0^1 t^{\mu+1}g''\left(\frac{t}{2}u + \frac{2-t}{2}v\right)dt + \int_0^1 t^{\mu+1}g''\left(\frac{2-t}{2}u + \frac{t}{2}v\right)dt\right],$$

   which is obtained by Tomar et al. [30].

2. $\psi(x) = x$ *and* $\mu = 1$, *we have*

$$\frac{1}{v-u}\int_u^v g(x)dx - g\left(\frac{u+v}{2}\right) = \frac{(v-u)^2}{16}\left[\int_0^1 t^2g''\left(\frac{t}{2}u + \frac{2-t}{2}v\right)dt + \int_0^1 t^2g''\left(\frac{2-t}{2}u + \frac{t}{2}v\right)dt\right],$$

   which is obtained by Sarikaya and Kiris [31].

Theorem 2.3. *Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I such that $u, v \in I$ with $0 \le u < v$. Suppose that $|g''|^q$ is convex on I, $\psi(x)$ is an increasing and positive function on I and $\psi'(x)$ is continuous on I. Then, for any $\mu \in (0, 1)$, we have*

$$\left| \sigma_{\mu,\psi}(g; u, v) \right| \le \frac{(v-u)^2}{8} \left(\frac{1}{\mu+2} \right)^{1-\frac{1}{q}} \left\{ \left[\frac{1}{2(\mu+3)} |g''(u)|^q + \left(\frac{1}{\mu+2} - \frac{1}{2(\mu+3)} \right) |g''(v)|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\left(\frac{1}{\mu+2} - \frac{1}{2(\mu+3)} \right) |g''(u)|^q + \frac{1}{2(\mu+3)} |g''(v)|^q \right]^{\frac{1}{q}} \right\}, \quad (9)$$

for $q \ge 1$.

Proof. Suppose that $q = 1$. By means of Lemma 2.1 and Definition 1.1, we get

$$\sigma_{\mu,\psi}(g; u, v) = \frac{2^{\mu-1}}{(v-u)^\mu} \left[\int_{\psi^{-1}\left(\frac{u+v}{2}\right)}^{\psi^{-1}(v)} \psi'(t_1)(v - \psi(t_1))^{\mu+1}(g'' \circ \psi)(t_1) dt_1 \right.$$
$$\left. + \int_{\psi^{-1}(u)}^{\psi^{-1}\left(\frac{u+v}{2}\right)} \psi'(t_2)(\psi(t_2) - u)^{\mu+1}(g'' \circ \psi)(t_2) dt_2 \right]. \quad (10)$$

Making the change of variables $x_1 = \frac{2(v-\psi(t_1))}{v-u}$ and $x_2 = \frac{2(\psi(t_2)-u)}{v-u}$ and then setting $t = x_1 = x_2$ into the resulting equality, we get

$$\sigma_{\mu,\psi}(g; u, v) = \frac{(v-u)^2}{8} \left[\int_0^1 t^{\mu+1} g'' \left(\frac{t}{2}u + \frac{2-t}{2}v \right) dt + \int_0^1 t^{\mu+1} g'' \left(\frac{2-t}{2}u + \frac{t}{2}v \right) dt \right],$$

that is

$$\left| \sigma_{\mu,\psi}(g; u, v) \right| \le \frac{(v-u)^2}{8} \left[\int_0^1 t^{\mu+1} \left(\left| g'' \left(\frac{t}{2}u + \frac{2-t}{2}v \right) \right| + \left| g'' \left(\frac{2-t}{2}u + \frac{t}{2}v \right) \right| \right) dt \right]. \quad (11)$$

By using the convexity of $|g''|$, then inequality (11) gives

$$\left| \sigma_{\mu,\psi}(g; u, v) \right| \le \frac{(v-u)^2}{8} \left[\int_0^1 t^{\mu+1} \left(\left| g'' \left(\frac{t}{2}u + \frac{2-t}{2}v \right) \right| + \left| g'' \left(\frac{2-t}{2}u + \frac{t}{2}v \right) \right| \right) dt \right]$$
$$\le \frac{(v-u)^2}{8} \left(|g''(u)| \int_0^1 \frac{1}{2} t^{\mu+2} dt + |g''(v)| \int_0^1 \frac{2-t}{2} t^{\mu+1} dt \right.$$
$$+ |g''(v)| \int_0^1 \frac{1}{2} t^{\mu+2} dt + |g''(u)| \int_0^1 \frac{2-t}{2} t^{\mu+1} dt \right)$$
$$= \frac{(v-u)^2}{8(\mu+2)} \left(|g''(u)| + |g''(v)| \right).$$

This gives (9) for $q = 1$.

