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Univalence Criteria for General Integral Operators Involving Normalized Dini Functions

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Abstract. In this paper our aim is to deduce some sufficient conditions for integral operators involving normalized Dini functions to be univalent in the open unit disc. The key tools in our proofs are the generalized versions of the well-known Ahlfor's and Becker's univalence criteria and some inequalities for the normalized Dini functions.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Geometric properties of special functions such as Hypergeometric functions, Bessel functions, Struve functions, Mittag Leffler functions, Wright functions and some other related functions is an ongoing part of research in geometric function theory. We refer for some geometric properties of special functions like Hypergeometric functions [8], Bessel functions [1, 2, 13, 15, 26], Struve functions [18, 20, 33], Lommel functions [9, 32], Mittag-Leffler function [29] and Wright function [24, 25] and references therein. Recently, many mathematicians have set the univalence criteria of several those integral operators which preserve the class $\mathcal{S}$. By using a variety of different analytic techniques, operators and special functions, several authors have studied univalence criterion, a few of them are as mentioned below. Kanas and Srivastava [16], and Deniz and Orhan [10–12] studied univalence criteria for analytic functions defined in $\mathcal{U}$ by using the Loewner chains method. Kiryakova, Saigo and Srivastava [17] obtained some univalence criteria for certain generalized fractional integral and derivatives, accompanying all the linear integro-differential operators. Frasin [15] studied the univalence criteria of some integral operators defined by Bessel functions of first kind. Geometric properties of these integral operators were discussed in [2, 15]. Deniz et al. [13] introduced certain integral operators by using Generalized Bessel functions and studied their univalence

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criteria. Further, Raza et al [26] discussed the convexity, starlikeness and uniformly convexity of these integral operators. Recently Al Kharsani et al. [1] investigated the sufficient conditions for linear fractional differential operators involving the normalized forms of the generalized Bessel functions of the first kind to be univalent. For further details of these univalence criterion, we refer the readers to [4–7, 19, 21–23, 27, 28, 30, 31].

The Bessel functions of the first kind J_ν is defined by

$$J_\nu(z) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \tag{2}$$

where Γ stands for Euler gamma function. It is a particular solution of the second order linear homogeneous differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0$$

where $\nu \in \mathbb{C}$. It is important to study their properties in many aspects. We consider the normalized Dini function $q_\nu : \mathcal{U} \to \mathbb{C}$ defined by

$$\begin{aligned} q_\nu(z) &= 2^{\nu-1}\Gamma(\nu+1) z^{1-\frac{\nu}{2}} \left((2-\nu) J_\nu\left(\sqrt{z}\right) + \sqrt{z} J_\nu'\left(\sqrt{z}\right)\right) \\ &= z + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)\Gamma(\nu+1)}{4^m m!\Gamma(\nu+m+1)} z^{m+1} \quad (z \in \mathcal{U}). \end{aligned} \tag{3}$$

Recently Baricz et al [3] studied the close-to-convexity of Dini functions. Further some geometric properties for the Dini functions are discussed in [14]. In this paper, we are mainly interested in the univalence of integral operators involving the normalized Dini functions of the form (3) defined by

$$F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{q_{\nu_i}(t)}{t}\right)^{\frac{1}{\alpha_i}} dt\right\}^{1/\beta}, \tag{4}$$

$$G_{\nu_1,\ldots,\nu_n,\alpha,n}(z) = \left\{(n\alpha+1) \int_0^z \prod_{i=1}^{n} \left(q_{\nu_i}(t)\right)^{\alpha} dt\right\}^{1/(n\alpha+1)}, \tag{5}$$

and

$$Q_{\nu,\lambda}(z) = \left\{\lambda \int_0^z t^{\lambda-1} \left(e^{q_\nu(t)}\right)^{\lambda} dt\right\}^{1/\lambda}. \tag{6}$$

More precisely, we would like to show that by using some inequalities for the normalized Dini functions the univalence of these integral operators involving normalized Dini functions can be derived easily via some well-known univalence criteria. In particular, we obtain simple sufficient conditions for some integral operators which involve the sin and cos functions. At the last section of this paper, we find the univalence of some integral operators for the bounds of normalized Dini functions.

2. Preliminary Results

In order to derive our results, we need the following lemmas.

Lemma 2.1. *[22] Let β and c be the complex numbers such that $\Re(\beta) > 0$ and $|c| \leq 1$, $c \neq -1$. If the function $f \in \mathcal{A}$ satisfies the inequality*

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1$$

for all $z \in \mathcal{U}$, then

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}} \tag{7}$$

is in the class $\mathcal{S}$.

Lemma 2.2. *[21] Let $\alpha \in \mathbb{C}$, such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies the inequality*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

for all $z \in \mathcal{U}$, then for all $\beta \in \mathbb{C}$ such that $\Re(\beta) \geq \Re(\alpha)$, the function F_β defined by (7) is in the class $\mathcal{S}$.

Lemma 2.3. *[23] Let $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ such that $\Re(\lambda) \geq 1$, $\alpha > 1$ and $2\alpha\,|\lambda| \leq 3\sqrt{3}$. If $f \in \mathcal{A}$ satisfies the inequality $\left|zf'(z)\right| \leq \alpha$ for all $z \in \mathcal{U}$, then the function $Q_\lambda : \mathcal{U} \to \mathbb{C}$, defined by*

$$Q_\lambda(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \left(e^{f(t)}\right)^\lambda dt \right\}^{1/\lambda}$$

is in the class $\mathcal{S}$.

