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On the Diameter of Compressed Zero-Divisor Graphs of Ore Extensions

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Abstract. This paper continues the ongoing effort to study the compressed zero-divisor graph over non-commutative rings. The purpose of our paper is to study the diameter of the compressed zero-divisor graph of Ore extensions and give a complete characterization of the possible diameters of $\Gamma_E\big(R[x;\alpha,\delta]\big)$, where the base ring R is reversible and also have the (α,δ)-compatible property. Also, we give a complete characterization of the diameter of $\Gamma_E\big(R[[x;\alpha]]\big)$, where R is a reversible, α-compatible and right Noetherian ring. By some examples, we show that all of the assumptions "reversiblity", "(α,δ)-compatiblity" and "Noetherian" in our main results are crucial.

1. Introduction

In this paper, the term "ring" (unless explicitly stated otherwise) means "associative ring with nonzero identity". We denote the set of all left zero divisors of R, the set of all right zero-divisors of R and the set $Z_l(R) \cup Z_r(R)$ by $Z_l(R)$, $Z_r(R)$ and $Z(R)$, respectively. For a nonzero element a of R, $l_R(a)$ and $r_R(a)$ denote the left annihilator and the right annihilator of a in R, respectively.

Let R be a ring, α an endomorphism of R and δ an α-derivation of R (so δ is an additive map satisfying $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$), the general (left) Ore extension $R[x;\alpha,\delta]$ is the ring of polynomials over R in the variable x, with term-wise addition and with coefficients written on the left of x, subject to the skew-multiplication rule $xr = \alpha(r)x+\delta(r)$ for $r \in R$. If α is an identity map on R or $\delta = 0$, then we denote $R[x;\alpha,\delta]$ by $R[x;\delta]$ and $R[x;\alpha]$, respectively. In [19], a ring R is called *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is said to be *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, we say that R is *(α,δ)-compatible*.

Following [13], a ring is *reversible* if $ab = 0$ implies that $ba = 0$ for each $a, b \in R$. Obviously, *reduced* rings (i.e. rings with no nonzero nilpotent elements) and commutative rings are reversible. In [27], Kim and Lee studied extensions of reversible rings and showed that polynomial rings over reversible rings need not to be reversible in general. Note that if R is a reversible ring, then $Z_l(R) = Z_r(R) = Z(R)$. Also, if R is a reversible ring and $a \in R$, then $l_R(a) = r_R(a)$ is an ideal of R. According to [27], a ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$.

For a graph G, $V(G)$ denotes the set of vertices of graph G. All the graphs considered in this article are undirected and connected. Recall that a graph is said to be *connected* if for each pair of distinct vertices u

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and v, there is a finite sequence of distinct vertices $v_1 = u, v_2, \ldots, v_n = v$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a *path* and for two distinct vertices a and b in the simple (undirected) graph Γ, the *distance* between a and b, denoted by $d(a, b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise we put $d(a, b) = \infty$. Recall that the *diameter* of a connected graph is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is *complete*; i.e., each pair of distinct vertices forms an edge.

The study of zero-divisor graphs was initiated by Istvan Beck [12], in 1988. He let all elements of R be vertices of the graph with vertices a and b joined by an edge when $ab = 0$ and was mainly interested in coloring. In 1999, Anderson and Livingston [7], redefined and studied the (undirected) zero-divisor graph $\Gamma(R)$, whose vertices are the nonzero zero-divisors of a ring such that distinct vertices a and b are adjacent if and only if $ab = 0$. Afterward, Redmond [34], defined a directed zero-divisor graph for non-commutative ring in a similar way. A directed graph is connected if there exists a directed path connecting any two distinct vertices. The distance and the diameter are defined in a similar way as well, having in mind that all paths in question are directed. Redmond, also defined an undirected zero-divisor graph of a non-commutative ring R, the graph $\Gamma(R)$, with vertices in the set $Z(R)^* = Z(R) \setminus \{0\}$ and such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$ or $ba = 0$. We will be concerned with this type of undirected zero-divisor graph of non-commutative rings. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings (cf. [1, 7, 25, 30–32, 34]).

As suggested by the vast literature, there is a considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The first such extensions that come to mind are those of polynomial and power series extensions. Axtell, Coykendall and Stickles [10], examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings. Also, Lucas [30], continued the study of the diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Moreover, Anderson and Mulay [8], studied the girth and diameter of zero-divisor graph of a commutative ring and investigated the girth and diameter of zero-divisor graphs of polynomial and power series rings over commutative rings.

For any elements a and b of R, define $a \sim b$ if and only if $ann_R(a) = ann_R(b)$, where $ann_R(a) = l_R(a) \cup r_R(a)$. Simply observed that $\sim$ is an equivalence relation on R. For any $a \in R$, let $[a]_R = \{b \in R \mid a \sim b\}$. For example, it is clear that $[0]_R = \{0\}$ and $[1]_R = R \setminus Z(R)$, and that $[a]_R \subseteq Z(R) \setminus \{0\}$ for every $a \in R \setminus \big([0]_R \cup [1]_R\big)$.

The graph $\Gamma_E(R)$ is a condensed version of $\Gamma(R)$, constructed in such a way as to reduce the "noise" produced by individual zero divisors (In [3], this is called the "compressed" zero-divisor graph). Accordingly, $\Gamma_E(R)$ is smaller and simpler than $\Gamma(R)$. *The compressed zero-divisor graph* $\Gamma_E(R)$ is the (undirected) graph whose vertices are the elements of $R_E \setminus \{[0]_R, [1]_R\}$ such that distinct vertices $[a]_R$ and $[b]_R$ are adjacent if and only if $ab = 0$ or $ba = 0$. Note that if a and b are distinct adjacent vertices in $\Gamma(R)$, then $[a]_R$ and $[b]_R$ are adjacent in $\Gamma_E(R)$ if and only if $[a]_R \neq [b]_R$. Clearly, $\operatorname{diam}\big(\Gamma_E(R)\big) \leq \operatorname{diam}\big(\Gamma(R)\big)$. Spiroff and Wickham [35], showed that $\Gamma_E(R)$ is connected with $\operatorname{diam}\big(\Gamma_E(R)\big) \leq 3$. They also studied relation between the associated primes of R and the vertices of $\Gamma_E(R)$, where R is a Noetherian ring. Anderson and LaGrange [6], determined the structure of $\Gamma_E(R)$ when it is a cyclic and the monoids R_E when $\Gamma_E(R)$ is a star graph.

In [15], the authors studied the diameter of $\Gamma_E(R)$ and gave a complete characterization for $\Gamma_E(R)$, where R is a commutative ring. They also characterized $\operatorname{diam}\big(\Gamma_E(R[x])\big)$ and $\operatorname{diam}\big(\Gamma_E(R[[x]])\big)$, where the base coefficient ring R is a commutative ring.

In line with [15], recently the authors extended the study of the diameter of the compressed zero-divisor graph of R to skew Laurent polynomial rings. They investigated the relationship between properties of R and Jordan extension $A = A(R, \alpha)$, and also characterized the diameter of $\Gamma_E\big(R[x, x^{-1}; \alpha]\big)$ (cf. [16]).

The present work aims to continue our study of the diameter of the compressed zero-divisor graph of skew polynomial rings $R[x; \alpha, \delta]$, where R is a reversible and (α, δ)-compatible ring. Also, we will give a complete characterization of the possible diameters of $\Gamma_E\big(R[[x; \alpha]]\big)$ in term of the diameter of $\Gamma_E(R)$, where R is a reversible right Noetherian ring and has α-compatible property. By some examples, we show that the assumptions "reversibility", "R is (α, δ)-compatibility" and "Noetherian" in main results are crucial.

