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Characterizations of an *MW*-Topological Rough Set Structure

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Abstract. Regarding the study of digital topological rough set structures, the present paper explores some mathematical and systemical structures of the Marcus-Wyse (*MW*-, for brevity) topological rough set structures induced by the *locally finite covering approximation* (*LFC*-, for brevity) space $(\mathbb{R}^2, \mathcal{C})$ (see Proposition 3.4 in this paper), where $\mathbb{R}^2$ is the 2-dimensional Euclidean space. More precisely, given the *LFC*-space $(\mathbb{R}^2, \mathcal{C})$, based on the set of adhesions of points in $\mathbb{R}^2$ inducing certain *LFC*-rough concept approximations, we systematically investigate various properties of the *MW*-topological rough concept approximations (D^-_M, D^+_M) derived from this *LFC*-space $(\mathbb{R}^2, \mathcal{C})$. These approaches can facilitate the study of an estimation of roughness in terms of an *MW*-topological rough set. In the present paper each of a universe U and a target set $X(\subseteq U)$ need not be finite and further, a covering $\mathcal{C}$ is locally finite. In addition, when regarding both an M-rough set and an *MW*-topological rough set in Sections 3, 4, and 5, the universe $U(\subset \mathbb{R}^2)$ is assumed to be the set $\mathbb{R}^2$ or a compact subset of $\mathbb{R}^2$ or a certain set containing the union of all adhesions of $x \in X$ (see Remark 3.6).

1. Introduction

Among several kinds of rough set structures induced by certain locally finite covering approximation spaces in [4–6], the present paper mainly concerns the Marcus-Wyse (*MW*-, for short) topological rough set structure [5, 6] which can facilitate the studies of object classifications and information geometry. To make this work more effective, we need to use several kinds of tools involving digital topological rough set structures. A recent paper [4] introduced the notion of *locally finite covering approximation* (*LFC*-, for brevity) space $(U, \mathcal{C})$ as a generalization of a (finite) covering approximation space. Further, for a subset X of the universe $\mathbb{R}^2$, a neighborhood system $M(X)$ [5] (or the so-called *LFC*-system) was established from a locally finite covering $\mathcal{C}$. Next, a paper [5] also established a quasi-discrete (or clopen) topology $T_{M(X)}$ on U, generated by the system $M(X)$ as a base. Then, a paper [5] further developed two types of *LFC*-rough set structures such as an M-rough set operator (M_*, M^*) (see Definition 3.7 in this paper) and an *MW*-topological rough set operator (D^-_M, D^+_M) (see Definition 3.10 in the paper). Besides, a paper [6] proposed certain measures of roughness of the concept approximations (M_*, M^*) and (D^-_M, D^+_M). Additionally, it also developed membership functions estimating roughness of these two concept approximations. Besides, it

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suggested several examples for explaining various properties of these operators and referred to certain utilities of them which can be used in applied science including information geometry, computational geometry, pattern recognition, finger print recognition, image processing, and so on.

In order to proceed with this work, we use various tools from rough set theory and topology, such as typical Pawlak's tools [13], granulations [14], neighborhood systems [10], related topologies, digitizations [8, 9] associated with the MW-topology [3, 5, 11], and so on. By using these tools, the present paper continues a series of studies of various properties associated with the concept approximations (M_*, M^*) and (D_M^-, D_M^+) in [5]. The present paper concerns theoretical properties of an MW-topological rough set operator, such as duality, limit conditions, comparability with union and intersection, increasingness *w.r.t.* set inclusion, idempotency, and so on. Besides, the paper further studies some relationships between the two rough set operators (M_*, M^*) and (D_M^-, D_M^+).

2. Preliminaries

Since the paper studies an M-rough set operator (M_*, M^*) and an MW-topological rough set operator (D_M^-, D_M^+), let us recall some notions and terminology related to these tools. The MW-topological plane, denoted by $(\mathbb{Z}^2, \gamma)$, and the study of its properties includes the papers [3, 5, 8, 11]. In order to make the present paper self-contained, we recall the MW-topology, as follows:

Definition 2.1. ([11]) The MW-topology on $\mathbb{Z}^2$, denoted by $(\mathbb{Z}^2, \gamma)$, is induced by the set $\{U_p \,|\, p \in \mathbb{Z}^2\}$ in (2.1) as a base, where for each point $p = (x, y) \in \mathbb{Z}^2$

$$U_p := \begin{cases} \{(x \pm 1, y), p, (x, y \pm 1)\} \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ otherwise.} \end{cases} \tag{2.1}$$

Regarding the further statement of a point in $\mathbb{Z}^2$, in this paper we call a point $p = (x_1, x_2)$ double even if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; even if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and odd if $x_1 + x_2$ is an odd number. In addition, these points are shown like the following symbols: The shape ■ means an even or a double even point, and • means an odd point in the present paper (see Fig.1).

We say that a topological space (X, T) is locally finite if each point $x(\in X)$ has a finite neighborhood [12]. In addition, it turns out that $(\mathbb{Z}^2, \gamma)$ is locally finite according to the property (2.1).