Now, suppose that $q > 1$. Using inequality of (11), convexity of $|g''|^q$ and the power–mean's inequality

for $q > 1$, we have

$$\int_0^1 t^{\mu+1} \left| g'' \left(\frac{t}{2} u + \frac{2-t}{2} v \right) \right| dt = \int_0^1 t^{\mu+1-\frac{\mu+1}{q}} \left[t^{\frac{\mu+1}{q}} \left| g'' \left(\frac{t}{2} u + \frac{2-t}{2} v \right) \right| \right] dt$$

$$\leq \left(\int_0^1 t^{\mu+1} \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\mu+1} \left| g'' \left(\frac{t}{2} u + \frac{2-t}{2} v \right) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{\mu+2} \right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{t^{\mu+2}}{2} |g''(u)|^q + \frac{2t^{\mu+1} - t^{\mu+2}}{2} |g''(v)|^q \right) dt \right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{\mu+2} \right)^{1-\frac{1}{q}} \left[\frac{1}{2(\mu+3)} |g''(u)|^q + \left(\frac{1}{\mu+2} - \frac{1}{2(\mu+3)} \right) |g''(v)|^q \right]^{\frac{1}{q}}. \tag{12}$$

In the same manner, we get

$$\int_0^1 t^{\mu+1} \left| g'' \left(\frac{2-t}{2} u + \frac{t}{2} v \right) \right| dt \leq \left(\frac{1}{\mu+2} \right)^{1-\frac{1}{q}} \left[\left(\frac{1}{\mu+2} - \frac{1}{2(\mu+3)} \right) |g''(u)|^q + \frac{1}{2(\mu+3)} |g''(v)|^q \right]^{\frac{1}{q}}. \tag{13}$$

Using (12) and (13) in (11) we obtain (9) for $q > 1$. Thus the proof of theorem 2.3 is completed. □

Corollary 2.4. *With similar assumptions of Theorem 2.3 if*

1. $\psi(x) = x$, *we have*

$$\left| \frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^\mu} \left[I^\mu_{\left(\frac{u+v}{2} \right)^+} g(v) + I^\mu_{\left(\frac{u+v}{2} \right)^-} g(u) \right] - (\mu+1)g \left(\frac{u+v}{2} \right) \right|$$

$$\leq \frac{(v-u)^2}{8} \left(\frac{1}{\mu+2} \right)^{1-\frac{1}{q}} \left\{ \left[\frac{1}{2(\mu+3)} |g''(u)|^q + \left(\frac{1}{\mu+2} - \frac{1}{2(\mu+3)} \right) |g''(v)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\left(\frac{1}{\mu+2} - \frac{1}{2(\mu+3)} \right) |g''(u)|^q + \frac{1}{2(\mu+3)} |g''(v)|^q \right]^{\frac{1}{q}} \right\},$$

   which is obtained by Tomar et al. [30].

2. $\psi(x) = x$ *and* $\mu = 1$, *we have*

$$\left| \frac{1}{v-u} \int_u^v g(x)dx - g \left(\frac{u+v}{2} \right) \right| \leq \frac{(v-u)^2}{48} \left[\left(\frac{3|g''(u)|^q + 5|g''(v)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{5|g''(u)|^q + 3|g''(v)|^q}{8} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{(v-u)^2}{6 \cdot 8^{\frac{1}{q}}} \left(|g''(u)| + |g''(v)| \right),$$

   which is obtained by Sarikaya et al. [32].

