Filomat 34:7 (2020), 2283–2292
https://doi.org/10.2298/FIL2007283A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Subordinating Results for Classes of Functions Defined by Sălăgean Type q Derivative Operator

Mohamed K. Aouf[a], **Adela O. Mostafa**[a]

[a]*Department of Mathematics, Faculty of Science, Mansoura 35516, Egypt*

Abstract. In this paper, we investigate several interesting some subordination results for classes of analytic functions defined by the Sălăgean type q-derivative operator.

1. Introduction

The class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k (z \in \mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}, \tag{1}$$

is denoted by $\mathfrak{A}$. Also, denote by $\mathcal{K}$ the subclass of functions $f \in \mathfrak{A}$ which are convex in $\mathbb{U}$. For functions f given by (1) and $g \in \mathfrak{A}$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g ($f \prec g$) if there exists an analytic function w, with $w(0) = 0$ and $|w(z)| < 1, z \in \mathbb{U}$, such that $f(z) = g(w(z))$. Furthermore, if g is univalent in $\mathbb{U}$, then (see [10] and [23]):

$$f(z) \prec g(z) \ \ (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f(z) \in \mathfrak{A}$ is in the class $\mathcal{UCV}(\alpha, \beta)$ of uniformly convex functions of order $\alpha(-1 \le \alpha < 1)$ and type $\beta \ge 0$ if it satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \ge \beta \left|\frac{zf''(z)}{f'(z)}\right|, \tag{2}$$

2010 *Mathematics Subject Classification*. 30C45
Keywords. Subordination, analytic functions, Sălăgean type-q-derivative operator.
Received: 01 July 2019; Revised: 01 January 2020; Accepted: 12 June 2020
Communicated by Hari M. Srivastava
Email addresses: mkaouf127@yahoo.com (Mohamed K. Aouf), adelaeg254@yahoo.com (Adela O. Mostafa)

and is in the corresponding class $\mathcal{SP}(\alpha,\beta)$ of uniformly starlike of order $\alpha(-1 \leq \alpha < 1)$ and type $\beta \geq 0$ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{3}$$

where the classes $\mathcal{S}P(\alpha,\beta)$ and $\mathcal{UCV}(\alpha,\beta)$ were introduced and studied by ([2], [9], [17] and [29]).

From (2) and (3), we have

$$f(z) \in \mathcal{UCV}(\alpha,\beta) \Leftrightarrow zf'(z) \in \mathcal{S}P(\alpha,\beta).$$

We note that:

i) $\mathcal{UCV}(0,1) = \mathcal{UCV}$ is the class of uniformly convex functions introduced and studied by Goodman [13];

ii) $\mathcal{UCV}(\alpha,1) = \mathcal{UCV}(\alpha)$, $\mathcal{S}P(\alpha,1) = \mathcal{S}P(\alpha)$ and $SP(0,1) = SP$ (see [25]);

iii) $\mathcal{UCV}(0,\beta) = \beta - \mathcal{UCV}$ and $\mathcal{S}P(0,\beta) = \beta - \mathcal{S}P$ (see [16], [18] and [19]).

One of the most common applications in number theory, especially in the theory of partitions is using the basic (or $q-$) series or polynomials has already observed in the monograph by Srivastava and Karlsson [35, pp. 350-351] and, more recently, in a survey-cum-expository review article by Srivastava [31] on the widespread usages of the q-analysis including in geometric function theory of complex analysis, our investigation here is believed to present another advance in the subject of the $(q-)$ calculus.

We now present a brief expository overview of the classical $(q-)$ analysis and the Salagean operator which will be used in this paper.

Definition 1.1. . *For $q \in (0,1)$, the q-number $[i]_q$ is defined by*

$$[i]_q = \begin{cases} \dfrac{1-q^i}{1-q} & i \in \mathbb{C} \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 ... & i = n \in \mathcal{N} = \{1,2,...\}. \end{cases}$$

We see that $[i]_q = \dfrac{1-q^i}{1-q} \to i$ as $q \to 1-$.