Let us now recall basic concepts in covering-based rough sets. Indeed, there are many types of covering-based approximation spaces (U, C) with a finite cardinality of U [2, 15, 16]. The paper [1] introduced the notion of covering approximation space, as follows: Let U be a nonempty universe of discourse and $C = \{C_i \,|\, C_i \neq \emptyset, C_i \subseteq X, i \in M\}$ a family of nonempty subsets of U. If $\bigcup_{i \in M} C_i = U$, then C is called a covering of U. We say that an ordered pair (U, C) is a covering approximation space. In addition, let $\emptyset$ be the empty set and for a set $A \subseteq U$, A^C denotes the complement of A in U.

3. Adhesions of Eelements of an *LFC*-Space on $\mathbb{R}^2$

A recent paper [4] defined the notion of *locally finite covering approximation* (*LFC*-, for brevity) space (U, C), where all related sets need not be finite, as follows:

Definition 3.1. ([4]) A covering $C = \{C_i \,|\, C_i \subseteq U, C_i \neq \emptyset, i \in M\}$ of a set U is called *locally finite* if every point $x(\in U)$ belongs only to a finite number of sets in C. We say that a covering approximation space (U, C) is locally finite if C is locally finite, where U and M need not be finite.

As mentioned earlier, we note again that a "locally finite covering approximation space" is called an *LFC*-space in the present paper. According to Definition 3.1, it is obvious that an *LFC*-space is a generalization of a finite covering approximation space.

Motivated by these approaches, a recent paper [4] introduced the notion of neighborhood of $x \in U$ and an adhesion of $x \in U$ for an *LFC*-space (U, C), as follows:

Definition 3.2. ([4]) Let (U, C) be an *LFC*-space, where $C = \{C_i \,|\, C_i \neq \emptyset, i \in M\}$ and M is the set of indices. For any $x \in U$, the neighborhood of x is defined by

$$n(x) = \cap\{C_i \in C \,|\, x \in C_i\}, \tag{3.1}$$

and the adhesion of $x \in U$ is defined as

$$P_x^C = \{y \in U \,|\, \forall C_i \in C (x \in C_i \Leftrightarrow y \in C_i)\}. \tag{3.2}$$

In (3.2), if there is no danger of confusion, we can omit the superscript C of P_x^C. In view of the above neighborhood $n(x)$ and the adhesion P_x^C in (3.1) and (3.2), it is obvious that whereas not every set $\{n(x) \,|\, x \in U\}$ is a partition of U [4], the set $\{P_x \,|\, x \in U\}$ clearly forms a partition of U. For each point $x \in \mathbb{Z}^2 (\subset \mathbb{R}^2)$, we obtain the sets $n(x)$ and $P_x (\subset \mathbb{R}^2)$ in (3.1) and (3.2) below according to the point x.

Let (U, C) be a covering approximation space and $X (\subseteq U)$ a nonempty set. Let us now propose a new neighborhood system of $U (\subseteq \mathbb{R}^2)$ related to $X (\subseteq U)$, called $M(X)$ (see Definition 3.7), derived from the following *LFC*-space C of $\mathbb{R}^2$.

Definition 3.3. ([5]) For $m, n \in \mathbb{Z}$, consider the sets $X_i (\subseteq \mathbb{R}^2), i \in \{1, 2, 3, 4\}$, as follows (see Fig.3 of [6]):

$$\begin{cases} X_1 = (2m - 1.5, 2m + 1.5) \times (2n - 1.5, 2n + 1.5); \\ X_2 = [p_1 - 0.5, p_1 + 0.5] \times [p_2 - 0.5, p_2 + 0.5], \\ \text{where } (p_1, p_2) \in \{(2m \pm 1, 2n \pm 1)\} \subset \mathbb{Z}^2; \\ X_3 = (2m - 0.5, 2m + 2.5) \times (2n - 0.5, 2n + 2.5); \text{ and} \\ X_4 := [p_1 - 0.5, p_1 + 0.5] \times [p_2 - 0.5, p_2 + 0.5], \text{ where} \\ (p_1, p_2) \in \{(2m + 2, 2n + 2), (2m + 2, 2n), (2m, 2n + 2), (2m, 2n)\} \subset \mathbb{Z}^2. \end{cases}$$

(Indeed, in Fig.1(2) of [5], the point with $(2m + 1, 2n + 1)$ should be marked with $(2m + 2, 2n + 1)$ instead.) Further, consider the four points in $\mathbb{R}^2$, such as $q_1 = (2m - 0.5, 2n + 0.5), q_2 = (2m - 0.5, 2n - 0.5), q_3 = (2m + 0.5, 2n + 0.5), q_4 = (2m + 0.5, 2n - 0.5)$. Assume

$$\begin{cases} C_1((2m, 2n)) = (X_1 \setminus X_2) \cup \{q_1, q_2, q_3, q_4\}, \text{and} \\ C_2((2m + 1, 2n + 1)) = X_3 \setminus X_4. \end{cases} \tag{3.3}$$

We then define the following *LFC*-space $(\mathbb{R}^2, C)$, where

$$C = \{C_1((2m, 2n)), C_2((2m + 1, 2n + 1)) \,|\, m, n \in \mathbb{Z}\}. \tag{3.4}$$

Hereafter, $(\mathbb{R}^2, C)$ means the *LFC*-space in (3.4).