3. $\psi(x) = x$ *and* $q = 1$, *we have*

$$\left| \frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^\mu} \left[I^\mu_{\left(\frac{u+v}{2} \right)^+} g(v) + I^\mu_{\left(\frac{u+v}{2} \right)^-} g(u) \right] - (\mu+1)g \left(\frac{u+v}{2} \right) \right| \leq \frac{(v-u)^2}{8(\mu+2)} \left(|g''(u)| + |g''(v)| \right),$$

   which is obtained by Tomar et al. [30].

4. $\psi(x) = x$, $\mu = 1$ *and* $q = 1$, *we have*

$$\left| \frac{1}{v-u} \int_u^v g(x)dx - g \left(\frac{u+v}{2} \right) \right| \leq \frac{(v-u)^2}{24} \left(\frac{|g''(u)| + |g''(v)|}{2} \right),$$

which is obtained by Sarikaya et al. [32].

Theorem 2.5. *Let* $g : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I *such that* $u, v \in I$ *with* $0 \le u < v$. *Suppose that* $|g''|^q$ *is convex on* I, $\psi(x)$ *is an increasing and positive function on* I *and* $\psi'(x)$ *is continuous on* I. *Then, for any* $\mu \in (0, 1)$, *we have*

$$\left|\sigma_{\mu,\psi}(g; u, v)\right| \le \frac{(v-u)^2}{8}\left(\frac{1}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\le \frac{(v-u)^2}{8}\left(\frac{4}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left(|g''(u)| + |g''(v)|\right), \quad (14)$$

such that $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Holder's inequality, we have

$$\int_0^1 t^{\mu+1}\left|g''\left(\frac{t}{2}u + \frac{2-t}{2}v\right)\right| dt \le \left(\int_0^1 t^{(\mu+1)p}\right)^{\frac{1}{p}}\left(\int_0^1\left|g''\left(\frac{t}{2}u + \frac{2-t}{2}v\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le \left(\frac{1}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{t}{2}|g''(u)|^q + \frac{2-t}{2}|g''(v)|^q\right)dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4}\right)^{\frac{1}{q}}. \quad (15)$$

Similarly, we have

$$\int_0^1 t^{\mu+1}\left|g''\left(\frac{2-t}{2}u + \frac{t}{2}v\right)\right| dt \le \left(\frac{1}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left(\frac{3|g''(u)|^q + |g''(v)|^q}{4}\right)^{\frac{1}{q}}. \quad (16)$$

Thus, the inequalities (11), (15) and (16) complete the proof of the first inequality of (14).

To prove the second inequality of (14), we apply the formula

$$\sum_{i=1}^n (c_i + d_i)^m \le \sum_{i=1}^n c_i^m + \sum_{i=1}^n +d_i^m, \quad 0 \le m < 1$$

for $c_1 = 3|g''(u)|^q, c_2 = |g''(u)|^q, d_1 = |g''(v)|^q, d_2 = 3|g''(v)|^q$ and $m = \frac{1}{q}$. Then, inequality (11) gives

$$\left|\sigma_{\mu,\psi}(g; u, v)\right| \le \frac{(v-u)^2}{8}\left(\frac{1}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\le \frac{(v-u)^2\left(3^{\frac{1}{q}}+1\right)}{32}\left(\frac{4}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left[|g''(u)| + |g''(v)|\right]$$

$$\le \frac{(v-u)^2}{8}\left(\frac{4}{(\mu+1)p+1}\right)^{\frac{1}{p}}\left(|g''(u)| + |g''(v)|\right).$$

Hence the proof of Theorem 2.5 is completed. $\square$

Corollary 2.6. *With the assumptions of Theorem 2.5, if*

1. $\psi(x) = x$, we have

$$\left| \frac{2^{\mu-1}\Gamma(\mu+2)}{(v-u)^{\mu}} \left[I^{\mu}_{\left(\frac{u+v}{2}\right)^+} g(v) + I^{\mu}_{\left(\frac{u+v}{2}\right)^-} g(u) \right] - (\mu+1)g\left(\frac{u+v}{2}\right) \right|$$

$$\leq \frac{(v-u)^2}{8} \left(\frac{1}{(\mu+1)p+1} \right)^{\frac{1}{p}} \left[\left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{(v-u)^2}{8} \left(\frac{4}{(\mu+1)p+1} \right)^{\frac{1}{p}} \left(|g''(u)| + |g''(v)| \right),$$

which is obtained by Tomar et al. [30].

