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Characterizations of Majorization on Summable Sequences

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Abstract. In this paper, we prove a necessary and sufficient condition for majorization on the summable sequence space. For this we redefine the notion of majorization on an infinite dimensional space and therein investigate properties of the majorization. We also prove the infinite dimensional Schur-Horn type and Hardy-Littlewood-Pólya type theorems.

1. Introduction

The theory of majorization arose while studying a number of apparently different unrelated topics such as wealth distribution, inequalities involving convex functions etc., around the early part of the twentieth century. The theory of majorization in finite dimensional space plays a vital role in mathematics [1, 2, 4, 5, 8, 13], statistics [15], quantum mechanics [7, 16] etc.

Let $\alpha_1^{\downarrow} \geq \alpha_2^{\downarrow} \geq \cdots \geq \alpha_n^{\downarrow}$ be the non-increasing rearrangement of the components of an element $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{R}^n$. Suppose $x, y \in \mathbb{R}^n$. Then x is said to be majorized by y (we denote it by $x \preceq y$) if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } 1 \leq k \leq n-1 \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

We recall well-known characterizations of majorization in $\mathbb{R}^n$.

Theorem 1.1. *Let $x, y \in \mathbb{R}^n$. Then*

1. ***Hardy, Littlewood and Pólya Theorem [10]*** *$x \preceq y$ if and only if $x = Dy$ for some doubly stochastic matrix D.*
2. ***Schur-Horn Theorem [11]*** *Given a self-adjoint $n \times n$ matrix H having eigenvalue list in y, there is a basis for which H has diagonal entries x if and only if $x \preceq y$.*
3. *$x \preceq y$ if and only if $\sum_{j=1}^{n} g(x_j) \leq \sum_{j=1}^{n} g(y_j)$ for any convex function g on $\mathbb{R}$ [10].*

By extending the notion of majorization to the space of all absolutely summable real sequences l^1, Markus *et. al.* [9, 14] proved Hardy-Littlewood-Pólya theorem and Schur-Horn theorem for monotonically

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decreasing sequences in l^1. Let $\alpha = \{\alpha_j\}$ and $\beta = \{\beta_j\}$ be two sequences in c_0, the space of all real sequences converging to zero. We say that $\alpha \ll \beta$ [9] if

$$\sup \sum_{m=1}^{k} \alpha_{\pi(m)} \leq \sup \sum_{m=1}^{k} \beta_{\pi(m)} \quad (k = 1, 2, 3, \dots),$$

where the supremum is taken over all permutations π on $\mathbb{N}$. Let $\alpha = \{\alpha_j\}$ and $\beta = \{\beta_j\}$ be two sequences in l^1. We say that $\alpha \preceq \beta$ [9] if

$$\alpha \ll \beta, \quad -\alpha \ll -\beta \text{ and } \sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} \beta_j.$$

Later in 1999, Nuemann, A [17], proved Horn type of theorems by using the notion of convex hull of all permutations of an infinite sequence. Recently, Arveson and Kadison [5] and Kaftal and Weiss [12] established infinite dimensional Schur-Horn theorem for sequences decreasing monotonically to zero. To avoid having to pass to decreasing sequences monotonically to zero, in this paper we will focus on sequences in l^1 that are neither decreasing nor increasing. In this stand point we redefine majorization to l^1 and investigate the nomenclature of majorization in l^1. We give a characterization of majorization in l^1 using convex functions. We also prove infinite dimensional Hardy-Littlewood-Pólya type theorem and Schur-Horn type theorem for such sequences.

2. MAJORIZATION ON l^1

Let $a, b \in \mathbb{R}$. Define $a \vee b = \max\{a, b\}$. The positive part of a (denoted by a^+) is $a \vee 0$, and the negative part of a (denoted by a^-) is $-a \vee 0$. Let $\xi = \{\xi_j\} \in l^1$, the positive part of the sequence ξ is $\xi^+ = (\xi_1^+, \xi_2^+, \dots)$ and the negative part of the sequence ξ is $\xi^- = (\xi_1^-, \xi_2^-, \dots)$. Let $\xi^{+\downarrow} = (\xi_1^{+\downarrow}, \xi_2^{+\downarrow}, \dots)$ and $\xi^{-\downarrow} = (\xi_1^{-\downarrow}, \xi_2^{-\downarrow}, \dots)$, where $\xi_1^{+\downarrow} \geq \xi_2^{+\downarrow} \geq \dots$ is the decreasing rearrangement of components of the sequence ξ^+ and $\xi_1^{-\downarrow} \geq \xi_2^{-\downarrow} \geq \dots$ is the decreasing rearrangement of components of the sequence ξ^-. Without loss of generality, in this paper, we redefine ξ^+ by $\xi^{+\downarrow}$ and ξ^- by $\xi^{-\downarrow}$.

