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An Evolution Equation Governed by a Quasi-Nonexpansive Mapping on Hadamard Manifolds and Its Backward Discretization

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Abstract. We study the asymptotic behavior of solutions to a first-order evolution equation governed by a locally Lipschitz quasi-nonexpansive mapping. We show that such solutions converge to a fixed point of the quasi-nonexpansive mapping as time goes to infinity. Time discretization of this system provides an iterative method to approximate a fixed point of quasi-nonexpansive mappings on Hadamard manifolds.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, induced norm $\|\cdot\|$ and identity operator I. A set-valued operator $A : H \to 2^H$ with the domain $D(A)$, containing all $x \in H$ such that $Ax \neq \emptyset$, is called monotone if for any two points $x, y \in D(A)$:

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall x^* \in Ax,\ y^* \in Ay.$$

A monotone operator A is maximal if the graph of A is not contained in the graph of any other monotone operator. The zero set of A, i.e. $\{x \in D(A) : 0 \in Ax\}$, is denoted by $A^{-1}(0)$.

$T : H \to H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

The function $\varphi : H \to (-\infty, +\infty]$ is called convex if

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y), \quad \forall x, y \in H,\ \forall\lambda \in (0,1).$$

We say φ is proper if there is $x \in H$ with $\varphi(x) < +\infty$. The function φ is lower semicontinuous at x if $\liminf_{y\to x}\varphi(y) \geq \varphi(x)$. The subdifferential of φ at x is defined by $\partial\varphi(x) := \{w \in H : \varphi(y) - \varphi(x) \geq \langle w, y - x\rangle,\ \forall y \in H\}$. The subdifferential of a proper, convex and lower semicontinuous function and $(I - T)$, where T is a nonexpansive mapping, are two important examples of maximal monotone operators. The zero sets of these operators are respectively the set of all minimizers of the convex function and the set of all fixed points of the mapping T.

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First-order evolution equations of the form

$$\begin{cases} -\dot{u}(t) \in Au(t), \\ u(0) = x \in \overline{D(A)}, \end{cases} \tag{1}$$

where A is a maximal monotone operator, were studied extensively in Hilbert and Banach spaces settings. These abstract equations model several concrete partial differential equations in mathematical physics and other disciplines in applied mathematics. If the maximal monotone operator has a zero, then the solutions of these equations (at least their averages) converge to a zero of the maximal monotone operator [5] (see also [14]). When the maximal monotone operator is in the form of $(I - T)$, where T is a nonexpansive mapping with a nonempty set of fixed points, or it is the subdifferential of a proper, convex and lower semicontinuous function with a minimizer, then solutions to (1) converge weakly toward a fixed point of T or a minimizer of the convex function, respectively. This fact was proved by Bruck [6]. The reader can consult [14] for more information about evolution equations of monotone type and the asymptotic behavior of their solutions.

The study of the asymptotic behavior of solutions to (1) was also extended to the cases that A is not a maximal monotone operator. In [10], the authors considered (1) when A is replaced with the nonmonotone operator $(I - T)$, where T is a Lipschitz quasi-nonexpansive mapping (see Definition 2.4), and they proved the weak and strong convergence of solutions to (1) toward a fixed point of the mapping T. For more results on the asymptotic behavior of solutions to (1) when A is not monotone see [7, 9].

In [1, 2, 15], the authors studied the asymptotic behavior of solutions to monotone type equations in Hadamard manifolds, which extends some classical results from Hilbert spaces to Hadamard manifolds. In this paper, we extend the main results of [10] to Hadamard manifolds. In the next section, some basic definitions and results of Riemannian geometry are presented which shall be needed in the sequel. In Section 3, we consider a counterpart of (1) in the setting of Hadamard manifolds, where $A = (I - T)$ and T is a locally Lipschitz quasi-nonexpansive mapping. We first establish the existence of the solutions to this system and then study the convergence of these solutions to a fixed point of the mapping T. Finally, in Section 4, a discrete version of the system is studied that provides an algorithm for approximating a fixed point of the quasi-nonexpansive mapping.

2. Some preliminaries on Hadamard manifolds

In this section, we recall some backgrounds on Riemannian manifolds from [8, 16].