Definition 1.2. . *(see [15], [1], ,[36] [39]) For $q \in (0,1)$, the q-derivative of $f \in \mathfrak{A}$, is given by (see [20] [21], [22], [27], [28], [30], [31], [32], [34], [36], [37], [38], [40] and [41])*

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & ,z \neq 0 \\ f'(0) & ,z = 0 \end{cases}. \tag{4}$$

From Definitions 1 and 2, note that (see [36])

$$\lim_{q\to 1-} D_q f(z) = \lim_{q\to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1) and (4) that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}. \tag{5}$$

Definition 1.3. . *For $f \in \mathfrak{A}$, Govindaraj and Sivasubramanian [14] (see also [24]) defined the Salagean q-derivative operator by*

$$\begin{aligned} D_q^0 f(z) &= f(z), \\ D_q^1 f(z) &= zD_q f(z) \\ D_q^n f(z) &= zD_q(D_q^{n-1} f(z)), n \in \mathcal{N}. \end{aligned}$$

It is easy to have

$$D_q^n f(z) = z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k (n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}). \tag{6}$$

We see that

$$\lim_{q \to 1^-} D_q^n f(z) = D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k (n \in \mathcal{N}_0),$$

where the differential operator D^n was introduced and studied by Salagean [26] (see also Aouf and Srivastava [7]).

Definition 1.4. . *For* $-1 \leq \alpha < 1$. $\beta \geq 0, 0 < q < 1$, $n \in \mathcal{N}_0$, $f(z) \in \mathfrak{A}$ *of the form (1),* $z \in \mathbb{U}$, *let* $\mathcal{S}_n(\alpha, \beta, q)$ *be the subclass of* $\mathfrak{A}$ *consisting of functions satisfying*

$$\operatorname{Re}\left\{ \frac{z D_q(D_q^n f(z))}{D_q^n f(z)} - \alpha \right\} > \beta \left| \frac{z D_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right| \tag{7}$$

and $C_n(\alpha, \beta, q)$ *be the subclass of* $\mathfrak{A}$ *consisting of functions satisfying*

$$\operatorname{Re}\left\{ \frac{D_q\left(z D_q(D_q^n f(z))\right)}{D_q(D_q^n f(z))} - \alpha \right\} > \beta \left| \frac{D_q\left(z D_q(D_q^n f(z))\right)}{D_q(D_q^n f(z))} - 1 \right|. \tag{8}$$

It follows from (7) and (8) that

$$D_q^n f(z) \in C_n(\alpha, \beta, q) \Leftrightarrow z D_q(D_q^n f(z)) \in \mathcal{S}_n(\alpha, \beta, q). \tag{9}$$

Note that:

i) $\mathcal{S}_0(\alpha, \beta, q) = \mathcal{S}(\alpha, \beta, q) = \left\{ f \in \mathfrak{A} : \operatorname{Re}\left\{ \frac{z D_q f(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{z D_q(f(z))}{f(z)} - 1 \right| \right\}$

and

$$\mathcal{S}(\alpha, 0, q) = \mathcal{S}(\alpha, q) = \operatorname{Re}\left\{ \frac{z D_q f(z)}{f(z)} \right\} > \alpha (0 \leq \alpha < 1);$$

ii) $C_0(\alpha, \beta, q) = C(\alpha, \beta, q)$

$$= \left\{ f \in \mathfrak{A} : \operatorname{Re}\left\{ \frac{D_q(z D_q f(z))}{D_q f(z)} - \alpha \right\} > \beta \left| \frac{D_q(z D_q f(z))}{D_q f(z)} - 1 \right| \right\}$$

and

$$C(\alpha, 0, q) = C(\alpha, q) = \operatorname{Re}\left\{ \frac{D_q(z D_q f(z))}{D_q f(z)} \right\} > \alpha (0 \leq \alpha < 1);$$