2. On the diameter of compressed zero-divisor graph of skew polynomial rings

Following Huckaba and Keller [24], a commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero-divisors, has a nonzero annihilator. Property (A) was originally studied by Quentel [33], where he used the term Condition (C) for Property (A). Using Property (A), Hinkle and Huckaba [22] extend the concept of Kronecker function rings from integral domains to rings with zero-divisors. The class of commutative rings with Property (A) is quite large. For example, the polynomial ring $R[x]$, rings whose classical ring of quotients are von Neumann regular [21], Noetherian rings [26, p. 56] and rings whose prime ideals are maximal [21], are examples of rings with Property (A). Kaplansky [26], proved that there are non-Noetherian rings such that do not have Property (A). Several papers are devoted to study the commutative rings with Property (A); see [11, 21, 24, 25, 30, 33].

Hong et al. [23], extended Property (A) to non-commutative setting as follows: A ring R has *right (left) Property* (A) if every finitely generated two-sided ideal of R consisting entirely of left (right) zero-divisors has a right (left) nonzero annihilator. A ring R is said to have *Property* (A) if R has right and left Property (A).

In this section, we proceed to characterize the diameter of $\Gamma_E\big(R[x;\alpha,\delta]\big)$, where R is reversible and (α,δ)-compatible. Since polynomial rings over reversible rings need not to be reversible by [27, Example 2.1], hence we can not use characterizations in [16, Theorem 2.2] for skew polynomial rings.

The following lemma, which is proved in [17, Theorem 2.6], will be helpful in our results.

Lemma 2.1. *Let R be a reversible and (α,δ)-compatible ring. Then $Z\big(R[x;\alpha,\delta]\big)$ is an ideal of $R[x;\alpha,\delta]$ if and only if $Z(R)$ is an ideal of R and R has right Property (A).*

Theorem 2.2. *Let R be a symmetric and (α,δ)-compatible ring which is not reduced. If there is a pair of zero-divisors $f(x), g(x) \in Z\big(R[x;\alpha,\delta]\big)$ such that $l_{R[x;\alpha,\delta]}\big(f(x)\big) \cap l_{R[x;\alpha,\delta]}\big(g(x)\big) = \{0\}$, then $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 3$.*

Proof. By [17, Theorem 2.2], there exist nonzero elements $\beta, \xi \in Z\big(R[x;\alpha,\delta]\big)^*$ such that $\beta\xi \neq 0 \neq \xi\beta$ and β, ξ don't have a nonzero mutual annihilator. Then $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 3$. □

Recall that if R is a reduced ring, then each minimal prime ideal of R is completely prime. Also each minimal prime ideal is a union of annihilators. Thus, if P is a minimal prime ideal of a reduced (α,δ)-compatible ring R, then $\alpha(P) \subseteq P$, $\delta(P) \subseteq P$, and so $P[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$. One can easily prove that $P[x;\alpha,\delta]$ is a minimal prime ideal of $R[x;\alpha,\delta]$.

For any $f \in R[x;\alpha,\delta]$, we denote by C_f the set of all coefficients of f. Also, the set of all nonzero coefficients of f is denoted by C_f^*.

Theorem 2.3. *Let R be a reversible and (α,δ)-compatible ring with $Z(R) \neq 0$. Then*

(1) $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 0$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R)\big) = 0$*;*

(2) $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 1$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R)\big) = 1$*;*

(3) $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 2$ *if and only if either (i) R has right Property (A), $Z(R)$ is an ideal of R with $(Z(R))^2 \neq 0$, and $Z(R) \neq ann_R(a)$, for each $a \in Z(R)^*$ or (ii) $Z(R) = ann_R(a)$, for some $a \in Z(R)^*$ and there exist two elements $b, c \in Z(R)^*$ such that $bc \neq 0$ and $[b]_R \neq [c]_R$;*

(4) $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and either R has not right Property (A) or $Z(R)$ is not an ideal of R.*

Proof. (1) For the forward direction, let $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 0$. Therefore $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 0$, since $\mathrm{diam}\Big(\Gamma_E(R)\Big) \le \mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big)$.

For the backward direction, let $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 0$. Thus $|\Gamma_E(R)| = 1$. Without loss of generality, we can consider $V\Big(\Gamma_E(R)\Big) = \Big\{[a]\Big\}$. Assume that $f \in Z\Big(R[x;\alpha,\delta]\Big)^*$. Since R is reversible and (α,δ)-compatible, there exist $r, s \in R$ such that $rf = 0 = fs$, by [17, Corollary 2.1], and so $C_f \subseteq Z(R)$. Hence $[a]_R = [a_i]_R$ for each $a_i \in C_f^*$. Since R is a reversible and (α,δ)-compatible ring, by [19, Lemma 2.1], we can easily prove $[a]_{R[x;\alpha,\delta]} = [f]_{R[x;\alpha,\delta]}$. Therefore $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 0$.

(2) For the forward direction, let $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 1$. Hence $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 1$, by statement (1).

For backward direction, let $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 1$. Then by [16, Theorem 2.2], either (i) R is a reduced ring with exactly two minimal prime ideals, or (ii) $|\Gamma_E(R)| = 2$ and $Z(R) = ann_R(a)$ for some $a \in Z(R)^*$.

First, assume that R is a reduced ring with exactly two minimal prime ideals P and Q, then $R[x;\alpha,\delta]$ is a reduced ring with exactly two minimal prime ideals $P[x;\alpha,\delta]$ and $Q[x;\alpha,\delta]$. Now, let $f, g \in Z\Big(R[x;\alpha,\delta]\Big)$. If $f, g \in P[x;\alpha,\delta]$, then $[f]_{R[x;\alpha,\delta]} = [g]_{R[x;\alpha,\delta]}$, since $fh = 0 = gh$ for each $h \in Q[x;\alpha,\delta]$ (since R is a reduced and (α,δ)-compatible ring). Similarly, if $f, g \in Q[x;\alpha,\delta]$, then $[f]_{R[x;\alpha,\delta]} = [g]_{R[x;\alpha,\delta]}$. Also, if $f \in P[x;\alpha,\delta]$ and $g \in Q[x;\alpha,\delta]$, then $fg = 0$. Therefore $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 1$.

Now, assume that $|\Gamma_E(R)| = 2$ and $Z(R) = ann_R(a)$, for some $a \in Z(R)^*$. Also let $V\Big(\Gamma_E(R)\Big) = \Big\{[a]_R, [b]_R\Big\}$. Consider $f \in Z\Big(R[x;\alpha,\delta]\Big)^*$. Since R is reversible and (α,δ)-compatible, there exists $0 \ne r, s \in R$ such that $rf = 0 = fs$, by [17, Corollary 2.1]. Hence $C_f^* \subseteq Z(R)^*$. We claim that $V\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = \Big\{[a]_{R[x;\alpha,\delta]}, [b]_{R[x;\alpha,\delta]}\Big\}$. We consider the following three cases:

Case 1. If for each $a_i \in C_f^*$, $[a_i]_R = [a]_R$, then $ann_R(a_i) = ann_R(a)$. Therefore $ann_{R[x;\alpha,\delta]}(f) = ann_{R[x;\alpha,\delta]}(a)$ and so $[a]_{R[x;\alpha,\delta]} = [f]_{R[x;\alpha,\delta]}$.