Definition 2.1. *Let $\xi = \{\xi_j\}$ and $\eta = \{\eta_j\}$ be two sequences in l^1. We say that ξ is majorized by η if $\xi^+ \ll \eta^+$, $\xi^- \ll \eta^-$ and $\sum_{j=1}^{\infty} \xi_j = \sum_{j=1}^{\infty} \eta_j$. We denote it by $\xi \prec \eta$.*

Fact 2.2. *Let $\xi = \{\xi_j\}$ and $\eta = \{\eta_j\}$ be in l^1. Then $\xi \preceq \eta \Rightarrow \xi \prec \eta$.*

Proof. For $k \in \mathbb{N}$, let us consider $N = \max\{1 \leq i \leq k : \xi_i^+ > 0\}$. Then

$$\begin{aligned}
\sum_{j=1}^{k} \xi_j^+ &= \sum_{j=1}^{N} \xi_j^+ \\
&= \sup \sum_{j=1}^{N} \xi_{\pi(j)} \\
&\leq \sup \sum_{j=1}^{N} \eta_{\pi(j)} \ \text{ (as } \xi \ll \eta) \\
&\leq \sum_{j=1}^{N} \eta_j^+ \\
&\leq \sum_{j=1}^{k} \eta_j^+ .
\end{aligned}$$

As k is arbitrary, we have $\xi^+ \ll \eta^+$. In a similar manner, one can show that $\xi^- \ll \eta^-$. Hence $\xi \prec \eta$. $\square$

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. One can contemplate x as a sequence of l^1 by setting $x_k = 0$ for all $k > n$.

Fact 2.3. *Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be two elements in $\mathbb{R}^n$. Then $x \preceq y$ if and only if $x \prec y$.*

Proof. Let m_1, m_2 be the number of non negative components in x and y respectively. Suppose $x \prec y$. For the case $m_1 \leq m_2$, as $x^+ \ll y^+$ and $x^- \ll y^-$, we have

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow} \text{ for } 1 \leq k \leq m_2 \tag{1}$$

and

$$\sum_{j=0}^{k} -x_{n-j}^{\downarrow} \leq \sum_{j=0}^{k} -y_{n-j}^{\downarrow} \text{ for } k = 0, 1, 2, \ldots, n - m_2 - 1. \tag{2}$$

Using (1), (2) and $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$, we get $\sum_{j=1}^{m} x_j^{\downarrow} \leq \sum_{j=1}^{m} y_j^{\downarrow}$ for $m = 1, 2, ..., n$. Therefore, $x \preceq y$. For the case $m_1 > m_2$, as $x^+ \ll y^+$, we have $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$ for $1 \leq k \leq m_2$. Also by $x^- \ll y^-$, we get $\sum_{j=0}^{k} -x_{n-j}^{\downarrow} \leq \sum_{j=0}^{k} -y_{n-j}^{\downarrow}$ for $0 \leq k \leq n - m_2 - 1$. Therefore, $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$ for $1 \leq k \leq n$. Hence $x \preceq y$.

Conversely, assume that $x \preceq y$. For the case $m_1 \leq m_2$, one may observe that $x^+ \ll y^+$. Also we have

$$\sum_{j=0}^{k} -x_{n-j}^{\downarrow} \leq \sum_{j=0}^{k} -y_{n-j}^{\downarrow} \text{ for } 0 \leq k \leq n - m_2 - 1 \tag{3}$$

and

$$\sum_{j=0}^{k} x_{n-j}^{\downarrow} \geq \sum_{j=0}^{n-m_2-1} y_{n-j}^{\downarrow} \text{ for } n - m_2 - 1 \leq k \leq n - m_1 - 1. \tag{4}$$

Now by (3) and (4), we get $x^- \ll y^-$. Hence $x \prec y$. Proceeding in the same manner, one can deduce that $x \prec y$, when $m_1 > m_2$. This completes the proof. $\square$

Fact 2.2 and Fact 2.3, show that the notation of majorization defined in Definition 2.1 is proper and well-defined.