iii) $\lim_{q \to 1-} \mathcal{S}_n(\alpha, \beta, q) = \mathcal{S}_n(\alpha, \beta)$

$$= \left\{ f \in \mathfrak{A} : \operatorname{Re}\left\{ \frac{z(D^n f(z))'}{D^n f(z)} - \alpha \right\} > \beta \left| \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right| \right\};$$

iv) $\lim_{q \to 1-} C_n(\alpha, \beta, q) = C_n(\alpha, \beta)$

$$= \left\{ f \in \mathfrak{A} : \operatorname{Re}\left\{ 1 + \frac{z(D^n f(z))''}{(D^n f(z))'} - \alpha \right\} > \beta \left| \frac{z(D^n f(z))''}{(D^n f(z))'} \right| \right\};$$

v) $\lim_{q \to 1-} \mathcal{S}(\alpha, \beta, q) = \mathcal{SP}(\alpha, \beta)$ and $\lim_{q \to 1-} C(\alpha, \beta, q) = \mathcal{UCV}(\alpha, \beta)$.

2. Main Results

Throughout this paper unless otherwise mentioned, we assume that $-1 \leq \alpha < 1, \beta \geq 0, q \in (0,1)$, $n \in \mathcal{N}_0$, $f(z) \in \mathfrak{A}$ of the form (1) and $z \in \mathbb{U}$.

To prove our main result we need the following definition and lemma.

Definition 2.1. *[42] . The sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1) analytic, univalent and convex in $\mathbb{U}$, we have*

$$\sum_{k=1}^{\infty} c_k a_k z^k \prec f(z) \;\; (a_1 = 1).$$

Lemma 2.2. *[42]. The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0.$$

We now prove the following lemmas.

Lemma 2.3. *. If $f(z)$ satisfies the following inequality*

$$\sum_{k=2}^{\infty} \left[[k]_q(1+\beta) - (\alpha+\beta)\right][k]_q^n a_k \leq 1 - \alpha, \tag{10}$$

then, $f(z) \in \mathcal{S}_n(\alpha, \beta, q)$.

Proof. Making use of (10), it is suffices to prove that

$$\beta\left|\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right| - \text{Re}\left\{\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right\} < 1 - \alpha.$$

We have

$$\begin{aligned} &\beta\left|\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right| - \text{Re}\left\{\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right\} \\ \leq\;& (1+\beta)\left|\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right| = (1+\beta)\left|\frac{\sum\limits_{k=2}^{\infty}([k]_q - 1)[k]_q^n a_k z^{k-1}}{1 + \sum\limits_{k=2}^{\infty}[k]_q^n a_k z^{k-1}}\right| \\ \leq\;& (1+\beta)\frac{\sum\limits_{k=2}^{\infty}([k]_q - 1)[k]_q^n |a_k| |z|^{k-1}}{1 - \sum\limits_{k=2}^{\infty}[k]_q^n |a_k| |z|^{k-1}} < (1+\beta)\frac{\sum\limits_{k=2}^{\infty}([k]_q - 1)[k]_q^n |a_k|}{1 - \sum\limits_{k=2}^{\infty}[k]_q^n |a_k|}. \end{aligned}$$

The last expression is bounded by $1 - \alpha$ if the inequality (10) holds.

From (9) and Lemma 2, we have □

Lemma 2.4. *. A function $f(z) \in C_n(\alpha, \beta, q)$ if it satisfies the following inequality*

$$\sum_{k=2}^{\infty} [k]_q \left[[k]_q(1+\beta) - (\alpha+\beta)\right][k]_q^n a_k \leq 1 - \alpha. \tag{11}$$

Let $\mathcal{S}_n^*(\alpha,\beta,q)$ and $C_n^*(\alpha,\beta,q)$ be the subclasses of $\mathfrak{A}$ whose coefficients satisfy the conditions (10) and (11), respectively. We note that $\mathcal{S}_n^*(\alpha,\beta,q) \subset \mathcal{S}_n(\alpha,\beta,q)$ and $C_n^*(\alpha,\beta,q) \subset C_n(\alpha,\beta,q)$.