3. Inequalities Involving In The Main Results

Lemma 3.1. *Let $\nu \in \mathbb{R}$ and consider the normalized Dini function $q_\nu(z) : \mathcal{U} \to \mathbb{C}$, defined by (3). Then the following inequalities hold for all $z \in \mathcal{U}$*

(i)

$$\left| q_\nu'(z) - \frac{q_\nu(z)}{z} \right| \leq \frac{4\nu + 9}{2\left(4\nu^2 + 11\nu + 7\right)} \quad (\nu > -1),$$

(ii)

$$\frac{4\nu^2 + 9\nu + 3}{4\nu^2 + 11\nu + 7} \leq \left| \frac{q_\nu(z)}{z} \right| \leq \frac{4\nu^2 + 13\nu + 11}{4\nu^2 + 11\nu + 7} \quad (\nu > -1),$$

(iii)

$$\left| \frac{zq_\nu'(z)}{q_\nu(z)} - 1 \right| \leq \frac{4\nu + 9}{2\left(4\nu^2 + 9\nu + 3\right)} \quad \left(\nu > \frac{-9 + \sqrt{33}}{8}\right),$$

(iv)

$$\frac{\nu^2 + \nu - 1}{(\nu + 1)^2} \leq \left| zq_\nu'(z) \right| \leq \frac{\nu^2 + 3\nu + 3}{(\nu + 1)^2} \quad (\nu > -1).$$

Proof. To prove the assertion (i) of the Lemma 3.1, we use the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

with the equality $\frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} = \frac{1}{(\nu+1)_m}$, $m \in \mathbb{N}$ and the result

$$4(m-1)!(\nu+2)_{m-1} \geq (m+1)(\nu+2)^{m-1}, \; m \in \mathbb{N} \setminus \{1\}.$$

Thus, we obtain

$$\begin{aligned}
\left| q_\nu'(z) - \frac{q_\nu(z)}{z} \right| &= \left| \sum_{m=1}^{\infty} \frac{(-1)^m m(m+1)\Gamma(\nu+1)}{4^m m! \Gamma(\nu+m+1)} z^m \right| \leq \sum_{m=1}^{\infty} \frac{(m+1)}{4^m (m-1)! (\nu+1)_m} \\
&= \frac{1}{\nu+1} \left[\frac{1}{2} + \sum_{m=2}^{\infty} \frac{m+1}{4(m-1)! 4^{m-1} (\nu+2)_{m-1}} \right] \\
&\leq \frac{1}{(\nu+1)} \left[\frac{1}{2} + \sum_{m=2}^{\infty} \left(\frac{1}{4(\nu+2)} \right)^{m-1} \right] \\
&= \frac{4\nu+9}{2(4\nu^2+11\nu+7)} \quad (z \in \mathcal{U}).
\end{aligned}$$

In order to prove the assertion (ii) of the Lemma 3.1, we use the triangle inequality and the following result:

$$2m!(\nu+2)_{m-1} \geq (m+1)(\nu+2)^{m-1} \quad (m \in \mathbb{N}). \tag{8}$$

We thus find that

$$\begin{aligned}
\left| \frac{q_\nu(z)}{z} \right| &= \left| 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)\Gamma(\nu+1)}{4^m m! \Gamma(\nu+m+1)} z^m \right| \leq 1 + \sum_{m=1}^{\infty} \frac{(m+1)}{4^m m! (\nu+1)_m} \\
&= 1 + \frac{1}{2(\nu+1)} \sum_{m=1}^{\infty} \frac{(m+1)}{2m! 4^{m-1} (\nu+2)_{m-1}} \leq 1 + \frac{1}{2(\nu+1)} \sum_{m=1}^{\infty} \left(\frac{1}{4(\nu+2)} \right)^{m-1} \\
&= \frac{4\nu^2+13\nu+11}{4\nu^2+11\nu+7} \quad (z \in \mathcal{U}).
\end{aligned}$$

Similarly, by using the reverse triangle inequality:

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

and the inequality (8) we have,

$$\begin{aligned}
\left| \frac{q_\nu(z)}{z} \right| &= \left| 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)\Gamma(\nu+1)}{4^m m! \Gamma(\nu+m+1)} z^m \right| \geq 1 - \sum_{m=1}^{\infty} \frac{(m+1)}{4^m m! (\nu+1)_m} \\
&= 1 - \frac{1}{2(\nu+1)} \sum_{m=1}^{\infty} \frac{(m+1)}{2m! 4^{m-1} (\nu+2)_{m-1}} \geq 1 - \frac{1}{2(\nu+1)} \sum_{m=1}^{\infty} \left(\frac{1}{4(\nu+2)} \right)^{m-1} \\
&= \frac{4\nu^2+9\nu+3}{4\nu^2+11\nu+7} \quad (z \in \mathcal{U}).
\end{aligned}$$

Now, by combining (i) and (ii), we get the assertion (iii) of Lemma 3.1.