Case 2. If for each $a_i \in C_f^*$, $[a_i]_R = [b]_R$, then $ann_R(a_i) = ann_R(b)$. Therefore $ann_{R[x;\alpha,\delta]}(f) = ann_{R[x;\alpha,\delta]}(b)$ and so $[b]_{R[x;\alpha,\delta]} = [f]_{R[x;\alpha,\delta]}$.

Case 3. If $f = f_1 + f_1$, where $f_1 \ne 0 \ne f_2$ and for each $a_i \in C_{f_1}^*$, $[a_i]_R = [a]_R$ and for each $b_j \in C_{f_2}^*$, $[b_j]_R = [b]_R$, then since $ann_R(b) \subseteq ann_R(a)$, one can easily show that $ann_{R[x;\alpha,\delta]}(f) = ann_{R[x;\alpha,\delta]}(b)$, and so $[b]_{R[x;\alpha,\delta]} = [f]_{R[x;\alpha,\delta]}$. Therefore $V\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = \Big\{[a]_{R[x;\alpha,\delta]}, [b]_{R[x;\alpha,\delta]}\Big\}$, which implies that $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 1$.

(3) For the forward direction, let $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 2$. Then $\mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big) = 2$ or 3, since $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) \le \mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big)$.

Case 1. Assume that $\mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big) = 2$. Then R has right Property (A) and $Z(R)$ is an ideal of R with $Z(R)^2 \ne 0$, by [17, Theorem 2.7]. This leads to either (i) $Z(R) \ne ann_R(a)$ for each $a \in Z(R)^*$ or (ii) $Z(R) = ann_R(a)$ for some $a \in Z(R)^*$.

(i) If $Z(R) \ne ann_R(a)$ for each $a \in Z(R)^*$, the result follows.

(ii) Assume that $Z(R) = ann_R(a)$ for some $a \in Z(R)^*$. Since $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 2$, $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 2$, by statements (1), (2). Hence by [16, Theorem 2.2(3)], there exist $b, c \in Z(R)^*$ such that $bc \ne 0$ and $[b]_R \ne [c]_R$, as desired.

Case 2. Assume that $\mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big) = 3$. Then by [17, Theorem 2.7], R is not a reduced ring with exactly two minimal primes and either R does not has right Property (A) or $Z(R)$ is not an ideal of R. By Theorem 2.2, $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 3$, which is a contradiction.

For the backward direction, assume that R has right Property (A), $Z(R)$ is an ideal of R with $\Big(Z(R)\Big)^2 \ne 0$. Then $Z\Big(R[x;\alpha,\delta]\Big)$ is an ideal of $R[x;\alpha,\delta]$, by Lemma 2.1, and so each pair of distinct zero-divisors of $R[x;\alpha,\delta]$ has a nonzero annihilator. Thus $\mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big) \le 2$. Since $\Big(Z(R)\Big)^2 \ne 0$, $\mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big) \ge 2$. Hence $\mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big) = 2$. Since $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) \le \mathrm{diam}\Big(\Gamma(R[x;\alpha,\delta])\Big)$, $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha,\delta])\Big) = 2$, by using statements (1) and (2).

Now let $Z(R) = ann_R(a)$ and there exist $b, c \in Z(R)^*$ such that $bc \neq 0$ and $[b]_R \neq [c]_R$. Thus $Z\big(R[x;\alpha,\delta]\big) = ann_{R[x;\alpha,\delta]}(a)$ and also $[b]_{R[x;\alpha,\delta]} \neq [c]_{R[x;\alpha,\delta]}$. Therefore $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 2$.

(4) For the forward direction, let $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 3$. We claim that $\big(Z(R)\big)^2 \neq 0$. To see this, let $\big(Z(R)\big)^2 = 0$. Then R is nonreduced. Thus by statement (1), $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 0$, which is a contradiction. Hence $\big(Z(R)\big)^2 \neq 0$. By statement (2), R is not a reduced ring with exactly two minimal primes and by statement (3), R has not right Property (A) or $Z(R)$ is not an ideal of R.

The backward direction follows from statements (2) and (3). □

Theorem 2.4. *Let R be a reversible and (α,δ)-compatible ring. The following cases describe all possibilities for the pair* $\mathrm{diam}\big(\Gamma_E(R)\big)$ *and* $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big)$. *Then*

(1) $\mathrm{diam}\big(\Gamma_E(R)\big) = 0$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 0$;

(2) $\mathrm{diam}\big(\Gamma_E(R)\big) = 1$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 1$;

(3) $\mathrm{diam}\big(\Gamma_E(R)\big) = \mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 2$ *if and only if either (i) R has property (A), $Z(R)$ is an ideal of R and $Z(R) \neq ann_R(a)$, for each $a \in Z(R)$ or (ii) $Z(R) = ann_R(a)$, for some $a \in Z(R)$ and there exist two elements $b, c \in Z(R)^*$ such that $bc \neq 0$ and $[b]_R \neq [c]_R$;*

(4) $\mathrm{diam}\big(\Gamma_E(R)\big) = 2$ *and* $\mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 3$ *if and only if $Z(R)$ is an ideal and each pair of distinct zero divisors has a nonzero annihilator, and R does not have property (A);*

(5) $\mathrm{diam}\big(\Gamma_E(R)\big) = \mathrm{diam}\big(\Gamma_E(R[x;\alpha,\delta])\big) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R.*

Proof. These follow from [16, Theorem 2.2] and Theorem 2.3. □

Taking $\alpha = Id_R$ and $\delta = 0$, so we have the following results.

Corollary 2.5. *Let R be a reversible ring. Then*

(1) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 0$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R)\big) = 0$;

(2) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 1$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R)\big) = 1$;

(3) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 2$ *if and only if either (i) R has right Property (A), $Z(R)$ is an ideal of R with $\big(Z(R)\big)^2 \neq 0$, and $Z(R) \neq ann_R(a)$, for each $a \in Z(R)^*$ or (ii) $Z(R) = ann_R(a)$, for some $a \in Z(R)^*$ and there exist two elements $b, c \in Z(R)^*$ such that $bc \neq 0$ and $[b]_R \neq [c]_R$;*

(4) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and either R has not right Property (A) or $Z(R)$ is not an ideal of R.*

Corollary 2.6. *Let R be a reversible ring. The following cases describe all possibilities for the pair* $\mathrm{diam}\big(\Gamma_E(R)\big)$, $\mathrm{diam}\big(\Gamma_E(R[x])\big)$. *Then*

(1) $\mathrm{diam}\big(\Gamma_E(R)\big) = \mathrm{diam}\big(\Gamma_E(R[x])\big) = 2$ *if and only if either (i) R has right Property (A), $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$, and $Z(R) \neq ann_R(a)$, for each $a \in Z(R)$ or (ii) $Z(R) = ann_R(a)$, for some $a \in Z(R)$ and there exist two elements $b, c \in Z(R)^*$ such that $bc \neq 0$ and $[b]_R \neq [c]_R$;*

(2) $\mathrm{diam}\big(\Gamma_E(R)\big) = 2$ *and* $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 3$ *if and only if $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator, $Z(R) \neq ann_R(a)$ for each $a \in Z(R)$ and R have not Property (A);*

(3) $\mathrm{diam}\big(\Gamma_E(R)\big) = \mathrm{diam}\big(\Gamma_E(R[x])\big) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R.*

The following example shows that the assumption "R is reversible" in Theorems 2.3 and 2.4 is crucial.