Let H be a self-adjoint operator on a separable Hilbert space K. As we are interested in the case that H is a non-positive operator (which is neither a positive operator nor a negative operator), we assume the following on the spectrum of H.

Definition 2.4. *Let $\eta \in l^1$. Then η is said to be pure if both η^- and η^+ are either in c_{00} or not in c_{00}, where c_{00} denotes the space of all finite sequences.*

It is to be observed that if η is the eigenspectrum of H and η is pure, then H is non-positive operator. Here we adopt techniques used in [6, 9]. In this case η can be rearranged such that $\eta_{2n} \geq 0$ and $\eta_{2n+1} \leq 0$ for all $n \in \mathbb{N}$. Also $\prec$ is independent on permuting the coordinates of vectors in l^1. Through out this paper, we assume $\eta_{2n} \geq 0$, $\eta_{2n+1} \leq 0$ for all $n \in \mathbb{N}$ and $\{\eta_{2n}\}$ is monotonically decreasing, $\{\eta_{2n+1}\}$ is monotonically increasing. Though the following theorem can be proved using the techniques employed in [5] and by the splitting the operator H as $H = H^+ - H^-$, we prove it in a different way, which turns out to be a simple and straight forwarded method.

Theorem 2.5. *Let H be a self-adjoint operator on a separable Hilbert space K and $\xi = \{\xi_j\} \in l^1$. Suppose $\eta = \{\eta_j\} \in l^1$ is the eigenspectrum of H and pure. If $\xi \prec \eta$, then there exists an orthonormal basis of K which is the union of $\{\phi_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{m}$ $(0 \le m \le \infty)$ such that $\langle H\phi_j, \phi_j\rangle = \xi_j$ for $j \in \mathbb{N}$ and $\langle Hf_j, f_j\rangle = 0$ for $j = 1, 2, 3, \dots, m$.*

Proof. Suppose $\{\psi_j : j \in \mathbb{N}\}$ is the system of orthonormal eigenvectors corresponding to the eigenvalues $\{\eta_j : j \in \mathbb{N}\}$. For $\xi_1 > 0$, as $\xi^+ \ll \eta^+$, there exists a unique k such that $0 \le \eta_{2(k+1)} \le \xi_1 \le \eta_{2k}$. Let S be the subspace spanned by $\{\psi_{2k}, \psi_{2(k+1)}\}$. As $\langle H\cdot, \cdot\rangle$ is continuous on the closed unit ball in S, we get a unit vector $\phi_1 \in S$ such that $\langle H\phi_1, \phi_1\rangle = \xi_1$. Let S_1 be the closed linear span by the vectors $\{\psi_j : j \in \mathbb{N}\}$. Suppose ψ is the unit vector orthogonal to ϕ_1 in S. Construct a new orthonormal basis $\{\psi'_j : j \in \mathbb{N}\}$ of S_1, by

$$\psi'_n := \begin{cases} \phi_1 & \text{if } n = 2k \\ \psi & \text{if } n = 2(k+1) \\ \psi_n & \text{otherwise .} \end{cases}$$

Then $\sum_{j=1}^{\infty}\langle H\psi'_j, \psi'_j\rangle = \sum_{j=1}^{\infty}\langle H\psi_j, \psi_j\rangle$. Hence $\langle H\psi, \psi\rangle = \eta_{2k} + \eta_{2(k+1)} - \xi_1$. Consider a new set of eigenvectors $\{\psi_j^{(1)} : j \in \mathbb{N}\}$ and a pair sequences $\{\eta_j^{(1)}\}, \{\xi_j^{(1)}\}$ in l^1 determined by

(a) $\xi_j^{(1)} = \xi_{j+1}$ for $j \ge 1$.

(b) $\eta_{2j+1}^{(1)} = \eta_{2j+1}$ and $\psi_{2j+1}^{(1)} = \psi'_{2j+1}$ for all j.

(c) $\eta_{2j}^{(1)} = \eta_{2j}$ for $j < k$, $\eta_{2k}^{(1)} = \eta_{2k} + \eta_{2(k+1)} - \xi_1$ and $\eta_{2j}^{(1)} = \eta_{2(j+1)}$ for $j \ge k+1$.

(d) $\psi_{2j}^{(1)} = \psi'_{2j}$ for $j < k$, $\psi_{2k}^{(1)} = \psi$ and $\psi_{2j}^{(1)} = \psi'_{2(j+1)}$ for $j \ge k+1$.