To prove the assertion (iv) of the Lemma 3.1, we use the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

with the equality $\frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} = \frac{1}{(\nu+1)_m}$, $m \in \mathbb{N}$ and the result

$$4^m \geq 2(m+1),\ 2m! \geq m+1,\ m \in \mathbb{N}. \tag{9}$$

Thus, we get

$$\begin{aligned}
\left|zq_\nu'(z)\right| &= \left|z + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)^2 \Gamma(\nu+1)}{4^m m! \Gamma(\nu+m+1)} z^m\right| \leq 1 + \sum_{m=1}^{\infty} \frac{(m+1)^2}{4^m m! (\nu+1)_m} \\
&= 1 + \frac{1}{\nu+1} \sum_{m=1}^{\infty} \left(\frac{2(m+1)^2}{4^m 2m! (\nu+2)_{m-1}}\right) \leq 1 + \frac{1}{\nu+1} \sum_{m=1}^{\infty} \left(\frac{1}{\nu+2}\right)^{m-1} \\
&= \frac{\nu^2+3\nu+3}{(\nu+1)^2} \quad (z \in \mathcal{U}).
\end{aligned}$$

Similarly, by using the reverse triangle inequality

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

and the inequalities used in (9) , we have

$$\begin{aligned}
\left|zq_\nu'(z)\right| &= \left|z + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)^2 \Gamma(\nu+1)}{4^m m! \Gamma(\nu+m+1)} z^m\right| \geq 1 - \sum_{m=1}^{\infty} \frac{(m+1)^2}{4^m m! (\nu+1)_m} \\
&= 1 - \frac{1}{\nu+1} \sum_{m=1}^{\infty} \frac{2(m+1)^2}{4^m 2m! (\nu+2)_{m-1}} \geq 1 - \frac{1}{\nu+1} \sum_{m=1}^{\infty} \left(\frac{1}{\nu+2}\right)^{m-1} \\
&= \frac{\nu^2+\nu-1}{(\nu+1)^2} \quad (z \in \mathcal{U}).
\end{aligned}$$

□

4. Univalence Of Integral Operators Involving Normalized Dini Functions

Our first main result is an application of Lemma 2.1 and contains sufficient conditions for an integral operator defined in (4) when the q_{ν_i} are the normalized Dini functions with parameters.

Theorem 4.1. *Let $\nu_1, \ldots, \nu_n > \frac{-9+\sqrt{33}}{8}$, where $n \in \mathbb{N}$ and $q_{\nu_i} : \mathcal{U} \to \mathbb{C}$ be defined in* (3)*. Suppose $\nu = \min\{\nu_1, \nu_2, ..., \nu_n\}$, $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, $c \in \mathbb{C}$ with $c \neq 1$ and α_i, $(i = 1, \ldots, n)$ be nonzero complex numbers and these numbers satisfy the relation*

$$|c| + \frac{4\nu+9}{2(4\nu^2+9\nu+3)} \sum_{i=1}^{n} \frac{1}{|\beta\alpha_i|} \leq 1, \tag{10}$$

then the function $F_{\nu_1,...,\nu_n,\alpha_1,...,\alpha_n,\beta} : \mathcal{U} \to \mathbb{C}$ defined by (4) *is in the class $\mathcal{S}$.*

Proof. We consider the function

$$f(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{q_{\nu_i}(t)}{t}\right)^{\frac{1}{\alpha_i}} dt.$$

First of all, since $q_{\nu_i} \in \mathcal{A}$ $(i = 1, 2, ..., n)$ then, we have that $f(z) \in \mathcal{A}$, that is,

$$f(0) = f'(0) - 1 = 0.$$

On the other hand, it is easy to see that

$$f'(z) = \prod_{i=1}^{n} \left(\frac{q_{\nu_i}(z)}{z} \right)^{\frac{1}{\alpha_i}} \tag{11}$$

and

$$\frac{zf''(z)}{f'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zq'_{\nu_i}(z)}{q_{\nu_i}(z)} - 1 \right).$$

By using assertion (iii) of Lemma 3.1, for each ν_i $(i = 1, 2, ..., n)$, we obtain

$$\begin{aligned} \left| \frac{zf''(z)}{f'(z)} \right| &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left| \frac{zq'_{\nu_i}(z)}{q_{\nu_i}(z)} - 1 \right| \\ &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \frac{4\nu_i + 9}{2\left(4\nu_i^2 + 9\nu_i + 3\right)}. \end{aligned}$$

Now as it is clear that the function

$$\phi(\nu) : \left(\frac{-9 + \sqrt{33}}{8}, \infty \right) \to \mathbb{R},$$

defined by

$$\phi(\nu) = \frac{4\nu + 9}{2\,(4\nu^2 + 9\nu + 3)}$$

is decreasing function. Therefore

$$\frac{4\nu_i + 9}{2\left(4\nu_i^2 + 9\nu_i + 3\right)} \leq \frac{4\nu + 9}{2\,(4\nu^2 + 9\nu + 3)}$$

and consequently

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{4\nu + 9}{2\,(4\nu^2 + 9\nu + 3)} \sum_{i=1}^{n} \frac{1}{|\alpha_i|}.$$

Finally, by using the triangle inequality and the assertion of Theorem 4.1, we get

$$\begin{aligned} &\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zf''(z)}{\beta f'(z)} \right| \\ \leq\ & |c| + \frac{4\nu + 9}{2\,(4\nu^2 + 9\nu + 3)} \sum_{i=1}^{n} \frac{1}{\left|\beta\alpha_i\right|} \leq 1, \end{aligned}$$

which, in view of Lemma 2.1, implies that $F_{\nu_1,...,\nu_n,\alpha_1,...,\alpha_n,\beta} \in \mathcal{S}$. This evidently completes the proof of Theorem 4.1. □

Choosing $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ in Theorem 4.1, we have the following results.