2. $\psi(x) = x$ and $\mu = 1$, we have

$$\left| \frac{1}{v-u} \int_u^v g(x)dx - g\left(\frac{u+v}{2}\right) \right| \leq \frac{(v-u)^2}{16(2p+1)^{\frac{1}{p}}} \left[\left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{(v-u)^2}{2^{1+\frac{2}{q}}(2p+1)^{\frac{1}{p}}} \left(|g''(u)| + |g''(v)| \right),$$

which is obtained by Sarikaya et al. [32].

Corollary 2.7. *From Corollaries 2.4 (2) and 2.6 (2), we obtain the following inequality for $q > 1$:*

$$\left| \frac{1}{v-u} \int_u^v g(x)dx - g\left(\frac{u+v}{2}\right) \right| \leq (v-u)^2 \min\{\delta_1, \delta_2\}\left(|g''(u)| + |g''(v)| \right),$$

where $\delta_1 = \frac{1}{6 \cdot 8^{\frac{1}{q}}}$ and $\delta_2 = \frac{1}{2^{1+\frac{2}{q}}(2p+1)^{\frac{1}{p}}}$ such that $p = \frac{q}{q-1}$.

3. Applications

In this section some applications are presented to demonstrate usefulness of our obtained results in the previous sections.

3.1. Applications to special means

Let u and v be two arbitrary positive real numbers, then consider the following special means:

(i) The arithmetic mean:

$$A = A(u, v) = \frac{u+v}{2}.$$

(ii) The inverse arithmetic mean:

$$H = H(u, v) = \frac{2}{\frac{1}{u} + \frac{1}{v}}, \quad u, v \neq 0.$$

(iii) The geometric mean:

$$G = G(u, v) = \sqrt{uv}.$$

(iv) The logarithmic mean:

$$L(u, v) = \frac{v - u}{\log(v) - \log(u)}, \quad u \neq v.$$

(v) The generalized logarithmic mean:

$$L_n(u, v) = \left[\frac{v^{n+1} - u^{n+1}}{(v - u)(n + 1)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}.$$

Proposition 3.1. *Let $|n| \geq 3$ and $u, v \in I \subseteq \mathbb{R}$ with $0 < u < v$, then we have*

$$\left| A^n(u, v) - L_n^n(u, v) \right| \leq \frac{(v - u)^2 |n(n - 1)|}{3 \cdot 4^{\frac{1}{q} + 2}} \left[A^{\frac{1}{q}} \left(3|u|^{(n-2)q}, 5|v|^{(n-2)q} \right) + A^{\frac{1}{q}} \left(5|u|^{(n-2)q}, 3|v|^{(n-2)q} \right) \right], \tag{17}$$

for $q \geq 1$.

Proof. Apply Corollary 2.4 part (2) for $g(x) = x^n$, where n as specified above. $\square$

Proposition 3.2. *Let $u, v \in I \subseteq \mathbb{R}$ with $0 < u < v$, then we have*

$$\left| A^{-1}(u, v) - L^{-1}(u, v) \right| \leq \frac{(v - u)^2}{3 \cdot 4^{\frac{1}{q} + 2}} \left[A^{\frac{1}{q}} \left(3|u|^{-3q}, 5|v|^{-3q} \right) + A^{\frac{1}{q}} \left(5|u|^{-3q}, 3|v|^{-3q} \right) \right], \tag{18}$$

for $q \geq 1$.