Assertion (a). $\xi^{(1)} \prec \eta^{(1)}$, where $\xi^{(1)} = \{\xi_j^{(1)}\}$ and $\eta^{(1)} = \{\eta_j^{(1)}\}$.

Proof of Assertion (a): It is easy to observe that $\xi^{(1)-} \ll \eta^{(1)-}$, $\sum_{j=1}^{\infty}\xi_j^{(1)} = \sum_{j=1}^{\infty}\eta_j^{(1)}$ and $\sum_{j=1}^{n}\xi_j^{(1)+} \le \sum_{j=1}^{n}\eta_j^{(1)+}$ for $n \le k-1$. Now for $n \ge k$

$$\begin{aligned} \sum_{j=1}^{n}\xi_j^{(1)+} &= \sum_{j=2}^{n+1}\xi_j^{+} \\ &= \sum_{j=1}^{n+1}\xi_j^{+} - \xi_1 \\ &\le \sum_{j=1}^{n+1}\eta_j^{+} - \xi_1 \\ &= \sum_{j=1}^{k-1}\eta_j^{+} + \sum_{j=k+2}^{n+1}\eta_j^{+} + \eta_k^{+} + \eta_{k+1}^{+} - \xi_1 \\ &= \sum_{j=1}^{k-1}\eta_j^{(1)+} + \sum_{j=k+1}^{n}\eta_j^{(1)+} + \eta_k^{(1)+} = \sum_{j=1}^{n}\eta_j^{(1)+}. \end{aligned}$$

Hence $\xi^{(1)+} \ll \eta^{(1)+}$. This completes the proof of the assertion (a).

For $\xi_1 < 0$, as $\xi^- \ll \eta^-$, there exists a unique k such that $\beta_{2k+3} \le \alpha_1 \le \beta_{2k+1} \le 0$. By applying similar argument as in the case $\xi_1 > 0$, we get ϕ_1 and ψ such that $\langle H\phi_1, \phi_1\rangle = \xi_1$ and $\langle H\psi, \psi\rangle = \eta_{2k+1} + \eta_{2k+3} - \xi_1$.

Let us consider a new set of eigenvectors $\{\psi_j^{(1)} : j \in \mathbb{N}\}$ and two sequences of real numbers $\eta^{(1)} = \{\eta_j^{(1)}\}$, $\xi^{(1)} = \{\xi_j^{(1)}\}$ in l^1 determined by

(e) $\xi_j^{(1)} = \xi_{j+1}$ for $j \geq 1$.

(f) $\eta_{2j}^{(1)} = \eta_{2j}$ and $\psi_{2j}^{(1)} = \psi_{2j}$ for all j.

(h) $\eta_{2j+1}^{(1)} = \eta_{2j+1}$ for $j < k$, $\eta_{2k+1}^{(1)} = \eta_{2k+1} + \eta_{2k+3} - \xi_1$ and $\eta_{2j+1}^{(1)} = \eta_{2j+3}$ for $j \geq k+1$.

(i) $\psi_{2j+1}^{(1)} = \psi_{2j+1}$ for $j < k$, $\psi_{2k+1}^{(1)} = \psi$ and $\psi_{2j+1}^{(1)} = \psi_{2j+3}$ for $j \geq k+1$.

By using the similar technique used in assertion (a), we get $\xi^{(1)} \prec \eta^{(1)}$. Since $\xi^{(1)} \prec \eta^{(1)}$, we get a unit vector ϕ_2 and the sequence of eigenvetors $\{\psi_j^{(2)}\}$ also a pair of sequences $\xi^{(2)}$, $\eta^{(2)}$ in l^1, such that $\langle H\phi_2, \phi_2\rangle = \xi_1^{(1)} = \xi_2$, $\langle H\psi_j^{(2)}, \psi_j^{(2)}\rangle = \eta_j^{(2)}$ and $\xi^{(2)} \prec \eta^{(2)}$.