Let M be a connected n-dimensional Riemannian manifold, with a Riemannian metric $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. For $p \in M$ the tangent space at p is denoted by T_pM and the tangent bundle of M is denoted by $TM = \bigcup_{p \in M} T_pM$. A vector field A is a mapping from M to TM which maps each point $p \in M$ to a vector $A(p) \in T_pM$. Let p and q be two points in M and $\gamma : [a, b] \to M$ be a piecewise smooth curve joining p to q. The length of γ is defined by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt,$$

and the Riemannian distance $d(p, q)$ is defined by

$$d(p, q) = \inf L(\gamma),$$

where the infimum is taken over all piecewise smooth curves $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. Notice that this distance induces the original topology on M.

Let ∇ be the Levi-Civita connection on M associated with the Riemannian metric $\langle \cdot, \cdot \rangle$, and γ be a smooth curve in M. A vector field A is said to be parallel along γ if $\nabla_{\dot{\gamma}} A = 0$. A smooth curve γ is a geodesic if $\dot{\gamma}$ itself is parallel along γ. If γ is a geodesic, then $\|\dot{\gamma}\|$ is constant. When $\|\dot{\gamma}\| = 1$, γ is said to be normalized.

A geodesic joining p to q in M is called minimal if its length is equal to $d(p,q)$. We use $P_{\gamma,.,.}$ to denote the parallel transport on the tangent bundle TM along γ with respect to ∇, which is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(v) = V(\gamma(b)) \qquad \forall a,b \in \mathbb{R}, \ \forall v \in T_{\gamma(a)}M,$$

where V is the unique vector field satisfying $\nabla_{\dot\gamma(t)}V = 0$ for all t and $V(\gamma(a)) = v$. For any $a,b \in \mathbb{R}$, the parallel transport $P_{\gamma,\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}M$ to $T_{\gamma(b)}M$. Note that for any $a,b,b_1,b_2 \in \mathbb{R}$, we have

$$P_{\gamma,\gamma(b_2),\gamma(b_1)} \circ P_{\gamma,\gamma(b_1),\gamma(a)} = P_{\gamma,\gamma(b_2),\gamma(a)}, \ \text{ and } \ P^{-1}_{\gamma,\gamma(b),\gamma(a)} = P_{\gamma,\gamma(a),\gamma(b)}.$$

A Riemannian manifold M is complete if for each $p \in M$ all geodesics emanating from p are defined on the whole of $\mathbb{R}$. If M is complete, then by the Hopf-Rinow theorem, any pair of points in M can be joined by a minimal geodesic.

Let M be a connected and complete Riemannian manifold. The exponential map $\exp_p : T_pM \to M$ at p is defined by $\exp_p(v) = \gamma_v(1)$ for each $v \in T_pM$, where $\gamma_v(.)$ is the geodesic with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. Then $\exp_p(tv) = \gamma_v(t)$ for each real number t.

Throughout the paper, we assume that M is a complete, simply connected Riemannian manifold of non-positive sectional curvature of dimension n, which is called a Hadamard manifold of dimension n.

Proposition 2.1. ([16, p.221]) *Let $p \in M$. Then $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p,q \in M$ there exists a unique normalized geodesic joining p to q, which is in fact a minimal geodesic.*

An immediate consequence of Proposition 2.1 is that $d(p,q) = \| \exp_p^{-1} q\|$, for any two points $p,q \in M$.

By the definition, a geodesic triangle $\Delta(p_1p_2p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics joining these points.

Proposition 2.2. ([16, p.223])(Comparison theorem for triangles) *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle. Denote by $\gamma_i : [0,l_i] \to M$ the geodesic joining p_i to p_{i+1}, and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\dot\gamma_i(0), -\dot\gamma_{i-1}(l_{i-1}))$, where $i = 1,2,3$ (mod 3). Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant \pi,$$

$$l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \leqslant l_{i-1}^2. \tag{2}$$