Imploying the technique of Attiya [8] and Srivastava and Attiya [33] also ([3], [4], [5], [6], and [11]), we obtain several subordination relations involving the function classes $\mathcal{S}_n^*(\alpha,\beta,q)$ and $C_n^*(\alpha,\beta,q)$.

Theorem 2.5. . *Let $f(z) \in \mathcal{S}_n^*(\alpha,\beta,q)$ and $g(z) \in \mathcal{K}$. Then*

$$\left(\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}\right)(f*g)(z) \prec g(z) \tag{12}$$

and

$$\operatorname{Re}(f(z)) > -\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}. \tag{13}$$

The constant factor $\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}$ in (12) cannot be replaced by a large one.

Proof. Let $f(z) \in \mathcal{S}_n^*(\alpha,\beta,q)$ and $g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{K}$. Then we have

$$\begin{aligned} &\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}(f*g)(z) \\ = \; &\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}(z+\sum_{k=2}^{\infty} c_k a_k z^k). \end{aligned} \tag{14}$$

Thus by Definition 1, the subordination result (12) will hold true if the sequence

$$\left\{\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}a_k\right\}_{k=1}^{\infty} \tag{15}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}a_k z^k\right\} > 0. \tag{16}$$

Since

$$\Phi(k) = [[k]_q(1+\beta)-(\alpha+\beta)][k]_q^n (k \geq 2; \beta \geq 0; -1 \leq \alpha < 1, n \in \mathcal{N}_0; 0 < q < 1)$$

is an increasing function of k, then, when $|z| = r < 1$, we have

$$\begin{aligned}
& \mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}a_k z^k\right\} \\
= \; & \mathrm{Re}\left\{1+\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}z\right. \\
+ \; & \left.\frac{\sum_{k=2}^{\infty}[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n a_k z^k}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}\right\} \\
\geq \; & 1-\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}r- \\
& \frac{\sum_{k=2}^{\infty}[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n a_k r^k}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)} \\
> \; & 1-\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}r-\frac{1-\alpha}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)}r \\
= \; & 1-r>0,
\end{aligned}$$

where we also used the assertion (10) of Lemma 2. Thus (16) holds in $\mathbb{U}$ and also, the subordination result (12) asserted by Theorem 1. The inequality (13) follows from (12) by taking the convex function $g(z) = z(1-z)^{-1} = z + \sum_{k=2}^{\infty} z^k$. To prove the sharpness of the constant

$$\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}},$$

we consider the function $f_0(z) \in \mathcal{S}_n^*(\alpha,\beta,q)$ given by

$$f_0(z) = z - \frac{1-\alpha}{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}z^2. \tag{17}$$

Thus, from (12), we have

$$\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}f_0(z) \prec \frac{z}{1-z}(z\in\mathbb{U}). \tag{18}$$

Moreover, it can easily be verivied for the function $f_0(z)$ that

$$\min_{|z|\leq r}\left\{\mathrm{Re}\left(\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}f_0(z)\right)\right\} = -\frac{1}{2}.$$

This shows that the constant

$$\frac{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][2]_q^n+(1-\alpha)\}}$$

is the best possible, which completes the proof.

Similarly, we can prove the following theorem for the class $C_n^*(\alpha,\beta,q)$.

Theorem 2.6. *. Let $f(z) \in \mathcal{C}_n^*(\alpha, \beta, q)$ and $g(z) \in \mathcal{K}$. Then*

$$\left(\frac{[[2]_q(1+\beta) - (\alpha+\beta)][2]_q[2]_q^n}{2\{[[2]_q(1+\beta) - (\alpha+\beta)][2]_q[2]_q^n + (1-\alpha)\}} \right) (f * g)(z) \prec g(z) \tag{19}$$

and

$$\mathrm{Re}(f(z)) > -\frac{[[2]_q(1+\beta) - (\alpha+\beta)][2]_q[2]_q^n + (1-\alpha)}{[[2]_q(1+\beta) - (\alpha+\beta)][2]_q[2]_q^n}. \tag{20}$$