Proposition 3.4. *[5] Consider the LFC-space $(\mathbb{R}^2, C)$. Then, for each point $x := (x_1, x_2) \in \mathbb{Z}^2$, P_x is obtained, as follows:*

$$P_x = \begin{cases} [x_1 - 0.5, x_1 + 0.5] \times [x_2 - 0.5, x_2 + 0.5] \\ \textit{if } x \textit{ is a double even point}; \\ [x_1 - 0.5, x_1 + 0.5] \times [x_2 - 0.5, x_2 + 0.5] \setminus \{(x_1 \pm 0.5, x_2 \pm 0.5)\} \\ \textit{if } x \textit{ is an even point; and} \\ (x_1 - 0.5, x_1 + 0.5) \times (x_2 - 0.5, x_2 + 0.5) \\ \textit{if } x \textit{ is an odd point}. \end{cases} \tag{3.5}$$

The shape of P_x of (3.5) is shown in Fig 3 of [6]. For any element $x (\in \mathbb{R}^2)$, since both $n(x)$ and P_x contains the element x, using these neighborhoods of x, we assume a reflexive neighborhood system of x. For $X \subseteq \mathbb{R}^2$, this paper uses the neighborhood system $(\mathbb{R}^2, M(X))$ (see Definition 3.5 below in this paper) inherited from the *LFC*-space $(\mathbb{R}^2, C)$ [5]. Indeed, $M(X)$ will contribute to the establishment of building blocks as equivalence classes for the construction of the lower and upper approximations for an *LFC*-space. Consequently, it is strongly used in establishing *LFC*-rough set structures (see Definition 3.7 below in the present paper). Let us now define the following $M(X)$ for the space X with finite or infinite cardinalities.

Definition 3.5. ([5]) Given the *LFC*-space $(\mathbb{R}^2, C)$ and a set $X(\subseteq \mathbb{R}^2)$, consider the set $M_1(X) = \{P_x \,|\, x \in U, P_x \cap X \neq \emptyset\}$. Then, a system $(\mathbb{R}^2, M(X))$ related to X is defined with

$$M(X) = \{\mathbb{R}^2 \setminus U^X\} \cup M_1(X), \tag{3.6}$$

where $U^X = \bigcup_{P_x \in M_1(X)} P_x$. Hereafter, $M(X)$ is said to be an *LFC*-system on $\mathbb{R}^2$, denoted by $(\mathbb{R}^2, M(X))$, inherited from the given *LFC*-space $(\mathbb{R}^2, C)$.

We observe that $M_1(X)$ is a neighborhood system of U^X in which each point $x(\in U^X)$ has only one neighborhood of x [5].

Indeed, the *LFC*-system of Definition 3.5 is equivalent to a family of sets derived from the *LFC*-space (U, C), where $U = \mathbb{R}^2$ or U is a compact plane or a big bounded subset of $\mathbb{R}^2$ containing the given target set X. In this case we can represent (3.6) as $M(X) = \{U \setminus U^X\} \cup M_1(X)$ [7]. Since each point $x \in U$ belongs to one of the element of $M_1(X)$, say $P_x, x \in U^X$ or $U \setminus U^X, x \in U \setminus U^X$ as in (3.6), we observe that the *LFC*-system $(U, M(X))$ induces a topological neighborhood system of each point $x \in U$ depending on the point x, as follows (for the details, see Remark 3.1 of [7]);

$$\mathcal{N}(x) := \{P_x\} \text{ or } \{U \setminus U^X\}. \tag{3.7}$$

Both the neighborhood system of (3.7) and the *LFC*-system $(\mathbb{R}^2, M(X))$ or $(U, M(X))$ play important roles in developing the quasi-discrete (or clopen) topology on $X(\subseteq \mathbb{R}^2)$ derived from the topology generated by the set $M(X)$ as a base [5]. More precisely, given the *LFC*-space $(\mathbb{R}^2, C)$, consider a set $X(\subseteq \mathbb{R}^2)$. Let $T_{M(X)}$ be the topology on $\mathbb{R}^2$ generated by the set $M(X)$ as a base. Then, $(\mathbb{R}^2, T_{M(X)})$ is a clopen (or quasi-discrete) topological space [5]. In view of the neighborhood system of (3.7), to sum up, we have the following:

Remark 3.6. (1) For a set $X(\subseteq \mathbb{R}^2)$ of Definition 3.5, as a further explanation of U mentioned above, we may consider a universe U as $\mathbb{R}^2$ or a compact subset $[l_1, l_1'] \times [l_2, l_2'](\supsetneq U^X)$ or $(l_1, l_1') \times (l_2, l_2')(\supsetneq U^X)$, and so on, where $l_1, l_1', l_2, l_2' \in \mathbb{Z}$.