Proof. Apply Corollary 2.4 part (2) for $g(x) = \frac{1}{x}, x \neq 0$. $\square$

Proposition 3.3. *Let $|n| \geq 3$ and $u, v \in I \subseteq \mathbb{R}$ with $0 < u < v$, then we have*

$$\left| H^{-n}(v, u) - L_n^n \left(v^{-1}, u^{-1} \right) \right|$$
$$\leq \frac{\left(v^{-1} - u^{-1} \right)^2 |n(n - 1)|}{3 \cdot 4^{\frac{1}{q} + 2}} \left[H^{\frac{-1}{q}} \left(\frac{1}{3}|u|^{(n-2)q}, \frac{1}{5}|v|^{(n-2)q} \right) + H^{\frac{-1}{q}} \left(\frac{1}{5}|u|^{(n-2)q}, \frac{1}{3}|v|^{(n-2)q} \right) \right], \tag{19}$$

and

$$\left| H(v, u) - L^{-1} \left(v^{-1}, u^{-1} \right) \right| \leq \frac{\left(v^{-1} - u^{-1} \right)^2}{3 \cdot 4^{\frac{1}{q} + 2}} \left[H^{\frac{-1}{q}} \left(\frac{1}{3}|u|^{-3q}, \frac{1}{5}|v|^{-3q} \right) + H^{\frac{-1}{q}} \left(\frac{1}{5}|u|^{-3q}, \frac{1}{3}|v|^{-3q} \right) \right], \tag{20}$$

for $q \geq 1$.

Proof. Observe that $A^{-1} \left(u^{-1}, v^{-1} \right) = H(v, u) = \frac{2}{\frac{1}{u} + \frac{1}{v}}$. So, making the change of variables $u \to v^{-1}$ and $v \to u^{-1}$ in the inequalities (17) and (18), we can deduce the desired inequalities (19) and (20), respectively. $\square$

Proposition 3.4. *Let $u, v \in I \subseteq \mathbb{R}$ with $0 < u < v$, then we have*

$$\left| G^{-2}(u, v) - A^{-2}(u, v) \right| \leq \frac{(b - a)^2}{2 \cdot 4^{\frac{1}{q} + 1}} \left[A^{\frac{1}{q}} \left(3|u|^{-4q}, 5|v|^{-4q} \right) + A^{\frac{1}{q}} \left(5|u|^{-4q}, 3|v|^{-4q} \right) \right], \tag{21}$$

for $q \geq 1$.

Proof. Apply Corollary 2.4 part (2) for $g(x) = x^{-2}$. $\square$

Now, we give an application to a midpoint formula. Let d be a partition $u = x_0 < x_1 < \cdots < x_{m-1} < x_m = v$ of the interval $[u, v]$ and consider the quadrature formula

$$\int_u^v g(x)dx = T(g, d) + E(g, d),$$

where

$$T(g, d) = \sum_{j=0}^{m-1} g\left(\frac{x_j + x_{j+1}}{2}\right)(x_{j+1} - x_j)$$

is the midpoint version and $E(g, d)$ denotes the associated approximation error. Here, we present some error estimates for the midpoint formula.

Proposition 3.5. *Let $g : \mathcal{I} \to \mathbb{R}$ be a twice differentiable mapping on $\mathcal{I}$ such that $u, v \in \mathcal{I}$ with $u < v$. Suppose that $|g''|^q, q \geq 1$ be a convex function, then for every partition of $\mathcal{I}$ the midpoint error satisfies*

$$\left|E(g, d)\right| \leq \min\{\delta_1, \delta_2\} \sum_{j=0}^{m-1} (x_{j+1} - x_j)^2 \left(|g''(x_j)| + |g''(x_{j+1})|\right). \tag{22}$$

Proof. From Corollary 2.7, we have

$$\left|\int_{x_j}^{x_{j+1}} g(x)dx - (x_{j+1} - x_j)g\left(\frac{x_j + x_{j+1}}{2}\right)\right| \leq \min\{\delta_1, \delta_2\}\, (x_{j+1} - x_j)^2\left(|g''(x_j)| + |g''(x_{j+1})|\right)$$