The constant factor $\frac{[[2]_q(1+\beta) - (\alpha+\beta)][2]_q[2]_q^n}{2\{[[2]_q(1+\beta) - (\alpha+\beta)][2]_q[2]_q^n + (1-\alpha)\}}$ *in (19) cannot be replaced by a large one.*

□

Putting $n = 0$ in Theorems 1 and 2, respectively, we have

Corollary 2.7. *. Let $f(z) \in \mathcal{S}^*(\alpha, \beta, q)$ and satisfies the condition*

$$\sum_{k=2}^{\infty} [[k]_q(1+\beta) - (\alpha+\beta)] \, |a_k| \leq 1 - \alpha,$$

then

$$\left(\frac{[2]_q(1+\beta) - (\alpha+\beta)}{2\{[2]_q(1+\beta) - (\alpha+\beta) + (1-\alpha)\}} \right) (f * g)(z) \prec g(z), g \in \mathcal{K}$$

and

$$\mathrm{Re}(f(z)) > -\frac{[2]_q(1+\beta) - (\alpha+\beta) + (1-\alpha)}{[2]_q(1+\beta) - (\alpha+\beta)}.$$

The constant factor $\frac{[2]_q(1+\beta) - (\alpha+\beta)}{2\{[2]_q(1+\beta) - (\alpha+\beta) + (1-\alpha)\}}$ *is the best estimate.*

Corollary 2.8. *. Let $f(z) \in \mathcal{C}^*(\alpha, \beta, q)$ and satisfies the condition*

$$\sum_{k=2}^{\infty} [k]_q[[k]_q(1+\beta) - (\alpha+\beta)] \, |a_k| \leq 1 - \alpha,$$

then

$$\left(\frac{[2]_q[[2]_q(1+\beta) - (\alpha+\beta)]}{2\{[2]_q[[2]_q(1+\beta) - (\alpha+\beta)] + (1-\alpha)\}} \right) (f * g)(z) \prec g(z), g \in \mathcal{K}$$

and

$$\mathrm{Re}(f(z)) > -\frac{[2]_q[[2]_q(1+\beta) - (\alpha+\beta)] + (1-\alpha)}{[2]_q[[2]_q(1+\beta) - (\alpha+\beta)]}.$$

The constant factor $\frac{[2]_q[[2]_q(1+\beta) - (\alpha+\beta)]}{2\{[2]_q[[2]_q(1+\beta) - (\alpha+\beta)] + (1-\alpha)\}}$ *is the best estimate.*

Putting $\beta = 0$ in Corollaries 1 and 2, respectively, we have

Corollary 2.9. . *Let $f(z) \in \mathcal{S}^*(\alpha, q)$ and satisfies the condition*

$$\sum_{k=2}^{\infty}([k]_q - \alpha)\,|a_k| \leq 1 - \alpha,$$

then

$$\left(\frac{[2]_q - \alpha}{2([2]_q + 1 - 2\alpha)}\right)(f * g)(z) \prec g(z), g \in \mathcal{K}$$

and

$$\mathrm{Re}(f(z)) > -\frac{[2]_q + 1 - 2\alpha}{[2]_q - \alpha}.$$

The constant factor $\frac{[2]_q - \alpha}{2([2]_q + 1 - 2\alpha)}$ *is the best estimate.*

Corollary 2.10. . *Let $f(z) \in \mathcal{C}^*(\alpha, q)$ and satisfies the condition*

$$\sum_{k=2}^{\infty}[k]_q([k]_q - \alpha)\,|a_k| \leq 1 - \alpha,$$

then

$$\left(\frac{[2]q\left([2]_q - \alpha\right)}{2\{[2]_q([2]_q - \alpha) + 1 - \alpha\}}\right)(f * g)(z) \prec g(z), g \in \mathcal{K}$$

and

$$\mathrm{Re}(f(z)) > -\frac{2\{[2]_q([2]_q - \alpha) + 1 - \alpha\}}{[2]_q([2]_q - \alpha)}.$$