(2) Unlike the neighborhood system established in (3.7), in U^X we may have another *LFC*-system of $x \in U^X$, denoted by $(U^X, M_1(X))$, and further, with the *LFC*-system $(U^X, M_1(X))$ we obtain a neighborhood system of $x \in U^X$ in the following way:

$$\mathcal{N}(x) := \{P_x\}.$$

Based on the neighborhood system $(U, M(X))$, we introduce the following lower and upper approximations for the *LFC*-space (U, C).

Definition 3.7. ([5]) Given an *LFC*-space (U, C), consider a set $X(\subseteq U)$. We define

$$M_*(X) = \bigcup_{P_x \subseteq X} P_x \text{ and } M^*(X) = \cup\{P_x \,|\, P_x \cap X \neq \emptyset\}. \tag{3.8}$$

If $M_*(X) \neq M^*(X)$, then we say that the pair $(M_*(X), M^*(X))$ is an M-rough set *w.r.t.* the *LFC*-space (U, C).

With an *LFC*-space (U, C), for a set $X(\subseteq U)$, it is obvious that $M_*(X) \subseteq X \subseteq M^*(X)$ [5]. Motivated by the typical Pawlak's rough set theory, we call $M_*(X)$ (*resp.* $M^*(X)$) the lower approximation (*resp.* the upper approximation) of the set X *w.r.t.* the *LFC*-space (U, C).

Definition 3.8. ([5]) With an *LFC*-space (U, C), for a set $X(\subseteq U)$, we define the set

$$D_M^-(X) = \{x \in \mathbb{Z}^2 \,|\, P_x \subseteq X\} \tag{3.9}$$

as a subspace induced by the clopen topological space $(U, T_{M(X)})$.

Definition 3.9. ([5]) With an *LFC*-space (U, C), for a set $X(\subseteq U)$, we define the set

$$D_M^+(X) = \{x \in \mathbb{Z}^2 \,|\, P_x \in M_1(X)\} \tag{3.10}$$

as a subspace induced by the clopen topological space $(U, T_{M(X)})$.

In view of Definitions 3.8 and 3.9, we observe that the subspaces $D_M^-(X)$ and $D_M^+(X)$ of (3.9) and (3.10) are obviously discrete topological spaces [5].

Definition 3.10. ([5]) Consider an *LFC*-space (U, C) and a set $X(\subseteq U)$. We say that the pair $(D_M^-(X), D_M^+(X))$ is an *MW*-topological rough set *w.r.t.* the *LFC*-space if $D_M^-(X) \neq D_M^+(X)$.

For the *LFC*-space (U, C) of Proposition 3.4 and $X \subseteq U$, we have the following [5]:

$$D_M^-(X) \subseteq X \cap \mathbb{Z}^2 \subseteq D_M^+(X).$$

4. Characterizations of the *MW*-Topological Rough Approximation Operators

Let us now explore certain properties of the concept approximations (D_M^-, D_M^+).

Lemma 4.1. *For an LFC-space (U, C) and subsets X, Y of U with $X \subseteq Y$,*

$$D_M^-(X) \subseteq D_M^-(Y) \text{ and } D_M^+(X) \subseteq D_M^+(Y). \tag{4.1}$$

Proof. Due to the property (3.5) and Definition 3.7, for $X \subseteq Y \subseteq U$, we obtain

$$M_*(X) \subseteq M_*(Y) \text{ and } M^*(X) \subseteq M^*(Y). \tag{4.2}$$

Further, due to Definitions 3.7, 3.8, and 3.9, we obtain

$$\begin{cases} (1)\, M_*(X) \cap \mathbb{Z}^2 = D_M^-(X), \text{ and} \\ (2)\, M^*(X) \cap \mathbb{Z}^2 = D_M^+(X). \end{cases} \tag{4.3}$$

Due to the properties (4.2) and (4.3), the proof is completed. □

In view of the property (4.1), we can represent the *MW*-topological rough set operators D_M^-, D_M^+, as follows: The functions

$$D_M^-, D_M^+ : P(U) \to P(\mathbb{Z}^2)$$

are defined as

$$D_M^-(X) = M_*(X) \cap \mathbb{Z}^2 \text{ and } D_M^+(X) = M^*(X) \cap \mathbb{Z}^2$$

w.r.t. the *LFC*-space (U, C).

Let us now investigate various properties characterizing the concept approximations (D_M^-, D_M^+). According to Definitions 3.7, 3.8 and 3.9, and Lemma 4.1, we now list the properties of the concept approximations (D_M^-, D_M^+) which are of interest in the rough set theory for *LFC*-spaces. Indeed, the following properties play essential roles in *MW*-topological rough set theory.