Corollary 4.2. *Let $\nu_1, \ldots, \nu_n > \frac{-9+\sqrt{33}}{8}$, where $n \in \mathbb{N}$ and $q_{\nu_i} : \mathcal{U} \to \mathbb{C}$ be defined in* (3)*. Suppose $\nu = \min\{\nu_1, \nu_2, \ldots, \nu_n\}$, $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, $c \in \mathbb{C}$ with $c \neq 1$ and α be a nonzero complex number and these numbers satisfy the relation*

$$|c| + \frac{4\nu + 9}{2\,(4\nu^2 + 9\nu + 3)} \frac{n}{\left|\beta\alpha\right|} \leq 1,$$

then the function $F_{\nu_1,...,\nu_n,\alpha,\beta} : \mathcal{U} \to \mathbb{C}$ defined by (4) *is in the class $\mathcal{S}$.*

It is observe that

$$q_{1/2}(z) = \frac{3}{2}\sqrt{z}\left(\sin\sqrt{z} + \sqrt{z}\cos\sqrt{z}\right)$$

and

$$q_{3/2}(z) = \frac{3}{2\sqrt{z}}\left((z-1)\sin\sqrt{z} + \sqrt{z}\cos\sqrt{z}\right).$$

Thus, taking $n = 1$ in Corollary 4.2, we immediately obtain the following result.

Corollary 4.3. *Let* $\nu_1, \ldots, \nu_n > \frac{-9+\sqrt{33}}{8}$, $\beta \in \mathbb{C}$ *with* $\Re(\beta) > 0$, $c \in \mathbb{C}$ *with* $c \neq 1$ *and* α *be a nonzero complex number and these numbers satisfy the relation*

$$|c| + \frac{4\nu + 9}{2(4\nu^2 + 9\nu + 3)}\frac{1}{|\beta\alpha|} \leq 1,$$

then the function $F_{\nu,\alpha,\beta} : \mathcal{U} \to \mathbb{C}$ *defined by*

$$F_{\nu,\alpha,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\left(\frac{q_\nu(t)}{t}\right)^{\frac{1}{\alpha}} dt\right\}^{1/\beta}$$

is in the class $\mathcal{S}$. *In particular, if* $|c| + \frac{11}{17|\beta\alpha|} \leq 1$, *then the function* $F_{\frac{1}{2},\alpha,\beta} : \mathcal{U} \to \mathbb{C}$ *defined by*

$$F_{\frac{1}{2},\alpha,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\left(\frac{\frac{3}{2}\left(\sin\sqrt{t} + \sqrt{t}\cos\sqrt{t}\right)}{\sqrt{t}}\right)^{\frac{1}{\alpha}} dt\right\}^{1/\beta}$$

is in the class $\mathcal{S}$. *Moreover, if* $|c| + \frac{15}{51|\beta\alpha|} \leq 1$, *then the function* $F_{\frac{3}{2},\alpha,\beta} : \mathcal{U} \to \mathbb{C}$ *defined by*

$$F_{\frac{3}{2},\alpha,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\left(\frac{3}{2t}\left((t-1)\sin\sqrt{t} + \sqrt{t}\cos\sqrt{t}\right)\right)^{\frac{1}{\alpha}} dt\right\}^{1/\beta}$$

is in the class $\mathcal{S}$.

The following results contains another sufficient conditions for an integral operator defined in (5). The key tool in the proof is the assertion (iii) of Lemma 3.1.

Theorem 4.4. *Let* $\nu_1, \ldots, \nu_n > \frac{-9+\sqrt{33}}{8}$, *where* $n \in \mathbb{N}$ *and* $q_{\nu_i} : \mathcal{U} \to \mathbb{C}$ *be defined in* (3). *Suppose* $\nu = \min\{\nu_1, \nu_2, \ldots, \nu_n\}$, $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$, *and suppose that these numbers satisfy the following inequality*

$$|\alpha| \leq \frac{2\left(4\nu^2 + 9\nu + 3\right)}{4\nu + 9}\frac{1}{n}\Re(\alpha).$$

Then the function $G_{\nu_1,\ldots,\nu_n,\alpha,n} : \mathcal{U} \to \mathbb{C}$ *defined by* (5) *is in the class* $\mathcal{S}$.

Proof. Consider the auxiliary function $g(z) : \mathcal{U} \to \mathbb{C}$ defined by

$$g(z) = \int_0^z \prod_{i=1}^n \left(\frac{q_{\nu_i}(t)}{t}\right)^{\alpha} dt.$$

We observe that $g(z) \in \mathcal{A}$, that is

$$g(0) = g'(0) - 1 = 0.$$

On the other hand, by using the assertion (iii) of Lemma 3.1, the assertion of Theorem 4.4 and the fact that

$$\frac{4\nu_i + 9}{2\left(4\nu_i^2 + 9\nu_i + 3\right)} \leq \frac{4\nu + 9}{2\left(4\nu^2 + 9\nu + 3\right)}$$

we have

$$\begin{aligned}\left(\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)\left|\frac{zg''(z)}{g'(z)}\right| &\leq \frac{|\alpha|}{\Re(\alpha)}\sum_{i=1}^{n}\left|\frac{zq_\nu'(z)}{q_\nu(z)} - 1\right| \\ &\leq \frac{n|\alpha|}{\Re(\alpha)}\frac{4\nu + 9}{2\left(4\nu^2 + 9\nu + 3\right)} \leq 1.\end{aligned}$$