As $\langle \phi_1, \psi_j^{(1)}\rangle = 0$, for all $j \in \mathbb{N}$, we have $\langle \phi_1, \phi_2\rangle = 0$. By repeating the same process, we get a system of orthonormal vectors $\{\phi_j\}_{j=1}^{\infty}$ such that $\langle H\phi_j, \phi_j\rangle = \alpha_j$ for all $j \in \mathbb{N}$. Let K_1 be the closed linear span of $\{\phi_j\}$. If $K_1 = K$, then $m = 0$. If not, we consider the subspace $K_1^{\perp}$, the orthogonal complement of K_1 in K. Let $\{g_j\}_{j=1}^{m}$ $(1 \leq m \leq \infty)$ be the orthonormal basis of $K_1^{\perp}$. As $\operatorname{tr}(H) = \sum_{j=1}^{\infty}\langle H\phi_j, \phi_j\rangle = \sum_{j=1}^{\infty}\eta_j = \sum_{j=1}^{\infty}\xi_j$, then $\sum_{j=1}^{m}\langle Hg_j, g_j\rangle = 0$. Hence, there is a unit vector $f_1 \in K_1^{\perp}$ such that $\langle Hf_1, f_1\rangle = 0$. Let K_2 be the orthogonal complement of the subspace spanned by the f_1 in $K_1^{\perp}$. Repeating this argument and using transfinite induction we get an orthonormal basis $\{f_j\}_{j=1}^{m}$ of $K_1^{\perp}$ with $\langle Hf_j, f_j\rangle = 0$ $(j = 1, 2, ..., m)$. Thus the union of $\{\phi_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{m}$ forms an orthonormal basis of K. □

Let $\xi = \{\xi_j\}, \eta = \{\eta_j\}$ be two sequences in l^1 and η is pure. Let K be a separable Hilbert space with an orthonormal basis $\{\psi_j : j \in \mathbb{N}\}$. Define an operator H on K by $H(x) = \sum_{j=1}^{\infty}\eta_j\langle x, \psi_j\rangle\psi_j$ for all $x \in K$. It is to be observed that H is bounded, self-adjoint, compact operator on K and $\{\eta_j : j \in \mathbb{N}\}$ is the eigenspectrum of H. Now if $\xi \prec \eta$, then by Theorem 2.5, there exists an orthonormal set $\{\phi_j : j \in \mathbb{N}\}$ in K such that $\langle H\phi_j, \phi_j\rangle = \xi_j$ for $j \in \mathbb{N}$. This states Horn type theorem for sequences in l^1, which essentially says that if $\alpha \prec \beta$, then there exists a compact self-adjoint operator H for which α is the diagonal vector and β is the eigenspectrum.

Theorem 2.6. *Let K be a separable Hilbert space and $\xi, \eta \in l_1$. Suppose η is pure. If $\xi \prec \eta$, then there exists an orthonormal basis of K which is the union of $\{\phi_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{m}$ $(0 \leq m \leq \infty)$ and a self-adjoint compact operator H on K such that $\{\eta_j : j \in \mathbb{N}\}$ is the eigenspectrum of H and $\langle H\phi_j, \phi_j\rangle = \xi_j$ for $j \in \mathbb{N}, \langle Hf_j, f_j\rangle = 0$ for $j = 1, 2, ..., m$.*

Remark 2.7. *In the above results, if the coordinates of η are positive, then coordinates of ξ are also non-negative and the self-adjoint operator H in the above theorem becomes a positive compact operator. In this case, m turns out to be 0. Thus $\{\phi_j : j \in \mathbb{N}\}$ becomes an orthonormal basis of K.*

Let $\xi = \{\xi_j\} \in l^1$. Denote a new sequence $\widehat{\xi} := \{\widehat{\xi_j}\}$ by including finite or infinite number of zeros as components in the sequence ξ. It is to be observed that $\xi \prec \eta \Leftrightarrow \widehat{\xi} \prec \eta$ for any $\eta \in l^1$. The following result is Hardy-Littlewood-Pólya type theorem.

Theorem 2.8. *Let $\xi = \{\xi_j\}, \eta = \{\eta_j\} \in l^1$ and η is pure. Then $\xi \prec \eta$ iff $\widehat{\xi} = M\eta$ for some infinite matrix $M = (m_{ij})$, with $m_{ij} \geq 0$ and $\sum_{j=1}^{\infty} m_{ij} = 1, \sum_{i=1}^{\infty} m_{ij} = 1$, for $i, j \in \mathbb{N}$, where $\widehat{\xi}$ is defined above.*