The constant factor $\frac{[2]_q\left([2]_q - \alpha\right)}{2\{[2]_q([2]_q - \alpha) + 1 - \alpha\}}$ *is the best estimate.*

Remark 2.11. . *I) Letting $q \to 1-$ in Theorems 1 and 2, respectively, we have the results obtained by Aouf and Mostafa [4, Corollaries 2.6 and 2.10, respectively];*

ii) Letting $q \to 1-$ in Corollies 1 and 2, respectively, we have the results obtained by Frasin [12, Corollaries, 2.2 and 2.5, respectively];

iii) Letting $q \to 1-$ in Corollies 3 and 4, respectively, we have the results obtained by Frasin [12, Corollaries, 2.3 and 2.6, respectively].

References

[1] M. H. Annaby Z. S. Mansour. q-Fractional Calculus and Equations, Lecture Notes in Math., 2056, Springer-Verlag Berlin Heidelberg, 2012.

[2] F. M. Al-Oboudi and K. A. Al-Amoudi, On classes of analytic functions related to conic domains, J. Math. Anal. Appl., 339(2008), 655–667 .

[3] M. K. Aouf, Subordination properties for certain class of analytic functions defined by Sălăgean operator, Appl. Math. Letters, 22 (2009), 1581-1585.

[4] M. K. Aouf and A. O. Mostafa, Some subordination results for classes of analytic functions defined by the Al-Oboudi–Al-Amoudi operator, Arch. Math. 92 (2009), 279–286.