Example 2.7. *Assume that $R = M_2(\mathbb{Z}_2)$. Clearly R is not reversible. In Example 2.8, we will show that* $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 2$. *Consider $\alpha = Id_R$ and $\delta = 0$. Since* $\mathrm{diam}\Big(\Gamma_E(R)\Big) \leq \mathrm{diam}\Big(\Gamma_E(R[x])\Big)$, $\mathrm{diam}\Big(\Gamma_E(R[x])\Big) = 2$ *or* 3. *We claim that* $\mathrm{diam}\Big(\Gamma_E(R[x])\Big) = 3$. *It is enough to show that there are two elements $f, g \in Z\Big(R[x]\Big)$ such that $fg \neq 0 \neq gf$ and f, g have not common nonzero annihilator. Consider elements $f(x) = A_0 + A_1x, g(x) = B_0 + B_1x \in Z\Big(R[x]\Big)$, where $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Obviously, $fg \neq 0 \neq gf$.*

First assume that $h(x) = C$ is a nonzero common annihilator of $f(x)$ and $g(x)$. If $h(x)f(x) = 0 = h(x)g(x)$, then $C \in l(A_0) \cap l(A_1) \cap l(B_0) \cap l(B_1)$. Hence $C = 0$, which is a contradiction. Now, if $h(x)f(x) = 0 = g(x)h(x)$, then $C \in l(A_0) \cap l(A_1) \cap r(B_0) \cap r(B_1)$, and so $C = 0$, which is also a contradiction. Similarly, if $h(x)g(x) = 0 = f(x)h(x)$ or $f(x)h(x) = 0 = g(x)h(x)$, then $C = 0$. Therefore $h(x)$ can not be in forms $h(x) = C$ or $h(x) = Cx^k$.

Now assume that $h(x) = \sum_{i=0}^{n} C_i x^i$, where $C_0 \neq 0 \neq C_n$ and $n > 0$. Also let $fh = 0 = gh$. Then we have $A_0C_0 = 0, A_0C_1 + A_1C_0 = 0, \ldots, A_0C_n + A_1C_{n-1} = 0$, $A_1C_n = 0$, $B_0C_0 = 0, B_0C_1 + B_1C_0 = 0, \ldots, B_0C_n + B_1C_{n-1} = 0$ and $B_1C_n = 0$. Hence $C_0 \in r(A_0) \cap r(B_0)$ and $C_n \in r(A_1) \cap r(B_1)$, and so $C_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $C_n \in \left\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right\}$. One can easily show that $A_0C_1 + A_1C_0 \neq 0$, for each $C_1 \in R$, which is a contradiction. By a similar method, we can show that the cases $fh = 0 = hg, hf = 0 = hg$ or $hf = 0 = gh$ can not occur. Thus f and g have not nonzero common annihilator, and so $\mathrm{diam}\Big(\Gamma_E(R[x])\Big) = 3$. *Therefore, assumption "R is reversible" in Theorem 2.4 can not be eliminated.*

The next example shows that the assumption "R is δ-compatible" in Theorems 2.3 and 2.4 is not superfluous.

Example 2.8. *Let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$, where $\bar{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Since $\bar{t}^2 = 0$ but $\bar{t}\delta(\bar{t}) \neq 0$, then R is not δ-compatible. It is obvious that $Z(R) = \Big\{0, \bar{t}\Big\}$, hence* $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 0$. *Now, consider the Ore extension $R[x; \delta]$. In [19, Example 2.10], was shown that $R[x; \delta] \cong M_2(\mathbb{Z}_2)[y]$. We put $R' = M_2(\mathbb{Z}_2)$. It is easy to check that $V\Big(\Gamma_E(R')\Big) = \Big\{[E_{11}]_{R'}, [E_{12}]_{R'}, [E_{21}]_{R'}, [E_{22}]_{R'}, [E_{11} + E_{12} + E_{21} + E_{22}]_{R'}, [E_{11} + E_{12}]_{R'}, [E_{11} + E_{21}]_{R'}, [E_{21} + E_{22}]_{R'}, [E_{12} + E_{22}]_{R'}\Big\}$, where E_{ij} denote the matrix units. Also, it can be seen that the multiplication of each pair of distinct zero divisors equal zero or has a nonzero annihilator. Hence* $\mathrm{diam}\Big(\Gamma_E(R')\Big) = 2$ *(see Figure 1). Thus* $\mathrm{diam}\Big(\Gamma_E(R'[y])\Big) \geq 2$, *and hence* $\mathrm{diam}\Big(\Gamma_E(R[x; \alpha])\Big) \geq 2$. *Therefore, the assumption "R is δ-compatible" in Theorem 2.3 is crucial.*

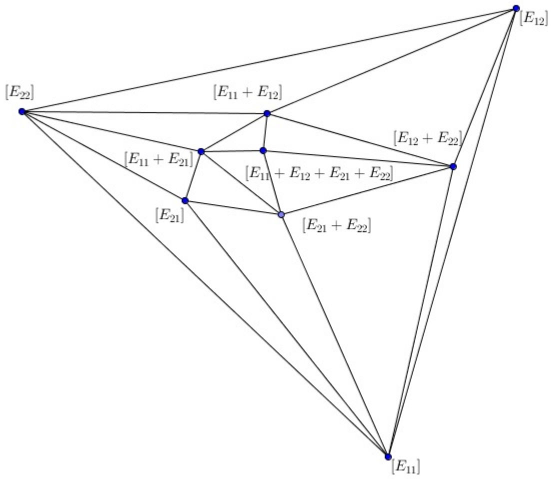

Figure 1:

The following example also shows that the assumption "R is α-compatible" in Theorems 2.3 and 2.4 is crucial.

Example 2.9. *Let $S = \mathbb{Z}_6$ and $R = S[y]$. Consider the endomorphism $\alpha : R \to R$ given by $\alpha(f(y)) = f(0)$. In [14, Example 3.8], it is shown that R is a reduced ring which is not α-compatible. One can see that $V\Big(\Gamma_E(R)\Big) = \Big\{[2]_R, [3]_R\Big\}$, and hence $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 1$. Consider the ring $R[x;\alpha]$. Now, we compute $Z(R[x;\alpha])$. Let $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^{n} f_i(y)x^i$ and $g(x) = \sum_{j=0}^{m} g_j(y)x^j$ are nonzero elements of $R[x;\alpha]$. We consider the following cases:*

Case (I): *Let $f_0(y) \neq 0$ and $g_t(y)$ be the first nonzero coefficient of $g(x)$. Since $f(x)g(x) = 0$, thus $f_0(y)g_t(y) = 0$, $f_1(y)\alpha(g_t(y)) + f_0(y)g_{t+1}(y) = 0$, $f_2(y)\alpha^2(g_t(y)) + f_1(y)\alpha(g_{t+1}(y)) + f_0(y)g_{t+2}(y) = 0, \ldots, f_n(y)\alpha^n(g_m(y)) = 0$. Since R is McCoy, so we have either $f_0(y) \in 2\mathbb{Z}_6[y]$, $g_t(y) \in 3\mathbb{Z}_6[y]$ or $f_0(y) \in 3\mathbb{Z}_6[y]$, $g_t(y) \in 2\mathbb{Z}_6[y]$. Without loss of generality, we may assume that $f_0(y) \in 2\mathbb{Z}_6[y]$ and $g_t(y) \in 3\mathbb{Z}_6[y]$. Now, we consider the following subcases:*