Since

$$\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2}\rangle = d(p_i,p_{i+1})d(p_{i+1},p_{i+2})\cos\alpha_{i+1},$$

hence the inequality (2) may be rewritten as follows

$$d^2(p_i,p_{i+1}) + d^2(p_{i+1},p_{i+2}) - 2\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2}\rangle \leqslant d^2(p_{i+2},p_i), \tag{3}$$

equivalently, we have

$$\| \exp^{-1}_{p_{i+1}} p_i - \exp^{-1}_{p_{i+1}} p_{i+2}\| \leq d(p_i,p_{i+2}). \tag{4}$$

We now collect some definitions which extend some notions of the monotonicity, from the corresponding notions in Hilbert spaces (see [11, 13]), to (possibly multi-valued) vector fields on Hadamard manifolds. Let M be an n-dimensional Hadamard manifold and $\mathcal{X}(M)$ denote the set of all (possibly multi-valued) vector fields $A : M \longrightarrow 2^{TM}$ such that $A(x) \subseteq T_xM$ for each $x \in M$. The domain of a vector field A which is a closed and convex subset of M is denoted by $D(A)$ and defined as follows

$$D(A) = \{x \in M : \ A(x) \neq \emptyset\}.$$

Definition 2.3 ([11]). *Let $A \in \mathcal{X}(M)$. Then A is said to be monotone if the following condition holds for any two points $x,y \in D(A)$:*

$$\langle u, \exp_x^{-1} y\rangle \leqslant \langle v, -\exp_y^{-1} x\rangle, \quad \forall u \in A(x) \quad \text{and} \quad \forall v \in A(y). \tag{5}$$

The gradient of a differentiable function $\varphi : M \to \mathbb{R}$, $\mathrm{grad}\varphi$, is the vector field metrically equivalent to the differential $d\varphi$. Thus

$$\langle \mathrm{grad}\varphi, X\rangle = d\varphi(X) = X\varphi,$$

where X is also a vector field. Let $p \in M$. The mapping $x \mapsto d^2(x, p)$ is a smooth map and

$$\frac{1}{2}\mathrm{grad}_x d^2(x, p) = -\exp_x^{-1} p, \tag{6}$$

where grad_x denotes the gradient with respect to x. (See for example, Proposition 4.8 of [16], page 108).

Definition 2.4. *Let M be a Hadamard manifold. A single-valued mapping $T : M \to M$ is called:*

(i) *Lipschitz if there exists $K > 0$ such that $d(Tx, Ty) \le Kd(x, y)$ for all $x, y \in M$ (the infimum of the K's that satisfy the above inequality is called Lipschitz constant of T).*

(ii) *locally Lipschitz near a point $x \in M$ if there exists $r > 0$ such that the mapping $T|_{\mathbb{B}(x;r)}$ is Lipschitz.*

(iii) *locally Lipschitz on M if it is locally Lipschitz near every point in M.*

(iv) *nonexpansive if it is Lipschitz with Lipschitz constant $K \le 1$.*

(v) *quasi-nonexpansive if $\mathrm{Fix}(T) := \{x \in M : Tx = x\} \ne \emptyset$ and $d(Tx, y) \le d(x, y)$ for each $x \in M$, and $y \in \mathrm{Fix}(T)$.*

In order to show the abundance of quasi-nonexpansive mappings on Hadamard manifolds we give the following example.

Example 2.5. *Assume that M is an n-dimensional Hadamard manifold and $x \in M$ is arbitrary and fixed. Take a quasi-nonexpansive mapping $f : T_xM \equiv \mathbb{R}^n \to T_xM \equiv \mathbb{R}^n$ with $\mathrm{Fix}(f) = \{0\}$ (there are a lot of mappings with this property, especially f can be taken nonexpansive with $\{0\}$ as the fixed point set). Consider the mapping $g : M \to M$, defined by $g(y) := \exp_x \circ f \circ \exp_x^{-1} y$. Clearly, $\mathrm{Fix}(g) = \{x\}$. Also, we have*

$$\begin{aligned} d(g(y), x) &= d(\exp_x \circ f \circ \exp_x^{-1} y, x) = \|f(\exp_x^{-1} y)\| \\ &= \|f(\exp_x^{-1} y) - 0\| \le \|\exp_x^{-1} y\| = d(y, x), \end{aligned}$$

which shows that g is quasi-nonexpansive but it is not necessarily nonexpansive.