Proposition 4.2. *For an LFC-space (U, C) and sets $X, Y(\subseteq U)$, the following hold:*
(1) $D_M^-(X) \subseteq X \cap \mathbb{Z}^2 \subseteq D_M^+(X)$.
(2) $D_M^-(\emptyset) = D_M^+(\emptyset) = \emptyset$ and $D_M^-(U) = D_M^+(U) = \mathbb{Z}^2 \cap U$.
(3) $D_M^+(X \cup Y) = D_M^+(X) \cup D_M^+(Y)$.
(4) $D_M^-(X \cup Y) \supseteq D_M^-(X) \cup D_M^-(Y)$.
(5) $D_M^-(X \cap Y) = D_M^-(X) \cap D_M^-(Y)$.
(6) $D_M^-(X) \subseteq D_M^-(Y)$ and $D_M^+(X) \subseteq D_M^+(Y)$ whenever $X \subseteq Y$.
(7) $D_M^+(X \cap Y) \subseteq D_M^+(X) \cap D_M^+(Y)$.
(8) $D_M^-(X^C) = (D_M^+(X))^C$ (duality) and $D_M^+(X^C) = (D_M^-(X))^C$.
(9) $D_M^-(D_M^-(X)) \neq D_M^-(X)$ and $D_M^+(D_M^-(X)) = D_M^-(X)$.
(10) $D_M^+(D_M^+(X)) = D_M^+(X) \neq D_M^-(D_M^+(X))$.

Proof. In view of Definitions 3.5, 3.8, 3.9, and Lemma 4.1, the proofs of (1), (2), and (6) are straightforward. Thus it suffices to prove the properties (3), (4), (5), (7), (8), (9), and (10).

(3) Due to the properties (4.1), (4.2) and (4.3), we clearly have the inclusion $D^+_M(X) \cup D^+_M(Y) \subseteq D^+_M(X \cup Y)$. Next, we now prove $D^+_M(X \cup Y) \subseteq D^+_M(X) \cup D^+_M(Y)$. Assume an arbitrary element $t \in D^+_M(X \cup Y) = M^*(X \cup Y) \cap \mathbb{Z}^2$. Then, there is $P_t \in M_1(X \cup Y)$ with $t \in \mathbb{Z}^2$ so that

$$\begin{cases} P_t \cap (X \cup Y) \neq \emptyset \\ \Leftrightarrow (P_t \cap X) \cup (P_t \cap Y) \neq \emptyset \\ \Leftrightarrow (P_t \cap X) \neq \emptyset \text{ or } (P_t \cap Y) \neq \emptyset \\ \Leftrightarrow t \in M^*(X) \text{ or } t \in M^*(Y) \\ \Leftrightarrow t \in M^*(X) \cap \mathbb{Z}^2 \text{ or } t \in M^*(Y) \cap \mathbb{Z}^2 \\ \Leftrightarrow t \in D^+_M(X) \cup D^+_M(Y), \end{cases}$$

which completes the proof.

(4) Due to the property (4.1), we obviously have the inclusion

$$D^-_M(X \cup Y) \supseteq D^-_M(X) \cup D^-_M(Y).$$

Next, we prove that $D^-_M(X \cup Y)$ need not be a subset of $D^-_M(X) \cup D^-_M(Y)$. For instance, with an *LFC*-space (U, C), where $U^V, U^W \subset U$(see Remark 3.6), let

$$\begin{cases} V = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \lneq 1\}, \text{ and} \\ W = \{(x, y) \in \mathbb{R}^2 : |x| + |y| = 1\} \cup \{(x, y) \in \mathbb{R}^2 : |x-1| + |y-1| \leq 1\}. \end{cases}$$

Whereas

$$D^-_M(V \cup W) = \{(0, 0), (1, 1)\}$$

we obtain

$$D^-_M(V) = \emptyset \text{ and } D^-_M(W) = \{(1, 1)\},$$

which completes the proof.

(5) Let us firstly prove the inclusion $D^-_M(X) \cap D^-_M(Y) \subseteq D^-_M(X \cap Y)$. Namely, take any element $t \in D^-_M(X) \cap D^-_M(Y)$. Then, we obtain

$$\begin{cases} t \in D^-_M(X) = M_*(X) \cap \mathbb{Z}^2 \text{ and } t \in D^-_M(Y) \\ \Rightarrow t \in (X \cap Y) \cap \mathbb{Z}^2 \text{ such that } t \in P_t \subseteq X \cap Y \\ \Rightarrow t \in M_*(X \cap Y) \cap \mathbb{Z}^2 \\ \Rightarrow t \in D^-_M(X \cap Y), \end{cases}$$

which completes the proof.

The proof of the converse is straightforward.