Now, since $\Re(n\alpha + 1) > \Re(\alpha)$ and the function $G_{\nu_1,\ldots,\nu_n,\alpha,n}(z)$ can be rewritten in the form:

$$G_{\nu_1,\ldots,\nu_n,\alpha,n}(z) = \left[(n\alpha + 1)\int_0^z t^{n\alpha}\prod_{i=1}^{n}\left(\frac{q_{\nu_i}(t)}{t}\right)^{\alpha} dt\right]^{\frac{1}{(n\alpha+1)}},$$

Lemma 2.2 would imply that $G_{\nu_1,\ldots,\nu_n,\alpha,n}(z) \in \mathcal{S}$, which completes the proof of Theorem 4.4. □

By setting $n = 1$ in Theorem 4.4, we get the required result.

Corollary 4.5. *Let $\nu > \frac{-9+\sqrt{33}}{8}$, where $n \in \mathbb{N}$ and $q_\nu : \mathcal{U} \to \mathbb{C}$ be defined in* (3)*. Suppose $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and suppose that these numbers satisfy the following inequality*

$$|\alpha| \leq \frac{2\left(4\nu^2 + 9\nu + 3\right)}{4\nu + 9}\Re(\alpha).$$

Then the function $G_{\nu,\alpha} : \mathcal{U} \to \mathbb{C}$ defined by

$$G_{\nu,\alpha}(z) = \left\{(\alpha + 1)\int_0^z (q_\nu(t))^{\alpha} dt\right\}^{1/(\alpha+1)}$$

is in the class $\mathcal{S}$.

In particular, if $|\alpha| \leq \left(\frac{17}{11}\right)\Re(\alpha)$, then the function $G_{1/2,\alpha} : \mathcal{U} \to \mathbb{C}$ defined by

$$G_{1/2,\alpha}(z) = \left\{(\alpha + 1)\int_0^z \left(\frac{3}{2}\sqrt{t}\left(\sin\sqrt{t} + \sqrt{t}\cos\sqrt{t}\right)\right)^{\alpha} dt\right\}^{1/(\alpha+1)}$$

is in the class $\mathcal{S}$. Moreover, if $|\alpha| \leq \left(\frac{51}{15}\right)\Re(\alpha)$, then the function $G_{3/2,\alpha} : \mathcal{U} \to \mathbb{C}$ defined by

$$G_{3/2,\alpha}(z) = \left\{(\alpha + 1)\int_0^z \left(\frac{3}{2\sqrt{t}}\left((t-1)\sin\sqrt{t} + \sqrt{t}\cos\sqrt{t}\right)\right)^{\alpha} dt\right\}^{1/(\alpha+1)}$$

is in the class $\mathcal{S}$.

Next, by applying the Lemma 2.3 and the assertion (iv) of Lemma 3.1, we easily get the required result.

Theorem 4.6. *Let $\lambda \in \mathbb{C}$, $\nu > -1$ and q_ν is the normalized Dini function. If $\Re(\lambda) \geq 1$ and*

$$|\lambda| \leq \frac{3\sqrt{3}(\nu+1)^2}{\nu^2+3\nu+3}.$$

Then the function $Q_{\nu,\lambda}(z) : \mathcal{U} \to \mathbb{C}$ defined by (6) is in the class $\mathcal{S}$.

Choosing $\nu = 1/2$ and $\nu = 3/2$ in the above Theorem, we obtain the following particular cases.

Corollary 4.7. *If $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \geq 1$ and $|\lambda| \leq \frac{27}{19}\sqrt{3}$, then $Q_{1/2,\lambda}(z) : \mathcal{U} \to \mathbb{C}$, defined by*

$$Q_{1/2,\lambda}(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \left(e^{\frac{3}{2}\sqrt{t}\left(\sin\sqrt{t} + \sqrt{t}\cos\sqrt{t}\right)} \right)^{\lambda} dt \right\}^{1/\lambda}$$

is in the class $\mathcal{S}$.

Corollary 4.8. *If $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \geq 1$ and $|\lambda| \leq \frac{25}{13}\sqrt{3}$, then $Q_{3/2,\lambda}(z) : \mathcal{U} \to \mathbb{C}$, defined by*

$$Q_{3/2,\lambda}(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \left(e^{\frac{3}{2\sqrt{t}}\left((t-1)\sin\sqrt{t} + \sqrt{t}\cos\sqrt{t}\right)} \right)^{\lambda} dt \right\}^{1/\lambda}$$

is in the class $\mathcal{S}$.

The univalence of the integral operators defined in Theorem 4.1, 4.4 and 4.6, can be improved by using the inequality (see [**?**])

$$(k)_m > k(k+\alpha_0)^{m-1}, \ m \in \mathbb{N} \setminus \{1,2\}, \tag{12}$$

where

$$\alpha_0 \simeq 1.302775637\cdots$$

is the greatest root of the following quadratic equation

$$\alpha^2 + \alpha - 3 = 0.$$

Thus, by using the inequality defined in (12) and the same steps used in Lemma 3.1, we will get some improved versions of Lemma 3.1 and Theorem 4.1, 4.4, 4.6.