Subcase (I-I): *If $\alpha(g_j(y)) = 0$ for each $t \leq j \leq m$, then $g_j(y) \in 3\mathbb{Z}_6[y]$ (since $f_0(y) \in 2\mathbb{Z}_6(y)$), and $f_i(y)$ are arbitrary, for each $1 \leq i \leq n$.*

Subcase (I-II): *Let $\alpha(g_j(y)) \neq 0$ for some $t \leq j \leq m$, also let s be the smallest index such that $\alpha(g_s(y)) \neq 0$. Then $f_0(y)g_t(y) = 0$, $f_0(y)g_{t+1}(y) = 0, \ldots, f_0(y)g_s(y) = 0$, $f_0(y)g_{s+1}(y) + f_1(y)\alpha(g_s(y)) = 0, \ldots, f_n(y)\alpha^n(g_m(y)) = 0$. Since $f_0(y) \in 2\mathbb{Z}_6[y]$, it is easy to see that $g_j(y) \in 3\mathbb{Z}_6[y]$, for each $t \leq j \leq s$. Now, by multiplying 3 to $f_0(y)g_{s+1}(y) + f_1(y)\alpha(g_s(y)) = 0$, we have $3f_1(y)\alpha(g_s(y)) = 0$. Hence $f_1(y) \in 2\mathbb{Z}_6(y)$, since $g_s(y) \in 3\mathbb{Z}_6[y]$. By continuing this process, we deduce that $f_i(y) \in 2\mathbb{Z}_6(y)$ for each $0 \leq i \leq n$, and $g_j(y) \in 3\mathbb{Z}_6[y]$ for each $t \leq j \leq m$.*

Now, if $f_0(y) \in 3\mathbb{Z}_6[y]$ and $g_t(y) \in 2\mathbb{Z}_6[y]$, then by a similar way as used in Subcase (I-I) and Subcase (I-II), we conclude that:

If $\alpha(g_j(y)) = 0$ for each $t \leq j \leq m$, then $g_j(y) \in 2\mathbb{Z}_6[y]$ (since $f_0(y) \in 3\mathbb{Z}_6(y)$), and $f_i(y)$ are arbitrary, for every $1 \leq i \leq n$.

If $\alpha(g_j(y)) \neq 0$, for some $t \leq j \leq m$ and s is the smallest index such that $\alpha(g_s(y)) \neq 0$, then $f_i(y) \in 3\mathbb{Z}_6(y)$ for every $i \geq 0$, and $g_j(y) \in 2\mathbb{Z}_6[y]$ for every $t \leq j \leq m$.

Case (II): *Let $f_s(y)$ and $g_t(y)$ be the first nonzero coefficient of $f(x)$ and $g(x)$ (for $s > 0$), respectively. Thus $f_s(y)\alpha^s(g_t(y)) = 0$, $f_s(y)\alpha^s(g_{t+1}(y)) + f_{s+1}(y)\alpha^{s+1}(g_t(y)) = 0, \ldots, f_n(y)\alpha^n(g_m(y)) = 0$, since $f(x)g(x) = 0$. Then the following subcases occur:*

Subcase (II-I): *Let $\alpha(g_j(y)) = 0$, for each $t \leq j \leq m$. Then $f_i(y)$ is arbitrary, for each $s \leq i \leq n$.*

Subcase (II-II): *Let $\alpha(g_t(y)) \neq 0$. Then either $f_s(y) \in 2\mathbb{Z}_6[y]$, $\alpha(g_t(y)) = 3$ or $f_s(y) \in 3\mathbb{Z}_6[y]$, $\alpha(g_t(y)) \in 2\mathbb{Z}_6[y]$ (since $f_s(y)\alpha^s(g_t(y)) = 0$ and $\alpha(g_t(y)) \neq 0$). First assume that $f_s(y) \in 2\mathbb{Z}_6[y]$ and $\alpha(g_t(y)) = 3$. By a similar way as used to Subcase (I-II), we have $f_i(y) \in 2\mathbb{Z}_6[y]$ for each $s \leq i \leq n$, $\alpha(g_t(y)) = 3$. Now assume that $f_s(y) \in 3\mathbb{Z}_6[y]$ and $\alpha(g_t(y)) \in 2\mathbb{Z}_6[y]$. Similarly, we conclude that $f_i(y) \in 3\mathbb{Z}_6[y]$ for each $s \leq i \leq n$, and $\alpha(g_t(y)) \in 2\mathbb{Z}_6[y]$.*

Hence $Z_l\Big(R[x;\alpha]\Big) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, *where*
$A_1 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid f_i(y) \in 2\mathbb{Z}_6[y] \text{ for each } i\Big\}$,
$A_2 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid f_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$,
$A_3 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid f_0(y) \neq 0,\ f_0(y) \in 2\mathbb{Z}_6[y] \text{ and } f_j(y) \notin 2\mathbb{Z}_6[y] \text{ for some } j\Big\}$,
$A_4 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid f_0(y) \neq 0,\ f_0(y) \in 3\mathbb{Z}_6[y] \text{ and } f_j(y) \notin 3\mathbb{Z}_6[y] \text{ for some } j\Big\}$
and $A_5 = \Big\{ \sum_{i=s}^{n} f_i(y)x^i \mid s > 0,\ \{f_i(y)\}_{i=s}^{n} \not\subseteq 2\mathbb{Z}_6[y] \text{ and } \{f_i(y)\}_{i=s}^{n} \not\subseteq 3\mathbb{Z}_6[y]\Big\}$, *and also* $Z_r\Big(R[x;\alpha]\Big) = B_1 \cup B_2 \cup B_3 \cup B_4$, *where*
$B_1 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid f_i(y) \in 2\mathbb{Z}_6[y] \text{ for each } i\Big\}$,
$B_2 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid f_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$,
$B_3 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid \alpha(f_i(y)) \in 2\mathbb{Z}_6 \text{ for each } i\Big\}$
and $B_4 = \Big\{ \sum_{i=0}^{n} f_i(y)x^i \mid \alpha(f_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i\Big\}$. *Therefore* $Z\Big(R[x;\alpha]\Big) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7$, *where* $A_6 = B_3$ *and* $A_7 = B_4$.

Now, we determine $ann_{R[x;\alpha]}(\beta(x))$, *for each* $\beta(x) = \sum_{i=0}^{n} f_i(y)x^i \in Z(R[x;\alpha])$. *If* $\beta(x) \in A_1$, *then we have the following cases:*

Case 1.1 *Let* $\alpha(f_i(y)) = 0$ *for each* $0 \leq i \leq n$. *Then we consider the following subcases:*

Subcase 1.1.1 *Let* $f_0(y) \neq 0$. *Then* $l(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_0(y) \in 3\mathbb{Z}_6[y]\Big\}$ *and* $r(\beta(x)) = \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_0(y) \in 3\mathbb{Z}_6[y]\Big\} \cup \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$.

Subcase 1.1.2 *Let* $f_0(y) = 0$ *and* $f_s(y)$ *be the first nonzero coefficient of* $\beta(x)$. *Then* $l(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_0(y) \in 3\mathbb{Z}_6[y]\Big\}$ *and* $r(\beta(x)) = \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=0}^{n} g_i(y)x^i \mid g_0(y) \in 3\mathbb{Z}_6[y]\Big\} \cup \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i\Big\}$.

Case 1.2 *Let* $\alpha(f_i(y)) \neq 0$ *for some* $0 \leq i \leq n$. *Then we consider the following subcases:*

Subcase 1.2.1 *Let* $f_0(y) \neq 0$. *Then* $l(\beta(x)) = r(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$.