(7) Due to the property (4.1), it suffices to prove that $D^+_M(X) \cap D^+_M(Y)$ need not be a subset of $D^+_M(X \cap Y)$. To be precise, we prove that not every $t \in D^+_M(X) \cap D^+_M(Y)$ satisfies $t \in D^+_M(X \cap Y)$. For instance, with an *LFC*-space (U, C), let

$$\begin{cases} X = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 2\} \text{ and} \\ Y = \{(x, y) \in \mathbb{R}^2 : |x-2| + |y-2| \lneq 2\}. \end{cases}$$

According to the property (3.10), we obtain

$$\begin{cases} D^+_M(X) = [-1, 1]^2_{\mathbb{Z}} \cup \{(\pm 2, 0), (0, \pm 2)\} \text{ and} \\ D^+_M(Y) = [1, 3]^2_{\mathbb{Z}} \cup \{(0, 2), (4, 2), (2, 0), (2, 4)\}. \end{cases}$$

Since $X \cap Y = \emptyset$, whereas $D^+_M(X \cap Y) = \emptyset$, we have

$$D^+_M(X) \cap D^+_M(Y) = \{(0,2),(1,1),(2,0)\} \neq \emptyset,$$

which completes the proof.

(8) Based on an LFC-space (U, C) and any set $X(\subseteq U)$, take an arbitrary element $t \in D^-_M(X^C) = M_*(X^C) \cap \mathbb{Z}^2$. Then, consider the following partition of U (see the property (3.6)),

$$\begin{cases} \text{in the case } U^X \neq U,\ \{P_x, U \setminus U^X \mid P_x \in M_1(X), U^X = \bigcup_{P_x \in M_1(X)} P_x\}, \text{and} \\ \text{in the case } U^X = U,\ \{P_x \mid P_x \in M_1(X)\}. \end{cases}$$

According to Definition 3.5 and the above partition of U (see the property (3.6)), we observe $U \setminus U^X = \bigcup_{P_x \subset (U^X)^C} P_x$. Therefore, by Definitions 3.5 and 3.7, we have the following:

$$\begin{cases} \text{(Case 1) in the case } U^X \neq U, \\ \forall x \in M_*(X^C) \Leftrightarrow x \notin \bigcup_{P_x \in M_1(X)} P_x \Leftrightarrow \forall x \in [M^*(X)]^C. \\ \text{(Case 2) in the case } U^X = U, \text{ according to Definition 3.7,} \\ M_*(X^C) = \emptyset \text{ and } M^*(X) = U. \end{cases}$$

Thus, we have

$$M_*(X^C) = [M^*(X)]^C. \tag{4.4}$$

Hence, according to the property (4.4), we obtain the following:

$$t \in M_*(X^C) \cap \mathbb{Z}^2 \Leftrightarrow t \notin M^*(X) \cap \mathbb{Z}^2 \Leftrightarrow t \in [M^*(X) \cap \mathbb{Z}^2]^C. \tag{4.5}$$

We can represent the property (4.5), as follows:

$$t \in D^-_M(X^C) \Leftrightarrow t \notin D^+_M(X) \Leftrightarrow t \in (D^+_M(X))^C,$$

which completes the proof.

(9) According to the property (4.3), although $D^-_M(X)$ need not be an empty set, $D^-_M(D^-_M(X))$ is an empty set. Next, according to Definition 3.9, we always have $D^+_M(D^-_M(X)) = D^-_M(X)$.

(10) According to Definition 3.9, we obtain $D^+_M(D^+_M(X)) = D^+_M(X)$. However, according to Definition 3.8, $D^-_M(D^+_M(X))$ is always an empty set. □

Owing to the properties (1), (2), (9), and (10) of Proposition 4.2, we obtain the following:

Corollary 4.3. *(1) D^-_M is not an interior operator.*
(2) D^+_M is not a closure operator.

5. Further Properties of an *M*-Rough Set and an *MW*-Topological Rough Set

For the LFC-space (U, C) (see Remark 3.6), it is obvious that for two sets $X, Y \subset U$ the two identities $M^*(X) = M^*(Y), M_*(X) = M_*(Y)$ need not imply $X' = Y'$, where $X' = X \cap \mathbb{Z}^2$ and $Y' = Y \cap \mathbb{Z}^2$.

Further, the nonidentity $X \neq Y$ need not imply

$$M^*(X) \neq M^*(Y), M_*(X) \neq M_*(Y).$$

According to Definition 3.7, we obtain the following:

Theorem 5.1. *Consider the LFC-space* $(\mathbb{R}^2, C)$ *and sets* X *and* Y *of* $\mathbb{R}^2$ *such that* $X \cap \mathbb{Z}^2 = Y \cap \mathbb{Z}^2$. *Then* $M_*(X)$ *(resp.* $M^*(X)$*) need not be equal to* $M_*(Y)$ *(resp.* $M^*(Y)$*).*