The set of all cluster points of $u(t)$, denoted by $\omega(u(t))$, is defined as follows

$$\omega(u(t)) := \{q \in M : \exists\{t_n\} \subseteq [0, \infty) \text{ s.t. } t_n \to \infty, u(t_n) \to q\}.$$

The set of all cluster points of a sequence u_n is defined similarly.

3. Convergence of the solutions

It is well known that if $\varphi : M \to \mathbb{R}$ is a convex differentiable function, then $\mathrm{grad}\varphi$ is a monotone vector field. Ahmadi and Khatibzadeh in [3] showed that if $T : M \to M$ is a nonexpansive mapping, then $-\exp_x^{-1} Tx$ is a monotone vector field. In [1, 4, 15], the authors studied (1) in Riemannian and Hadamard manifolds for $A = \mathrm{grad}\varphi$ and $A = -\exp_x^{-1} Tx$, where T is a nonexpansive mapping, and proved the convergence of solutions to a minimum point of φ and a fixed point of T, respectively. In this section, we replace the monotone vector field A with $-\exp_x^{-1} Tx$, where $T : M \to M$ is a locally Lipschitz and quasi-nonexpansive mapping.

In fact, we consider the following first-order system

$$\begin{cases} \dot{u}(t) = \exp_{u(t)}^{-1} Tu(t), \\ u(0) = x \in M. \end{cases} \tag{7}$$

where $T : M \to M$ is a locally Lipschitz and quasi-nonexpansive mapping. First, we prove the existence of solutions to (7) when T is locally Lipschitz. For this purpose, we prove that the vector field $x \mapsto \exp_x^{-1} Tx$ is Lipschitz on bounded sets.

Proposition 3.1. *If $T : M \to M$ is locally Lipschitz, then $\psi(x) := \exp_x^{-1} Tx$ is Lipschitz on bounded sets.*

Proof. Let $C \subseteq M$ be bounded and $x, y \in C$. Set $l = d(x, y)$ and let $\gamma : [0, l] \to M$ be a geodesic curve in M with endpoints x and y. By (4), we have

$$\begin{aligned}\|P_{\gamma(l),\gamma(0)}\psi(\gamma(l)) - \psi(\gamma(0))\|_{\gamma(0)} &= \|P_{y,x} \exp_y^{-1} Ty - \exp_x^{-1} Tx\|_x \\ &= \|P_{y,x} \exp_y^{-1} Ty - \exp_x^{-1} Ty + \exp_x^{-1} Ty - \exp_x^{-1} Tx\|_x \\ &\leq \|P_{y,x} \exp_y^{-1} Ty - \exp_x^{-1} Ty\|_x + \|\exp_x^{-1} Ty - \exp_x^{-1} Tx\|_x \\ &\leq \|P_{\gamma(l),\gamma(0)} \exp_{\gamma(l)}^{-1} T\gamma(l) - \exp_{\gamma(0)}^{-1} T\gamma(l)\|_{\gamma(0)} + Kd(x, y),\end{aligned}$$

where the last inequality is obtained by using the comparison theorem for triangles and the Lipschitz property of T on bounded sets. Now define $h : [0, l] \subseteq \mathbb{R} \to \mathbb{R}^n$ by

$$h(t) := P_{\gamma(t),\gamma(0)} \exp_{\gamma(t)}^{-1} T\gamma(l).$$

Clearly, h is a smooth function and hence $\dot{h}$ admits its extremum points on the compact set $[0, l]$ and therefore $\dot{h}$ is bounded. Now a corollary of the mean value theorem implies that

$$\|P_{\gamma(l),\gamma(0)} \exp_{\gamma(l)}^{-1} T\gamma(l) - \exp_{\gamma(0)}^{-1} T\gamma(l)\|_{\gamma(0)} = \|h(l) - h(0)\| \leq Ml,$$

where $M := \max_{0 \leq t \leq \mathrm{diam} C} \|\dot{h}(t)\|$, and $\mathrm{diam}\, C = \sup\{d(u, v) : u, v \in C\}$. □