Summing over j from 0 to $m-1$ and taking into account that $|g''|$ is convex, we obtain, by the triangle inequality, that

$$\left|\int_a^b g(x)dx - T(g, d)\right| = \left|\sum_{j=0}^{m-1}\left[\int_{x_j}^{x_{j+1}} g(x)dx - (x_{j+1} - x_j)g\left(\frac{x_j + x_{j+1}}{2}\right)\right]\right|$$

$$\leq \sum_{j=0}^{m-1}\left|\int_{x_j}^{x_{j+1}} g(x)dx - (x_{j+1} - x_j)g\left(\frac{x_j + x_{j+1}}{2}\right)\right|$$

$$\leq \min\{\delta_1, \delta_2\} \sum_{j=0}^{m-1} (x_{j+1} - x_j)^2\left(|g''(x_j)| + |g''(x_{j+1})|\right).$$

This ends the proof. $\square$

3.2. Modified Bessel functions

Let the function $\mathrm{I}_p : \mathbb{R} \to [1, \infty)$ be defined by

$$\mathrm{I}_p(x) = 2^p \Gamma(p + 1)x^{-v}\mathbf{I}_p(x), \quad x \in \mathbb{R}.$$

For this we recall the modified Bessel function of the first kind $\mathbf{I}_p$ which is defined as [33]:

$$\mathbf{I}_p(x) = \sum_{n \geq 0} \frac{\left(\frac{x}{2}\right)^{p+2n}}{n!\Gamma(p + n + 1)}.$$

The first and the nth order derivative formula of $I_p(x)$ are, respectively, given by [34]:

$$I'_p(x) = \frac{x}{2(p+1)} I_{p+1}(x), \tag{23}$$

$$\frac{\partial^n I_p(x)}{\partial x^n} = 2^{n-2p} \sqrt{\pi} x^{p-n} \Gamma(p+1)\, _2F_3\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{p+1-n}{2}, \frac{p+2-n}{2}, p+1; \frac{x^2}{4}\right), \tag{24}$$

where $_2F_3(\cdot,\cdot;\cdot,\cdot,\cdot;\cdot)$ is the hypergeometric function defined by [34]:

$$_2F_3\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{p+1-n}{2}, \frac{p+2-n}{2}, p+1; \frac{x^2}{4}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{p+1}{2}\right)_k \left(\frac{p+2}{2}\right)_k}{\left(\frac{p-2}{2}\right)_k \left(\frac{p-1}{2}\right)_k (p+1)_k} \frac{x^{2k}}{4^k\,(k)!}, \tag{25}$$

where, for some parameter ν, the Pochhammer symbol $(\nu)_k$ is defined as

$$(\nu)_0 = 1, \quad (\nu)_k = \nu(\nu+1)\cdots(\nu+k-1), \quad k = 1,2,\dots$$

Proposition 3.6. *Let $u, v \in \mathbb{R}$ with $0 < u < v$, then for each $p > -1$ we have*

$$\left| \frac{I_p(v) - I_p(u)}{v-u} - \frac{u+v}{4(p+1)} I_{p+1}\left(\frac{u+v}{2}\right) \right| \le (v-u)^2 \min\{\delta_1, \delta_2\}\, 2^{3-2p} \sqrt{\pi}\, \Gamma(p+1)$$

$$\times \left(|u|^{p-3} \left| _2F_3\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{p+1-n}{2}, \frac{p+2-n}{2}, p+1; \frac{u^2}{4}\right) \right| \right.$$

$$\left. + |v|^{p-3} \left| _2F_3\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{p+1-n}{2}, \frac{p+2-n}{2}, p+1; \frac{v^2}{4}\right) \right| \right). \tag{26}$$

Proof. Let $g(x) = I'_p(x)$. Note that the function $x \mapsto I'''_p(x)$ is convex on the interval $[0, \infty)$ for each $p > -1$. Using Corollary 2.7 and (23)–(24), we obtain the desired inequality (26) immediately. $\square$

Remark 3.7. *Assuming $p > -1$ in Proposition 3.6 is due to $\frac{1}{(0)_k}$ is undefined for each natural number k.*

4. Conclusion

In this paper, we established some new integral inequalities of midpoint type for convex functions with respect to increasing functions involving Riemann–Liouville fractional integrals. It can be observed from Corollaries 2.2, 2.4, 2.6 that our results are generalizations of those in [30–32].

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