Subcase 1.2.2 *Let* $f_0(y) = 0$ *and* $f_s(y)$ *be the first nonzero coefficient of* $\beta(x)$. *Then* $l(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\}$ *and* $r(\beta(x)) = \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\Big\} \cup \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i\Big\}$.

Therefore $A_1 = [2y] \cup [2yx] \cup [2] \cup [2x]$.

If $\beta(x) \in A_2$, *then by a similar argument as used in Cases 1.1 and 1.2, one can easily show that* $A_2 = [3y] \cup [3yx] \cup [3] \cup [3x]$.

If $\beta(x) \in A_3$, *then we can write* $\beta(x) = \beta_1(x) + \beta_2(x)$, *where* $\beta_1(x) = \sum_{i=0}^{t} f_{1i}(y)x^{m_i}$ *and* $\beta_2(x) = \sum_{j=0}^{l} f_{2j}(y)x^{k_j}$ *such that* $f_{1i}(y) \in 2\mathbb{Z}_6[y]$ *and* $f_{2j}(y) \notin 2\mathbb{Z}_6[y]$, *for each* $0 \leq i \leq t$ *and* $0 \leq j \leq l$. *Hence we have the following cases:*

Case 3.1 *Let* $\alpha(f_{1i}(y)) = 0 = \alpha(f_{2j}(y))$ *for each* $0 \leq i \leq t$ *and* $0 \leq j \leq l$. *Then* $l(\beta(x)) = \Big\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i\Big\}$ *and* $r(\beta(x)) = \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i\Big\} \cup \Big\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\Big\}$.

Case 3.2 *Let* $\alpha(f_{1i}(y)) = 0$ *for each* $0 \leq i \leq t$, $\alpha(f_{2j}(y)) \neq 0$ *for some* $0 \leq j \leq l$. *We consider the following subcases:*

Subcase 3.2.1 *Assume* $\alpha(f_{2j}(y)) \in \{1, 5\}$, *for some* $0 \leq j \leq l$. *Then* $l(\beta(x)) = 0$ *and* $r(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(g_i(y)) = 0 \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(g_i(y)) = 0 \text{ for each } i\Big\}$.

Subcase 3.2.2 *Assume* $\Big\{\alpha(f_{2j}(y))\Big\}_{j=0}^{l} = \{2, 3\}$. *Then* $l(\beta(x)) = 0$ *and* $r(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(g_i(y)) = 0 \text{ for each } i\Big\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \Big\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(g_i(y)) = 0 \text{ for each } i\Big\}$.

Subcase 3.2.3 *Assume* $\left\{\alpha(f_{2j}(y))\right\}_{j=0}^{l} = \{3\}$. *Then* $l(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 2\mathbb{Z}_6[y] \text{ for each } i\right\}$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 2\mathbb{Z}_6[y] \text{ for each } i\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$.

Subcase 3.2.4 *Assume* $\left\{\alpha(f_{2j}(y))\right\}_{j=0}^{l} = \{2,4\}$. *Then* $l(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\}$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$.

Case 3.3 *Let* $\alpha(f_{1i}(y)) \neq 0$ *for some* $0 \leq i \leq t$. *Then we have the following subcases:*

Subcase 3.3.1 *Assume* $\alpha(f_{2j}(y)) = 0$ *for each* $0 \leq j \leq l$. *Then* $l(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\}$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$. *Hence* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$.

Subcase 3.3.2 *Assume* $\alpha(f_{2j}(y)) \neq 0$ *for some* $0 \leq j \leq l$, *and* $\alpha(f_{2j}(y)) \notin 2\mathbb{Z}_6$. *Then* $l(\beta(x)) = 0$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(g_i(y)) = 0 \text{ for each } i\right\}$. *Hence* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(g_i(y)) = 0 \text{ for each } i\right\}$.

Subcase 3.3.3 *Let* $\alpha(f_{2j}(y)) \neq 0$ *for some* $0 \leq j \leq l$, *and* $\left\{\alpha(f_{2j}(y))\right\}_{j=0}^{l} = 2\mathbb{Z}_6$. *Then* $l(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\}$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$. *Hence* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i\right\}$.

Therefore $A_3 = [2y + yx] \cup [2y + x] \cup [2y + 3x] \cup [2y + (2 + 3y)x]$.

If $\beta(x) \in A_4$, *then by a similar way as used in Cases 3.1-3.3, one can show that* $A_4 = [3y + yx] \cup [3y + x] \cup [3y + 2x] \cup [3y + (3 + 2y)x]$.

If $\beta(x) \in A_5$, *then we have the following cases:*

Case 5.1 *Let* $\alpha(f_i(y)) \neq 0$ *for some* $s \leq i \leq n$. *Also let* $\alpha(f_i(y)) \neq 3$ *and* $\alpha(f_i(y)) \notin 2\mathbb{Z}_6$ *for some* $s \leq i \leq n$. *Then* $l(\beta(x)) = 0$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid \alpha(g_i(y)) = 0 \text{ for each } i\right\}$. *Hence* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid \alpha(g_i(y)) = 0 \text{ for each } i\right\}$.

Case 5.2 *Let* $\alpha(f_i(y)) = 0$ *for each* $s \leq i \leq n$. *Then* $l(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i\right\}$ *and* $r(\beta(x)) = \left\{\sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) = 0 \text{ for each } i\right\}$. *Hence* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) = 0 \text{ for each } i\right\}$.

Therefore $A_5 = [x] \cup [yx]$.

If $\beta(x) \in A_6$, *then we have the following cases:*

Case 6.1 *Let* $\alpha(f_i(y)) = 0$ *for each* $0 \leq i \leq n$. *Thus we consider the following subcases:*

Subcase 6.1.1 *Let* $f_i(y) \in 2\mathbb{Z}_6[y]$ *for each* $0 \leq i \leq n$. *Then we have the following subcases:*

Subcase 6.1.1.1 *If* $f_0(y) \neq 0$, *then* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid g_0(y) \in 3\mathbb{Z}_6[y]\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i\right\}$, *by Subcase 1.1.1.*

Subcase 6.1.1.2 *If* $f_0(y) = 0$, *then* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=0}^{n} g_i(y)x^i \mid g_0(y) \in 3\mathbb{Z}_6[y]\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i\right\}$, *by Subcase 1.1.2.*

Subcase 6.1.2 *Let* $f_i(y) \notin 2\mathbb{Z}_6[y]$ *for some* $0 \leq i \leq n$. *Then we consider the following subcases:*

Subcase 6.1.2.1 *If* $f_i(y) \in 3\mathbb{Z}_6[y]$ *for each* $0 \leq i \leq n$, *then we have the following subcases:*

Subcase 6.1.2.1.1 *Let* $f_0(y) = 0$. *Then by a similar way as used in Subcase 1.1.2, we have* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 2\mathbb{Z}_6[y] \text{ for each } i\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 2\mathbb{Z}_6 \text{ for each } i\right\}$.

Subcase 6.1.2.1.2 *Let* $f_0(y) \neq 0$. *Then by a similar way as used in Subcase 1.1.1, we have* $ann_{R[x;\alpha]}(\beta(x)) = \left\{\sum_{i=0}^{m} g_i(y)x^i \mid g_0(y) \in 2\mathbb{Z}_6[y]\right\} \cup \left\{\sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 2\mathbb{Z}_6[y] \text{ for each } i\right\}$.