Lemma 4.9. *Let $\nu > -1$ and consider the normalized Dini function $q_\nu(z) : \mathcal{U} \to \mathbb{C}$, defined in (3). Then the following inequalities hold for all $z \in \mathcal{U}$*

(i)

$$\left| q_\nu'(z) - \frac{q_\nu(z)}{z} \right| < \frac{3 + 16(\nu+1)(\nu+2)\Phi(\nu+1)}{16(\nu+1)(\nu+2)},$$

(ii)

$$\frac{8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\} - 1}{8(\nu+1)(\nu+2)} < \left| \frac{q_\nu(z)}{z} \right| < \frac{1 + 8(\nu+1)(\nu+2)\{1+\Phi(\nu+1)\}}{8(\nu+1)(\nu+2)},$$

(iii)

$$\left| \frac{zq_\nu'(z)}{q_\nu(z)} - 1 \right| < \frac{3 + 16(\nu+1)(\nu+2)\Phi(\nu+1)}{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\} - 1\right]},$$

(iv)

$$\frac{(\nu+1)(\nu+\alpha_0)-(\nu+1+\alpha_0)}{(\nu+1)(\nu+\alpha_0)} \leq \left|zq_\nu'(z)\right| \leq \frac{(\nu+1)(\nu+\alpha_0)+(\nu+1+\alpha_0)}{(\nu+1)(\nu+\alpha_0)},$$

where $\Phi(\nu+1)$ *is defined by*

$$\Phi(\nu+1) = \frac{1}{2(\nu+1)}\left[1+\frac{1}{4(\nu+1+\alpha_0)\{4(\nu+1+\alpha_0)-1\}}\right].$$

Proof. To prove the assertion (i) of the Lemma 4.9, we use the well-known triangle inequality

$$|z_1+z_2| \leq |z_1|+|z_2|$$

with the equality $\frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} = \frac{1}{(\nu+1)_m}$, $m \in \mathbb{N}$ and the results

$$m! \geq \frac{m(m+1)}{2} \text{ and } (\nu+1)_m > (\nu+1)(\nu+1+\alpha_0)^{m-1},\ m \in \mathbb{N}\setminus\{1,2\}.$$

Thus, we obtain

$$\begin{aligned}
\left|q_\nu'(z)-\frac{q_\nu(z)}{z}\right| &= \left|\sum_{m=1}^{\infty}\frac{(-1)^m m(m+1)\Gamma(\nu+1)}{4^m m!\Gamma(\nu+m+1)}z^m\right| \leq \sum_{m=1}^{\infty}\frac{m(m+1)}{4^m m!(\nu+1)_m}\\
&= \frac{1}{2(\nu+1)}+\frac{3}{16(\nu+1)(\nu+2)}+\sum_{m=3}^{\infty}\frac{m(m+1)}{4^m m!(\nu+1)_m}\\
&< \frac{1}{2(\nu+1)}+\frac{3}{16(\nu+1)(\nu+2)}+\frac{1}{2(\nu+1)}\sum_{m=3}^{\infty}\left(\frac{1}{4(\nu+1+\alpha_0)}\right)^{m-1}\\
&= \frac{1}{2(\nu+1)}+\frac{3}{16(\nu+1)(\nu+2)}+\frac{1}{2(\nu+1)}\frac{1}{4(\nu+1+\alpha_0)\{4(\nu+1+\alpha_0)-1\}}\\
&= \frac{3}{16(\nu+1)(\nu+2)}+\Phi(\nu+1)\\
&= \frac{3+16(\nu+1)(\nu+2)\Phi(\nu+1)}{16(\nu+1)(\nu+2)} \quad (\nu>-1).
\end{aligned}$$

In order to prove the assertion (ii) of the Lemma 4.9, we use the triangle inequality and the following results:

$$m! \geq \frac{(m+1)}{2},\ m \in \mathbb{N} \text{ and } (\nu+1)_m > (\nu+1)(\nu+1+\alpha_0)^{m-1},\ m \in \mathbb{N}\setminus\{1,2\}. \tag{13}$$

We thus find that

$$\begin{aligned}
\left|\frac{q_\nu(z)}{z}\right| &= \left|1+\sum_{m=1}^{\infty}\frac{(-1)^m(m+1)\Gamma(\nu+1)}{4^m m!\Gamma(\nu+m+1)}z^m\right| < 1+\sum_{m=1}^{\infty}\frac{2}{4^m(\nu+1)_m}\\
&= 1+\frac{1}{2(\nu+1)}+\frac{1}{8(\nu+1)(\nu+2)}+\sum_{m=3}^{\infty}\frac{2}{4^m(\nu+1)_m}\\
&< 1+\frac{1}{2(\nu+1)}+\frac{1}{8(\nu+1)(\nu+2)}+\frac{1}{2(\nu+1)}\sum_{m=3}^{\infty}\left(\frac{1}{4(\nu+1+\alpha_0)}\right)^{m-1}\\
&= 1+\frac{1}{8(\nu+1)(\nu+2)}+\frac{1}{2(\nu+1)}+\frac{1}{2(\nu+1)}\frac{1}{4(\nu+1+\alpha_0)\{4(\nu+1+\alpha_0)-1\}}\\
&= 1+\frac{1}{8(\nu+1)(\nu+2)}+\Phi(\nu+1)\\
&= \frac{1+8(\nu+1)(\nu+2)\{1+\Phi(\nu+1)\}}{8(\nu+1)(\nu+2)} \quad (\nu>-1).
\end{aligned}$$