Proof. As an example, with the *LFC*-space $(\mathbb{R}^2, C)$ in (3.4), consider two sets $X := [-2.3, 2.8] \times [0.1, 2.4]$ and $Y := [-2.7, 2.3] \times [0.2, 2.3]$ in Fig.1. Whereas $X \cap \mathbb{Z}^2 = Y \cap \mathbb{Z}^2 = [-2, 2]_{\mathbb{Z}} \times [1, 2]_{\mathbb{Z}}$, we obtain

$$\begin{cases} M_*(X) = [-1.5, 2.5) \times [0.5, 1.5] \setminus X_1, \\ \text{where } X_1 := [(-0.5, 0.5) \cup (1.5, 2.5)] \times \{0.5, 1.5\} \text{ (see (a) of Fig.1)}, \\ M^*(X) = [-2.5, 3.5] \times [-0.5, 2.5] \setminus X_2 \\ \text{where } X_2 := [[(-1.5, -0.5) \cup (0.5, 1.5) \cup (2.5, 3.5)] \times \{-0.5, 2.5\}] \\ \cup [\{-2.5\} \times (0.5, 1.5)] \cup [\{3.5\} \times (-0.5, 0.5) \cup (1.5, 2.5)] \text{ (see (b) of Fig.1)}, \\ M_*(Y) = (-2.5, 1.5] \times [0.5, 1.5] \setminus Y_1, \\ \text{where } Y_1 := [(-2.5, -1.5) \cup (-0.5, 0.5)] \times \{0.5, 1.5\} \text{ (see (c) of Fig.1), and,} \\ M^*(Y) = [-3.5, 2.5] \times [-0.5, 2.5] \setminus Y_2, \text{ where} \\ \text{where } Y_2 := ([(-3.5, -2.5) \cup (-1.5, -0.5) \cup (0.5, 1.5)] \times \{-0.5, 2.5\}) \\ \cup (\{-3.5\} \times [(-0.5, 0.5) \cup (1.5, 2.5)]) \cup [\{2.5\} \times (0.5, 1.5)] \text{ (see (d) of Fig.1)}, \end{cases}$$

which completes the proof. □

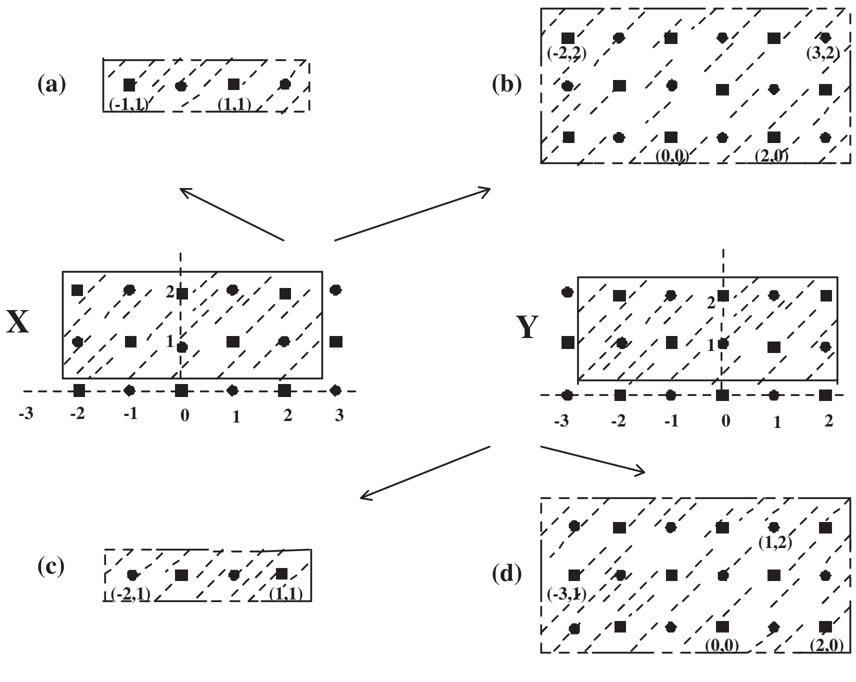

Figure 1: Configuration of the concept approximations of X and Y: (a) $M_*(X)$, (b) $M^*(X)$, (c) $M_*(Y)$, and (d) $M^*(Y)$.

In view of the property (4.3), based on Remark 3.6, we obtain the following:

Corollary 5.2. *Consider an LFC-space* (U, C) *and sets* X *and* Y *of* U *such that* $X \cap \mathbb{Z}^2 = Y \cap \mathbb{Z}^2$. *Then* $D^-_M(X)$ *(resp.* $D^+_M(X)$*) need not be equal to* $D^-_M(Y)$ *(resp.* $D^+_M(Y)$*).*

Motivated by Theorem 5.1, we may pose the following query. Under what conditions are the rough approximations with respect to X and Y equal in Theorem 5.1?

Theorem 5.3. *Let us consider two subsets* X, Y *in an LFC-space* (U, C) *such that* $X \cap \mathbb{Z}^2 = Y \cap \mathbb{Z}^2$ *and* $M_*(X) \neq \emptyset \neq M_*(Y)$. *If there are no elements* $t_1, t_2 \in \mathbb{Z}^2$ *such that* $P_{t_1} \subseteq X$ *and* $P_{t_1} \nsubseteq Y$, *and* $P_{t_2} \subseteq Y$ *and* $P_{t_2} \nsubseteq X$, *then* $M_*(X) = M_*(Y)$, *i.e.* $D_M^-(X)$ *is equal to* $D_M^-(Y)$. *The converse holds.*

Before proving this theorem, we need to mention the condition $M_*(X) \neq \emptyset \neq M_*(Y)$. In the case $M_*(X) = \emptyset$ or $M_*(Y) = \emptyset$, the assertion is trivial.