Subcase 6.1.2.2 *If* $f_j(y) \notin 3\mathbb{Z}_6[y]$ *for some* $0 \leq j \leq n$, *then we have the following subcases:*

Subcase 6.1.2.2.1 *Let* $f_0(y) = 0$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) = 0 \text{ for each } i \right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) = 0 \text{ for each } i \right\}$.

Subcase 6.1.2.2.2 *Let* $f_0(y) \neq 0$. *Then we have the following subcases:*

Subcase 6.1.2.2.2.1 *If* $i = j = 0$, *then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = 0$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$.

Subcase 6.1.2.2.2.2 *If* $i = j \neq 0$, *then we consider the following subcases:*

Subcase 6.1.2.2.2.2.1 *Let* $f_0(y) \in 2\mathbb{Z}_6[y]$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$.

Subcase 6.1.2.2.2.2.2 *Let* $f_0(y) \in 3\mathbb{Z}_6[y]$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 2\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 2\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$.

Subcase 6.1.2.2.2.2.3 *Let neither* $f_0(y) \in 2\mathbb{Z}_6[y]$ *nor* $f_0(y) \in 3\mathbb{Z}_6[y]$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = 0$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in \mathbb{Z}_6[y] \text{ for each } i \right\}$.

Subcase 6.1.2.2.2.3 *If* $i = 0, j \neq 0, i \neq 0, j = 0$ *or* $i \neq j \neq 0$, *then one of the Subcases 6.1.2.2.2.2.1-6.1.2.2.2.2.3 appears.*

Case 6.2 *Let* $\alpha(f_i(y)) \neq 0$ *for some* $0 \leq i \leq n$. *Thus we consider the following subcases:*

Subcase 6.2.1 *Let* $f_i(y) \in 2\mathbb{Z}_6[y]$ *for each* $0 \leq i \leq n$. *Then we have the following subcases:*

Subcase 6.2.1.1 *Let* $f_0(y) = 0$. *Then* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) \in 3\mathbb{Z}_6 \text{ for each } i \right\}$, *by Subcase 1.2.2.*

Subcase 6.2.1.2 *Let* $f_0(y) \neq 0$. *Then* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=0}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$, *by Subcase 1.2.1.*

Subcase 6.2.2 *Let* $f_j(y) \notin 2\mathbb{Z}_6[y]$ *for some* $0 \leq j \leq n$. *Then we have the following subcases:*

Subcase 6.2.2.1 *Let* $f_0(y) = 0$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) = 0 \text{ for each } i \right\}$. *Hence* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid \alpha(h_i(y)) = 0 \text{ for each } i \right\}$.

Subcase 6.2.2.2 *Let* $f_0(y) \neq 0$. *Then we consider the following subcases:*

Subcase 6.2.2.2.1 *If* $i = j = 0$, *then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = 0$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$.

Subcase 6.2.2.2.2 *If* $i = j \neq 0$, *then we consider following subcases:*

Subcase 6.2.2.2.2.1 *Let* $f_0(y) \in 2\mathbb{Z}_6[y]$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 3\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$.

Subcase 6.2.2.2.2.2 *Let* $f_0(y) \in 3\mathbb{Z}_6[y]$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 2\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\} \cup \left\{ \sum_{i=0}^{m} h_i(y)x^i \mid h_i(y) \in 2\mathbb{Z}_6[y] \text{ and } \alpha(h_i(y)) = 0 \text{ for each } i \right\}$.

Subcase 6.2.2.2.2.3 *Let neither* $f_0(y) \in 2\mathbb{Z}_6[y]$ *nor* $f_0(y) \in 3\mathbb{Z}_6[y]$. *Then* $l(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$ *and* $r(\beta(x)) = 0$. *Thus* $ann_{R[x;\alpha]}(\beta(x)) = \left\{ \sum_{i=1}^{m} g_i(y)x^i \mid g_i(y) \in 3\mathbb{Z}_6[y] \text{ for each } i \right\}$.

Subcase 6.2.2.2.3 *If* $i = 0, j \neq 0, i \neq 0, j = 0$ *or* $i \neq j \neq 0$, *then one of the Subcases 6.2.2.2.2.1-6.2.2.2.2.3 appears.*

Therefore $A_6 = [2y] \cup [2yx] \cup [3yx] \cup [3y] \cup [yx] \cup [y] \cup [2y + (2 + 3y)x] \cup [2y + yx] \cup [3y + yx] \cup [2x] \cup [2] \cup [(2 + y)x] \cup [2 + y] \cup [3y + 2x]$.

If $\beta(x) \in A_7$, then by a similar way as used in Cases 6.1 and 6.2, one can show that $A_7 = [2y] \cup [2yx] \cup [3yx] \cup [3y] \cup [yx] \cup [y] \cup [3y + (3 + 2y)x] \cup [2y + yx] \cup [3y + yx] \cup [3x] \cup [3] \cup [(3 + y)x] \cup [3 + y] \cup [2y + 3x]$. Therefore

$$\begin{aligned} V\Big(\Gamma_E(R[x;\alpha])\Big) = \Big\{ & [2y], [2], [3y + yx], [2x], [3y], [3], [3yx], [3x], [2y + yx], [2y + x], \\ & [2y + (2 + 3y)x], [2yx], [3y + x], [3y + 2x], [3y + (3 + 2y)x], \\ & [x], [yx], [y], [2y + 3x], [2 + y], [3 + y], [(2 + y)x], [(3 + y)x]\Big\}. \end{aligned}$$

One can easily check that the distinct vetices $[3]$ and $[2y + x]$ have not nonzero common annihilator, and also $[3][2y + x] \neq 0 \neq [2y + x][3]$. Hence $\mathrm{diam}\Big(\Gamma_E(R[x;\alpha])\Big) = 3$ (see Figure 2). Therefore, assumption α-compatiblity

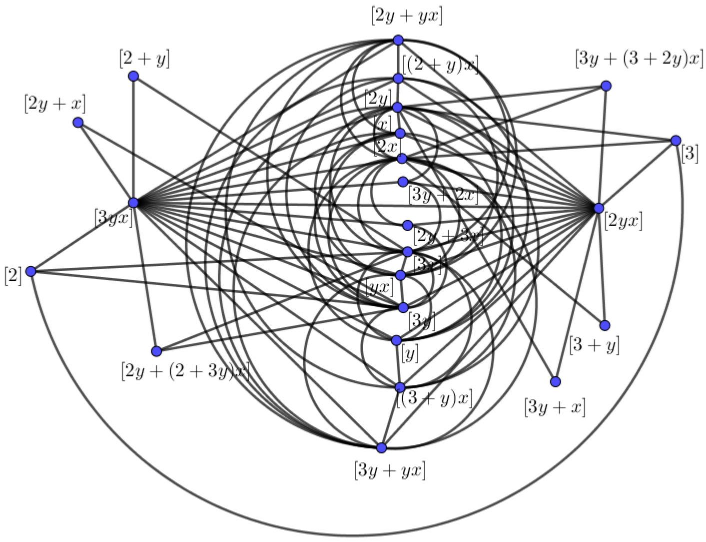

Figure 2:

in Theorem 2.3 can not be eliminated.

3. On the diameter of compressed zero-divisor graph of skew power series rings

Yang, Song and Liu in [36], introduced the concept of power-serieswise McCoy. A ring R is said to be *right power-serieswise McCoy* if whenever power series $f(x), g(x) \in R[[x]] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then there exists $0 \neq r \in R$ such that $f(x)r = 0$. *Left power-serieswise McCoy* can be defined similarly. If ring R is both right and left power-serieswise McCoy, we say that R is *power-serieswise McCoy*.