Similarly, by using the reverse triangle inequality:

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

and the results (13) we have

$$\begin{aligned}
\left|\frac{q_\nu(z)}{z}\right| &= \left|1 + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)\Gamma(\nu+1)}{4^m m!\Gamma(\nu+m+1)} z^m\right| \geq 1 - \sum_{m=1}^{\infty} \frac{2}{4^m (\nu+1)_m} \\
&= 1 - \left\{\frac{1}{2(\nu+1)} + \frac{1}{8(\nu+1)(\nu+2)} + \sum_{m=3}^{\infty} \frac{2}{4^m (\nu+1)_m}\right\} \\
&\geq 1 - \left\{\frac{1}{2(\nu+1)} + \frac{1}{8(\nu+1)(\nu+2)} + \frac{1}{2(\nu+1)} \sum_{m=3}^{\infty} \left(\frac{1}{4(\nu+1+\alpha_0)}\right)^{m-1}\right\} \\
&= 1 - \frac{1}{8(\nu+1)(\nu+2)} - \left[\frac{1}{2(\nu+1)} + \frac{1}{2(\nu+1)} \frac{1}{4(\nu+1+\alpha_0)\{4(\nu+1+\alpha_0)-1\}}\right] \\
&= 1 - \frac{1}{8(\nu+1)(\nu+2)} - \Phi(\nu+1) \\
&= \frac{8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\} - 1}{8(\nu+1)(\nu+2)} \quad (\nu > -1).
\end{aligned}$$

Now, by combining (i) and (ii), we get the assertion (iii) of Lemma 4.9

$$\left|\frac{zq_\nu'(z)}{q_\nu(z)} - 1\right| \leq \frac{3 + 16(\nu+1)(\nu+2)\Phi(\nu+1)}{2[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\} - 1]} \quad (\nu > -1).$$

To prove the assertion (iv) of the Lemma 4.9 we will use the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

with the equality $\frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} = \frac{1}{(\nu+1)_m}$, $m \in \mathbb{N}$ and the results

$$\begin{aligned}
4^m &\geq 2(m+1),\ 2m! \geq m+1,\ m \in \mathbb{N}, \\
(\nu+1)_m &> (\nu+1)(\nu+1+\alpha_0)^{m-1},\ m \in \mathbb{N}\setminus\{1,2\}.
\end{aligned} \tag{14}$$

Thus, we get

$$\begin{aligned}
\left|zq_\nu'(z)\right| &= \left|z + \sum_{m=1}^{\infty} \frac{(-1)^m (m+1)^2 \Gamma(\nu+1)}{4^m m!\Gamma(\nu+m+1)} z^m\right| \leq 1 + \sum_{m=1}^{\infty} \frac{(m+1)^2}{4^m m!(\nu+1)_m} \\
&= 1 + \sum_{m=1}^{\infty} \left(\frac{2(m+1)^2}{4^m 2m!(\nu+1)_m}\right) < 1 + \frac{1}{\nu+1} \sum_{m=1}^{\infty} \left(\frac{1}{\nu+1+\alpha_0}\right)^{m-1} \\
&= \frac{(\nu+1)(\nu+\alpha_0) + (\nu+1+\alpha_0)}{(\nu+1)(\nu+\alpha_0)} \quad (\nu > -1,\ z \in \mathcal{U}).
\end{aligned}$$

Similarly, by using the reverse triangle inequality

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

and the inequalities used in (14) , we get the required

$$\begin{aligned}\left|zq_\nu'(z)\right| &= \left|z+\sum_{m=1}^{\infty}\frac{(-1)^m(m+1)^2\Gamma(\nu+1)}{4^m m!\Gamma(\nu+m+1)}z^m\right| \geq 1-\sum_{m=1}^{\infty}\frac{(m+1)^2}{4^m m!(\nu+1)_m}\\ &= 1-\sum_{m=1}^{\infty}\left(\frac{2(m+1)^2}{4^m 2m!(\nu+1)_m}\right) > 1-\frac{1}{\nu+1}\sum_{m=1}^{\infty}\left(\frac{1}{\nu+1+\alpha_0}\right)^{m-1}\\ &= \frac{(\nu+1)(\nu+\alpha_0)-(\nu+1+\alpha_0)}{(\nu+1)(\nu+\alpha_0)} \quad (\nu>-1,\ z\in\mathcal{U}).\end{aligned}$$

Therefore, the proof of Theorem 4.1 is completed. □

Theorem 4.10. *Let $\nu_1,,\ldots\nu_n > -1$, $n \in \mathbb{N}$ and $q_{\nu_i}:\mathcal{U}\to\mathbb{C}$ be defined in (3). Suppose $\nu=\min\{\nu_1,\nu_2,\ldots\nu_n\}$, $\beta\in\mathbb{C}$ with $\Re(\beta)>0$, $c\in\mathbb{C}$ with $c\neq 1$ and α_i, $(i=1,\ldots,n)$ be nonzero complex numbers and these numbers satisfy the relation*

$$|c|+\frac{3+16(\nu+1)(\nu+2)\Phi(\nu+1)}{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\}-1\right]}\sum_{i=1}^{n}\frac{1}{|\beta\alpha_i|}\leq 1,$$

then the function $F_{\nu_1,\ldots\nu_n,\alpha_1,\ldots\alpha_n,\beta}:\mathcal{U}\to\mathbb{C}$ defined by (4) is in the class $\mathcal{S}$.