Remark 3.2. *Proposition 3.1 guarantees the existence and uniqueness of a global solution to* (7)*. More precisely, since ψ is Lipschitz on bounded sets, by the Cauchy-Lipschitz theorem, for any initial point $u(0) = x \in M$, there exists a unique local solution to the Cauchy problem* (7)*. Let $u(t)$ denote the corresponding maximal solution which is defined on some interval $[0, \mathcal{T})$ with $0 < \mathcal{T} \leq \infty$. If $\mathcal{T} \neq \infty$, an argument similar to what is represented in the proof of Theorem 3.3 shows that u is bounded, and therefore $\lim_{t\to\mathcal{T}} u(t) := u_{\mathcal{T}}$ exists. Now considering* (7) *with the initial condition $u(\mathcal{T}) = u_{\mathcal{T}}$ and applying the Cauchy-Lipschitz theorem we obtain a solution on an interval which is larger than $[0, \mathcal{T})$. This is a contradition with the maximality of $\mathcal{T}$. Hence u is infinitely extendible to the right.*

The following theorem is the main result of this section. To prove this theorem we adapt the proof of a theorem due to Bruck [6] to the setting of Hadamard manifolds.

Theorem 3.3. *Assume that $T : M \to M$ is a quasi-nonexpansive mapping and $u(t)$ is a solution to the system* (7)*. Then $u(t)$ converges to a fixed point of T.*

Proof. By the comparison theorem for triangles, we have

$$d^2(Tx, x) \leq 2\langle \exp_x^{-1} Tx, \exp_x^{-1} y\rangle, \quad \forall x \in M, \forall y \in \mathrm{Fix}(T). \tag{8}$$

Let $y \in \mathrm{Fix}(T)$ be fixed. The above inequality yields:

$$\frac{d}{dt} d^2(u(t), y) = \langle \mathrm{grad}_{u(t)} d^2(u(t), y), \dot{u}(t)\rangle = -2\langle \exp_{u(t)}^{-1} y, \exp_{u(t)}^{-1} Tu(t)\rangle \leq 0.$$

Therefore $\lim_{t\to\infty} d^2(u(t), y)$ exists. By (8), we get:

$$\begin{aligned}\|\dot{u}(t)\|^2 = \|\exp_{u(t)}^{-1} Tu(t)\|^2 = d^2(Tu(t), u(t)) &\leq 2\langle \exp_{u(t)}^{-1} Tu(t), \exp_{u(t)}^{-1} y\rangle \\ &= -\frac{d}{dt} d^2(u(t), y).\end{aligned}$$

The above inequality implies that $\|\dot{u}(t)\| \in L^2(0, +\infty; M)$ and hence $h(t) := \langle \exp_{u(t)}^{-1} Tu(t), \exp_{u(t)}^{-1} y\rangle$ belongs to $L^1(0, \infty; M)$. Let $q \in \omega(u(t))$. There exists a sequence t_k such that $t_k \to +\infty$ and $u(t_k) \to q$. Since $h \in L^1(0, +\infty; M)$ for all $\varepsilon > 0$, we have

$$\mu(\{t \in \mathbb{R} : h(t) > \varepsilon\}) < +\infty,$$

where μ is the Lebesgue measure. On the other hand, for each $\varepsilon > 0$ there exist $s > 0$ and $k > 0$ sufficiently large such that $h(s) < \varepsilon$ and $|t_k - s| < \varepsilon^2$. Besides, by the Cauchy-Schwarz inequality, we have

$$d(u(t_k), u(s)) \leq \int_s^{t_k} \|\dot{u}(t)\| dt \leq \Big(\int_0^{\infty} \|\dot{u}(t)\|^2 dt \Big)^{\frac{1}{2}} \sqrt{t_k - s} \leq M\varepsilon,$$