Let α be an endomorphism of a ring R. According to [2], a ring R is called *right α-power-serieswise McCoy*, if whenever power series $f(x), g(x) \in R[[x;\alpha]] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then there exists $0 \neq c \in R$ such that $f(x)c = 0$. *Left α-power-serieswise McCoy* can be defined similarly. If ring R is both right and left α-power-serieswise McCoy, we say that R is *α-power-serieswise McCoy*.

In this section, we proceed to characterize the diameter of $\Gamma_E(R[[x;\alpha]])$, where R is a reversible, α-compatible and right Noetherian ring.

Remark 3.1. *If R is a reversible, α-compatible and right Noetherian ring, then R is α-power-serieswise McCoy by [20, Corollary 2.7].*

Remark 3.2. *Let R be a reduced and α-compatible ring. Then $R[[x;\alpha]]$ is reduced by [19].*

Theorem 3.3. *Let R be a reversible and α-compatible ring. Then $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 0$ if and only if R is not reduced with $Z(R)^2 = 0$.*

Proof. For forward direction, suppose that $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 0$. Hence $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 0$, since $\mathrm{diam}\Big(\Gamma_E(R)\Big) \leq \mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big)$. Therefore R is not a reduced with $Z(R)^2 = 0$, by [16, Theorem 2.2].

For backward direction, let R don't be a reduced ring with $Z(R)^2 = 0$. By [20, Theorem 2.21], $\mathrm{diam}\Big(\Gamma(R[[x;\alpha]])\Big) = 1$. Thus $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 0$ or 1. We claim that $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 0$. Otherwise, there exist $f, g \in Z\Big(R[[x;\alpha]]\Big)^*$ such that $fg = 0$ but $[f] \neq [g]$. Hence there is $h \in ann_{R[[x;\alpha]]}(f)$ with $hg \neq 0$. This is a contradiction. □

In [20, Theorem 2.17], the authors showed that if R is a reversible, α-compatible and right Noetherian ring, then $Z\Big(R[[x;\alpha]]\Big)$ is an ideal of $R[[x;\alpha]]$ if and only if $Z(R)$ is an ideal of R. This will be useful in the following results.

Theorem 3.4. *Let R be a reversible, α-compatible and right Noetherian ring with $Z(R)^2 \neq 0$. Then*

(1) $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 1$ *if and only if* $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 1$;

(2) $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 2$ *if and only if* $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 2$;

(3) $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 3$ *if and only if* $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 3$.

Proof. (1) For forward direction, suppose that $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 1$. By Theorem 3.3, $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 1$, Since $\mathrm{diam}\Big(\Gamma_E(R)\Big) \leq \mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big)$.

For backward direction, let $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 1$. By [16, Theorem 2.2], either (i) R is reduced with exactly two minimal prime ideals P and Q with $|Z(R)| \geq 3$ or (ii) $|\Gamma_E(R)| = 2$ and there exists $a \in Z(R)^*$ such that $Z(R) = ann_R(a)$.

If (i) holds, by Remark 3.2, $R[[x;\alpha]]$ is a reduced ring, and also $P[[x;\alpha]]$ and $Q[[x;\alpha]]$ are the exactly two minimal primes of $R[[x;\alpha]]$. Assume that $f, g \in Z\Big(R[[x;\alpha]]\Big)$. If both f and g belong to $P[[x;\alpha]]$, then $[f]_{R[[x;\alpha]]} = [g]_{R[[x;\alpha]]}$, since $fh = 0 = gh$ for each $h \in Q[[x;\alpha]]$ (since R is an α-compatible and R is reduced ring). Similarly, if $f, g \in Q[[x;\alpha]]$, then $[f]_{R[[x;\alpha]]} = [g]_{R[[x;\alpha]]}$. Now if $f \in P[[x;\alpha]]$ and $g \in Q[[x;\alpha]]$, then $fg = 0$. Therefore $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 1$.

If (ii) holds, by Remark 3.1, one can show that $Z\Big(R[[x;\alpha]]\Big) = ann_{R[[x;\alpha]]}(a)$. Hence $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 1$.

(2) For forward direction, assume that $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 2$. Since $\mathrm{diam}\Big(\Gamma_E(R)\Big) \leq \mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big)$ and by statement (1), $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 2$.

For backward direction, let $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 2$. By [16, Theorem 2.2], either (i) $Z(R)$ is an ideal of R whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator and $Z(R) \neq ann_R(a)$ for every $a \in Z(R)$ or (ii) $Z(R) = ann_R(a)$ for some $a \in Z(R)^*$ and there exist $b, c \in Z(R)^*$ such that $bc \neq 0$ and $[b]_R \neq [c]_R$.

If (i) holds, since R is a reversible, α-compatible right Noetherian ring and $Z(R)$ is ideal, then $Z(R[[x;\alpha]])$ is an ideal of $R[[x;\alpha]]$ (by [20, Theorem 2.17]). Hence each pair of distinct zero divisors of $Z(R[[x;\alpha]])$ has a nonzero annihilator, and so $\mathrm{diam}\Big(\Gamma(R[[x;\alpha]])\Big) = 2$. Since $\mathrm{diam}\Big(\Gamma_E(R)\Big) \leq \mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big)$, therefore $\mathrm{diam}\Big(\Gamma_E(R[[x;\alpha]])\Big) = 2$.

If (ii) holds, we can easily show that $Z\Big(R[[x;\alpha]]\Big) = ann_{R[[x;\alpha]]}(a)$ and $[b]_{R[[x;\alpha]]} \neq [c]_{R[[x;\alpha]]}$. Then the result follows.

(3) It follows from statements (1) and (2). □

We have the following corollary, if $\alpha = Id_R$.

Corollary 3.5. *Let R be a reversible and right Noetherian ring. Then*

(1) $\mathrm{diam}\Big(\Gamma_E(R[[x]])\Big) = 0$ *if and only if* $\mathrm{diam}\Big(\Gamma_E(R)\Big) = 0$;

(2) $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 1$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R)\big) = 1$;

(3) $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 2$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R)\big) = 2$;

(4) $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R)\big) = 3$.

The following example shows that the assumption "R is Noetherian" in Theorem 3.4 is not superfluous.

Example 3.6. *Let K be a field and $D = K[w, y, z]_M$, where w, y and z are algebraically independent indeterminates. Clearly D is a domain. Let $\mathcal{P}$ denote the height two primes of D and Q be the maximal ideal of D. Also let $B = \sum F_\gamma$ where $F_\gamma = qf(D/P_\gamma)$ for each $P_\gamma \in \mathcal{P}$. Let $R = D(+)B$ be the idealization of B over D, and $\alpha = Id_R$. Clearly, R is not Noetherian. Lucas [30, Example 5.2] showed that each two generated ideal contained in $Z(R)$ has a nonzero annihilator but R have not Property (A), and $\operatorname{diam}\big(\Gamma(R)\big) = 2$ but $\operatorname{diam}\big(\Gamma(R[x])\big) = \operatorname{diam}\big(\Gamma(R[[x]])\big) = 3$. Therefore $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$, by [16, Theorem 2.2]. Since R have not Property (A) and $\operatorname{diam}\big(\Gamma(R)\big) = 2$, hence $\operatorname{diam}\big(\Gamma_E(R)\big) = 2$, by [16, Theorem 2.2]. Thus assumption "R is Noetherian" in Theorem 3.4 is not superfluous.*

Example 2.9 also shows that the assumption "R is α-compatible" in Theorem 3.4 is crucial.

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