[5] M. K. Aouf, A. Shamandy, A. O. , Mostafa and E. A. Adwan, Subordination results for certain class of analytic functions defined by convolution, Rend. del Circolo Math. di Palermo, (2011), no. 60, 255-262.
[6] M. K. Aouf, A. Shamandy, A. O. , Mostafa and E. A. Adwan, Subordination theorem of analytic functions defined by convolution, Complex Anal. Oper. Theory, (2013), no. 7, 111-1126.
[7] M. K. Aouf and H. M. Srivastava, Some families of starlike functions with negative coefficients, J. Math. Anal. Appl., 203(1996), 762-790.
[8] A. A. Attiya, On some application of a subordination theorems, J. Math. Anal. Appl., 311(2005), 489–494.
[9] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamakang J. Math., 28(1997), no. 1, 17-32.
[10] T. Bulboacă, Differential Subordinations and Superordnations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.
[11] S. Bulut and M. K. Aouf, Subordination properties for a certain class of analytic functions of complex order, Le Matimatiche. , 69(2014), 117-128.
[12] B. A. Frasin, Subordination results for a class of analytic functions defined by a linear operator, J. Inequal. Pure Appl. Math., 7(2006), no. 4, Art. 134, 1-7.
[13] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math., 56 (1991), 87-92.
[14] M. Govindaraj, and S. Sivasubramanian, On a class of analytic function related to conic domains involving q-calculus, Analysis Math., 43(2017), no. 3, 475–487.
[15] F. H. Jackson, On q-definite integrals, Quart. J. Pure Appl. Math. 41 (1910), 193–203.
[16] S. Kanas, Coefficient estimates in subclasses of the Caratheodory class related to conic domains, Acta Math. Univ. Comenian. (N. S.), 74(2005), 149-161.
[17] S. Kanas and D. Răducanu, Some class of af analytic functions related to conic domains, Math. Solvaca, 64(2014), no. 5, 1183-1196.
[18] S. Kanas and H. M. Srivastava, Linear operators associated with k-uniformly convex functions, Integral Transforms Spec. Funct., 9(2000), 121-132.
[19] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, J. Comput. Appl. Math., 105(1999), 327-336.
[20] Q. Khan, M. Arif, M. Raza, G. Srivastava, H. Tang, S. U. Rehman and B. Ahmad, Some applications of a new integral operator in q-analog for multivalent functions, Mathematics, 7 (2019), Article ID 1178, 1-13.
[21] S. Mahmood, M. Jabeen, S. N. Malik, H. M. Srivastava, R. Manzoor and S. M. Jawwad Riaz, Some coefficient inequalities of q-starlike functions associated with conic domain defined by q-derivative, Journal of Function Spaces, 2018(2018), Article ID 8492072, 1-13.
[22] S. Mahmood, N. Raza, E. S. A. Abujarad, G. Srivastava , H. M. Srivastava and S. N. Malik, Geometric properties of certain classes of analytic Functions Associated with a q-Integral Operator, Symmetry 11 (2019), Article ID 719, 1-14.
[23] S.S. Miller and P.T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.
[24] G. Murugusundaramoothy and K. Vijaya, Subclasses of bi-univalent functions defined by Sălăgean type q-difference operator, arXiv:1710.00143v1 [math. CV] 30 Sept. 2017.
[25] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Soc., 118 (1993), 189-196.
[26] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag) 1013, 362–372 (1983).
[27] T. M. Seoudy and M. K. Aouf, Convolution properties for certain classes of analytic functions defined by q-derivative operator, Abstract Appl. Anal., Vol 2014, Art. ID 846719, 1-7.
[28] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, J. Math. Inequal., 10(2016), no. 1, 135-145.
[29] S. Shams, S. R. Kulkarni and J. M. Jahangiri, Classes of uniformly starlike and convex functions, Internat. J. Math. Math. Sci., 55(2004), 2959-2961.
[30] L. Shi, Q. Khan, G. Srivastava, J.-L. Liu and M. Arif, A study of multivalent q-starlike functions connected with circular domain, Mathematics, 7 (2019), Article ID 670, 1-12
[31] H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol Trans. Sci., 44(2020), 327–344.
[32] H. M. Srivastava, S. Altinkaya and S. Yalcin, Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q-derivative operator, Filomat, 32 (2018), no. 2, 503–516.
[33] H. M. Srivastava and A. A. Attiya, Some subordination results associated with certain subclasses of analytic functions, J. Inequal. Pure Appl. Math., 5 (2004), Art. 82, 1–6.
[34] H. M. Srivastava and D. Bansal, Close to convexity of a certain family of q-Mittag-Leffler functions, J. Nonlinear Var. Anal., 1(2017), no. 1, 61-69.
[35] H. M. Srivastava and Karlsson, Multiple Gaussian hypergeometric series. Wiley, New York, 1985.
[36] H. M Srivastava, B. Khan, N. Khan and Q.Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, Hokkaido Mathematical Journal, 48(2019), no. 2, 407-425.
[37] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional q-calculus and associated Fekete-Szego problem for p-valently q–starlike functions and p-valently q-convex functions of complex order, Miskolc Mathematical Notes 20(2019), no. 1, 489-509.
[38] H. M. Srivastava, N. Raza, E. S. A. Abujarad, G. Srivastava and M. H. Abujarad, Fekete-Szego inequality for classes of (p, q)-Starlike and (p, q)-convex functions, Rev. Real Acad. Cienc: Natur. Ser. A Mat., 113(2019), 3563–3584.
[39] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad and N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, Symmetry, 11(2019), no. 2, ID 292, 1-14.
[40] H. M. Srivastava, M. Tahir, B. Khanc, Q. Z. Ahmad , N. Khan, Some general families of q-starlike functions associated with the Janowski functions, Filomat, 33(2019), no. 9, 2613–2626.

[41] H. Tang, Q. Khan, M. Arif, M. Raza, G. Srivastava, S. U. Rehman and B. Ahmad,Some applications of a new integral operator in q-analog for multivalent functions, Mathematics 7 (2019), Article ID 1178, 1-13.

[42] H. S. Wilf, Subordinating factor sequences for convex maps of the unit circle, Proc. Amer. Math. Soc. 12, 689–693 (1961).