where $M := \|\dot{u}(t)\|_2$. Taking $\varepsilon = \frac{1}{j}$, there exist sequences s_j and k_j such that $s_j \to +\infty$ and $k_j \to +\infty$ and $h(s_j) < \frac{1}{j}$ and $|t_{k_j} - s_j| < \frac{1}{j^2}$. Therefore $d(u(t_{k_j}), u(s_j)) \leq \frac{M}{j}$. This means $\langle \exp^{-1}_{u(s_j)} Tu(s_j), \exp^{-1}_{u(s_j)} y \rangle \to 0$ and $u(s_j) \to q$ as j tends to infinity. Now [11, Lemma 2.4 (i)] yields that $\langle \exp^{-1}_q Tq, \exp^{-1}_q y \rangle = 0$. The comparison theorem for triangles and the fact that T is quasi-nonexpansive imply that $d^2(Tq, q) + d^2(y, q) \leq d^2(Tq, y) \leq d^2(q, y)$. Therefore $q \in \mathrm{Fix}(T)$. The proof is now complete because for each $y \in \mathrm{Fix}(T)$, $\lim_{t\to+\infty} d^2(u(t), y)$ exists. □

4. Discrete case

As in the linear spaces there are two approaches for the discretization of (7): the forward and backward discretizations. The backward discretization induces the well-known proximal point algorithm for vector field $-\exp^{-1}_x Tx$. In [2, 11, 13], the authors studied this algorithm in the case that T is nonexpansive. In fact, they considered a more general case and studied the proximal point algorithm for monotone vector fields. The forward discretization yields the Mann iteration method. In [12], the authors studied the convergence of the Mann iteration for vector field $-\exp^{-1}_x Tx$ when T is nonexpansive. It is easy to see that the same reasoning also works for a Lipschitz and quasi-nonexpansive mapping. Therefore, this section is devoted to the convergence of the backward discretization of (7) i.e. the proximal point algorithm for the vector field $\exp^{-1}_x Tx$ on a Hadamard manifold M, where $T : M \to M$ is a Lipschitz quasi-nonexpansive mapping. Let C be a nonempty closed convex subset of M and $T : C \to C$ be a quasi-nonexpansive mapping. The backward discretization of (7) is formulated as follows

$$\begin{cases} -\exp^{-1}_{u_n} u_{n-1} = \lambda_n \exp^{-1}_{u_n} Tu_n, \\ u_0 \in C. \end{cases} \tag{9}$$

Theorem 4.1. *Let C be a nonempty closed convex subset of M, and $T : C \to C$ be a quasi-nonexpansive mapping. Also suppose λ_n is a sequence of real numbers such that* $\liminf \lambda_n > 0$. *If $u_0 \in C$ is arbitrary and u_n iteratively defined by* (9), *then $u_n \to p \in$* Fix(T).

Proof. Let $y \in \mathrm{Fix}(T)$ be arbitrary and fixed. By the comparison theorem for triangles, we see that

$$-2\langle \exp^{-1}_{u_n} u_{n-1}, \exp^{-1}_{u_n} y \rangle \leq d^2(u_{n-1}, y) - d^2(u_n, y). \tag{10}$$

Substituting from (9) in the above inequality, and then using (8) we conclude that the sequence u_n is Fejér convergent to Fix(T). Now by [11, Lemma 4.6] it is enough to prove that $\omega(u_n) \subseteq \mathrm{Fix}(T)$. For this purpose, let $q \in \omega(u_n)$. There exists a subsequence u_{n_k} of u_n such that $u_{n_k} \to q$. Using (8), we get

$$\begin{aligned} d^2(Tu_n, u_n) &\leq 2\langle \exp^{-1}_{u_n} Tu_n, \exp^{-1}_{u_n} y \rangle = -\frac{2}{\lambda_n} \langle \exp^{-1}_{u_n} u_{n-1}, \exp^{-1}_{u_n} y \rangle \\ &\leq \frac{1}{\lambda_n} \{ d^2(u_{n-1}, y) - d^2(u_n, y) \}. \end{aligned}$$

Summing up both sides of the above inequality from $n = 1$ to $n = N$, we have

$$\sum_{n=1}^{N} d^2(Tu_n, u_n) \leq M d^2(u_0, y),$$

where $M := \sup \frac{1}{\lambda_n}$. Tending N to infinity yields $d(Tu_n, u_n) \in l^2$ which yields $d(Tu_n, u_n) \to 0$ and hence $d(Tu_{n_k}, u_{n_k}) \to 0$. Now, the continuity of T completes the proof. □

The recent theorem shows the convergence of the sequence u_n generated by (9) to a fixed point of T. It remains to prove the existence of the sequence given by (9). On the other hand, since (9) is an implicit equation, it is not easy to compute u_n. In the following proposition we show a way to compute (or at least approximate) u_n by the Picard iterations of a contraction on the Hadamard manifold M.