Proof. (⇒) With the hypothesis, in case $P_{t_1} \subseteq X$, we should have $P_{t_1} \subseteq Y$ and further, in the case $P_{t_1} \subseteq Y$, we are required to have $P_{t_1} \subseteq X$. Hence, it is obvious that if $P_{t_1} \subseteq X$, then $P_{t_1} \subseteq X \cap Y$ and further, if $P_{t_2} \subseteq Y$, then $P_{t_2} \subseteq X \cap Y$, which implies that $M_*(X) = M_*(Y)$. In addition, by the property (4.3), $D_M^-(X)$ is prove to be equal to $D_M^-(Y)$.

(⇐) With the hypothesis, consider an arbitrary element $t' \in M_*(X)(= M_*(Y))$. Then, there are elements t in $X \cap \mathbb{Z}^2$ and $Y \cap \mathbb{Z}^2$ such that $t' \in P_t \subseteq M_*(X \cap Y)$. Therefore, there are no elements $t_1, t_2 \in \mathbb{Z}^2$ such that $P_{t_1} \subseteq X$ and $P_{t_1} \nsubseteq Y$, and $P_{t_2} \subseteq Y$ and $P_{t_2} \nsubseteq X$. □

As referred to in Theorem 5.1, using a method similar to Theorem 5.3, we observe the following:

Remark 5.4. Consider the sets X and Y in the *LFC*-space $(\mathbb{R}^2, C)$ in Fig.1. Whereas $X \cap \mathbb{Z}^2 = Y \cap \mathbb{Z}^2$, comparing $M^*(X)$(see Fig.1(b)) and $M^*(Y)$(see Fig.1(d)), we obtain

$$D_M^+(X) \neq D_M^+(Y).$$

Motivated by Remark 5.4, let us now explore the condition making the rough approximations $M^*(X)$ and $M^*(Y)$ with respect to X and Y equal in Theorem 5.1.

Theorem 5.5. *Let us consider two subsets* X, Y *in an LFC-space* (U, C) *such that* $X \cap \mathbb{Z}^2 = Y \cap \mathbb{Z}^2$. *Let* $B = \{t \in \mathbb{Z}^2 \,|\, P_t \cap X \neq \emptyset\} \neq \emptyset$ *and* $C = \{t \in \mathbb{Z}^2 \,|\, P_t \cap Y \neq \emptyset\} \neq \emptyset$. *Then we obtain the following:*
$B = C$ *if and only if* $M^*(X) = M^*(Y)$, *i.e.* $D_M^+(X)$ *is equal to* $D_M^+(Y)$

Before proving this theorem, we strongly need to recall that the universe U of this theorem is the set $\mathbb{R}^2$ or a compact subset of $\mathbb{R}^2$ containing the sets U^X and U^Y of Definition 3.5 (see Remark 3.6). In particular, in case $B = \emptyset = C$, the assertion is trivially proved.

Proof. If $B = C$, we obviously obtain the identity $M^*(X) = M^*(Y)$.

Conversely, with the hypothesis, consider any element $z \in M^*(X)(= M^*(Y))$. Then, since each of $M^*(X)$ and $M^*(Y)$ is not empty, there is an element $w \in \mathbb{Z}^2$ such that $P_w \cap X \neq \emptyset$ and $P_w \cap Y \neq \emptyset$. Further, for any element $w \in B$ we obtain $w \in C$ and further, for any element $w \in C$ we obtain $w \in B$, which implies $B = C$. □

6. Conclusions and a Further Work

We have explored theoretical properties of *MW*-digital topological rough set structures (or *MW*-rough topological concept approximations). We can use these results in the fields of pattern recognition, image classifications, and so on. Particularly, it turns out that the function D_M^- is not an interior operator and D_M^+ is not a closure operator either, which can characterize the *MW*-digital topological rough concept approximations. Based on this approach, we can try to establish certain new *LFC*-rough set structures in $\mathbb{R}^n, n \in \mathbb{N}$ with the following issues.

- Accuracies of various types of *LFC*-rough sets.
- Extensions of the utilities of *LFC*-rough sets.
- Developing new topologies on $\mathbb{Z}^n$ or $\mathbb{R}^n$ with a locally finite topology.
- After developing efficient rough set structures, we can apply them to the field of object classifications from the viewpoint mathematical morphology.

In addition, based on the *MW*-rough topological concept approximations, various types of rough membership functions can be considered. As a further work, based on the obtained results in this paper, we can further study some of the following: Topological data analysis, geographical modeling, finger-print recognition, establishments of new types of rough set structures, and so forth.

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