Proof. From proof of theTheorem 4.1 we know that

$$\frac{zf''(z)}{f'(z)}=\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zq_\nu'(z)}{q_\nu(z)}-1\right).$$

By using assertion (iii) of Lemma 4.9, for each ν_i $(i=1,2,...,n)$, we obtain

$$\begin{aligned}\left|\frac{zf''(z)}{f'(z)}\right| &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{zq_\nu'(z)}{q_\nu(z)}-1\right|\\ &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\frac{3+16(\nu_i+1)(\nu_i+2)\Phi(\nu_i+1)}{2\left[8(\nu_i+1)(\nu_i+2)\{1-\Phi(\nu_i+1)\}-1\right]}.\end{aligned}$$

Now as it is shown that the function

$$\psi(\nu):(-1,\infty)\to\mathbb{R},$$

defined by

$$\psi(\nu)=\frac{3+16(\nu+1)(\nu+2)\Phi(\nu+1)}{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\}-1\right]} \tag{15}$$

is decreasing and, consequently, that

$$\frac{3+16(\nu_i+1)(\nu_i+2)\Phi(\nu_i+1)}{2\left[8(\nu_i+1)(\nu_i+2)\{1-\Phi(\nu_i+1)\}-1\right]}\leq\frac{3+16(\nu+1)(\nu+2)\Phi(\nu+1)}{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\}-1\right]}.$$

Finally, by using the triangle inequality and the assertion of Theorem 4.10, we obtain

$$\begin{aligned}&\left|c|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right|\\ \leq\ & |c|+\frac{3+16(\nu+1)(\nu+2)\Phi(\nu+1)}{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\}-1\right]}\sum_{i=1}^{n}\frac{1}{|\beta\alpha_i|}\leq 1.\end{aligned}$$

which, in view of Lemma 2.1, implies that $F_{\nu_1,\ldots\nu_n,\alpha_1,\ldots\alpha_n,\beta}\in\mathcal{S}$. This evidently completes the proof of Theorem 4.10. □

Theorem 4.11. *Let $\nu_1, , \ldots \nu_n > -1$, $n \in \mathbb{N}$ and $q_{\nu_i} : \mathcal{U} \to \mathbb{C}$ be defined in* (3). *Suppose $\nu = \min\{\nu_1, \nu_2, \ldots \nu_n\}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and suppose that these numbers satisfy the following inequality*

$$|\alpha| \leq \frac{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\}-1\right]}{3+16(\nu+1)(\nu+2)\Phi(\nu+1)} \frac{1}{n} \Re(\alpha).$$

Then the function $G_{\nu_1,\ldots\nu_n,\alpha,n} : \mathcal{U} \to \mathbb{C}$ defined by (5) *is in the class $\mathcal{S}$.*

Proof. We consider the function $g(z) : \mathcal{U} \to \mathbb{C}$ defined in the proof of Theorem 4.4. By using the assertion (iii) of Lemma 4.9 and the decreasing function $\psi(\nu) : (-1, \infty) \to \mathbb{R}$ given by (15) we have

$$\begin{aligned}\left(\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)\left|\frac{zg''(z)}{g'(z)}\right| &\leq \frac{|\alpha|}{\Re(\alpha)} \sum_{i=1}^{n} \left|\frac{zq'_\nu(z)}{q_\nu(z)} - 1\right| \\ &\leq \frac{n|\alpha|}{\Re(\alpha)} \frac{3+16(\nu+1)(\nu+2)\Phi(\nu+1)}{2\left[8(\nu+1)(\nu+2)\{1-\Phi(\nu+1)\}-1\right]} \leq 1.\end{aligned}$$

Now, since $\Re(n\alpha+1) > \Re(\alpha)$ and the function can be rewritten in the form:

$$G_{\nu_1,\ldots\nu_n,\alpha,n}(z) = \left[(n\alpha+1)\int_0^z t^{n\alpha} \prod_{i=1}^{n} \left(\frac{q_{\nu_i}(t)}{t}\right)^{\alpha} dt\right]^{\frac{1}{(n\alpha+1)}}.$$

which, in view of Lemma 2.2, implies that $G_{\nu_1,\ldots\nu_n,\alpha,n}(z) \in \mathcal{S}$. This evidently completes the proof of Theorem 4.11. □

Next, by applying the Lemma 2.3 and the inequality (iv) of Lemma 4.9, we easily get the required result.

Theorem 4.12. *Let $\lambda \in \mathbb{C}$, $\nu > -1$ and q_ν is the normalized Dini function. If $\Re(\lambda) \geq 1$ and*

$$|\lambda| \leq \frac{3(\nu+1)(\nu+\alpha_0)\sqrt{3}}{\nu^2+3\nu+3}.$$

Then the function $Q_{\nu,\lambda}(z) : \mathcal{U} \to \mathbb{C}$ defined by (6) *is in the class $\mathcal{S}$.*

Remark 4.13. *Similarly, some corollaries can also be obtained by using some particular values as used above.*

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