Proposition 4.2. *Let C be a nonempty closed convex subset of M, and $T : C \to C$ be a Lipschitz mapping with Lipschitz constant $K > 1$. If $\lambda_n < \frac{1}{K-1}$, for all $n \in \mathbb{N}$, then u_n in* (9) *is the unique fixed point of the contraction mapping*

$$\varphi(x) := \frac{1}{1+\lambda_n} u_{n-1} \oplus \frac{\lambda_n}{1+\lambda_n} Tx. \tag{11}$$

Proof. First we show that u_n lies on the unique geodesic segment joining u_{n-1} and $T(u_n)$. It is equivalent to show that

$$d(u_{n-1}, Tu_n) = d(u_{n-1}, u_n) + d(u_n, Tu_n). \tag{12}$$

By the comparison theorem for triangles, we have

$$d^2(Tu_n, u_n) + d^2(u_{n-1}, u_n) - 2\langle \exp_{u_n}^{-1} Tu_n, \exp_{u_n}^{-1} u_{n-1}\rangle \le d^2(Tu_n, u_{n-1}).$$

Using (9) and the Cauchy-Schwarz inequality, we get

$$(1+\lambda_n)d(Tu_n, u_n) \le d(Tu_n, u_{n-1}).$$

On the other hand, we have

$$\begin{aligned} d(Tu_n, u_{n-1}) &\le d(Tu_n, u_n) + d(u_n, u_{n-1}) = d(Tu_n, u_n) + \| \exp_{u_n}^{-1} u_{n-1}\| \\ &= d(Tu_n, u_n) + \|\lambda_n \exp_{u_n}^{-1} Tu_n\| = d(Tu_n, u_n) + \lambda_n d(Tu_n, u_n) \\ &= (1+\lambda_n)d(Tu_n, u_n). \end{aligned}$$

Therefore

$$d(Tu_n, u_{n-1}) = (1+\lambda_n)d(Tu_n, u_n), \tag{13}$$

which together with (9) implies (12). Also by (13) we have $d(u_{n-1}, u_n) = (1 - \frac{1}{1+\lambda_n})d(u_{n-1}, Tu_n)$ and $d(u_n, Tu_n) = \frac{1}{1+\lambda_n} d(u_{n-1}, Tu_n)$. On the other hand, φ defined in (11), can also be rewritten as $\varphi(x) = \exp_{u_{n-1}} \frac{\lambda_n}{1+\lambda_n} \exp_{u_{n-1}}^{-1} Tx$. For every $x, y \in C$, we have $d(u_{n-1}, \varphi(x)) = \frac{\lambda_n}{1+\lambda_n} d(u_{n-1}, Tx)$ and $d(u_{n-1}, \varphi(y)) = \frac{\lambda_n}{1+\lambda_n} d(u_{n-1}, Ty)$. This together with (4) implies that

$$\begin{aligned} d(\varphi(x), \varphi(y)) &\le \frac{\lambda_n}{1+\lambda_n} \|\exp_{u_{n-1}}^{-1} Tx - \exp_{u_{n-1}}^{-1} Ty\| \\ &\le \frac{\lambda_n}{1+\lambda_n} d(Tx, Ty) \\ &\le \frac{\lambda_n}{1+\lambda_n} K d(x, y), \end{aligned}$$

where K is the Lipschitz constant of T. Now, the contraction mapping principle concludes the desired result. □

The above discussion yields the subsequent conclusion.

Theorem 4.3. *Let C be a nonempty closed convex subset of M, and $T : C \to C$ be a quasi-nonexpansive and Lipschitz mapping with Lipschitz constant $K > 1$. Also suppose λ_n is a sequence of real numbers such that $\lambda_n < \frac{1}{K-1}$, for all $n \in \mathbb{N}$, and $\liminf \lambda_n > 0$. If $u_0 \in C$ is arbitrary, then the sequence u_n generated by* (9) *exists and $u_n \to p \in \mathrm{Fix}(T)$.*

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