Proof. First, assume that $\xi \prec \eta$. Then by Theorem 2.6, for any separable Hilbert space K with an orthonormal basis $\{\psi_j : j \in \mathbb{N}\}$, there exists a self-adjoint operator H defined by $H(x) = \sum_{j=1}^{\infty} \eta_j \langle x, \psi_j \rangle \psi_j$ and an orthonormal basis $\{\phi_j\}_{j=1}^{\infty} \cup \{f_j\}_{j=1}^{m}$ such that $\{\eta_j : j \in \mathbb{N}\}$ is the eigenspectrum of H and $\langle H\phi_j, \phi_j \rangle = \xi_j$ for $j \in \mathbb{N}$, $\langle Hf_j, f_j \rangle = 0$ for $j = 1, 2, ..., m$. Define $\{\phi_j^{'}\} = \{\phi_j\} \cup \{f_j\}_{j=1}^{m}$ and a unitary operator U on K by $U(\psi_j) = \phi_j^{'}$ for $j \in \mathbb{N}$. Now for any $j \in \mathbb{N}$,

$$
\begin{aligned}
\widehat{\xi_j} = \langle H\phi_j^{'}, \phi_j^{'} \rangle &= \langle H(U\psi_j), U\psi_j \rangle \\
&= \Big\langle \sum_{k=1}^{\infty} \eta_k \langle U\psi_j, \psi_k \rangle \psi_k, U\psi_j \Big\rangle \\
&= \sum_{k=1}^{\infty} \eta_k \langle U\psi_j, \psi_k \rangle \langle \psi_k, U\psi_j \rangle \\
&= \sum_{k=1}^{\infty} \eta_k \mid \langle U\psi_j, \psi_k \rangle \mid^2 .
\end{aligned}
$$

Set $m_{jk} = \mid \langle U\psi_j, \psi_k \rangle \mid^2$ for $j, k \in \mathbb{N}$. Then $m_{jk} \geq 0$ and $\widehat{\xi} = M\eta$, where $M = (m_{ij})$. As U is a unitary operator, $\sum_{j=1}^{\infty} m_{jk} = \sum_{j=1}^{\infty} \mid \langle U\psi_j, \psi_k \rangle \mid^2 = 1$ for all $k \in \mathbb{N}$. In a similar fashion, we have $\sum_{k=1}^{\infty} m_{jk} = 1$ for $j \in \mathbb{N}$.

Conversely, assume that $\widehat{\xi} = M\eta$. For $n \in \mathbb{N}$, fix $N = \text{Max}\{1 \leq i \leq n : \widehat{\xi_i}^{+} > 0\}$. Now we rearrange the coordinates of $\widehat{\xi}$ in such a way that the first N components of $\widehat{\xi}$ are the same as the first N components of $\widehat{\xi}^{+}$. Therefore,

$$
\begin{aligned}
\sum_{j=1}^{n} \widehat{\xi_j}^{+} &= \sum_{j=1}^{N} \widehat{\xi_j}^{+} = \sum_{j=1}^{N} \xi_j = \sum_{j=1}^{N} \sum_{k=1}^{\infty} m_{jk}\eta_k \\
&\leq \sum_{j=1}^{N} \sum_{k=1}^{\infty} m_{j2k}\eta_k^{+} = \sum_{k=1}^{\infty} \sum_{j=1}^{N} m_{j2k}\eta_k^{+} \\
&= \sum_{k=1}^{\infty} S_k\eta_k^{+}, \qquad \text{where } S_k = \sum_{j=1}^{N} m_{j2k} \\
&= \sum_{k=1}^{N-1} S_k\eta_k^{+} + \sum_{k=N}^{\infty} S_k\eta_k^{+} \\
&\leq \sum_{k=1}^{N-1} S_k\eta_k^{+} + \sum_{k=N}^{\infty} S_k\eta_N^{+} \\
&\leq \sum_{k=1}^{N-1} S_k\eta_k^{+} + \Big(N - \sum_{k=1}^{N-1} S_k\Big)\eta_N^{+} \\
&= \sum_{k=1}^{N-1} S_k(\eta_k^{+} - \eta_N^{+}) + N\eta_N^{+} \\
&\leq \sum_{k=1}^{N-1} (\eta_k^{+} - \eta_N^{+}) + N\eta_N^{+}
\end{aligned}
$$

$$\leq \sum_{k=1}^{N} \eta_k^+ = \sum_{k=1}^{n} \eta_k^+.$$

As n is arbitrary, we have $\widehat{\xi}^+ \ll \eta^+$. By repeating the same one can derive that $\widehat{\xi}^- \ll \eta^-$ Also $\sum_{j=1}^{\infty} \widehat{\xi}_j = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} m_{jk}\eta_k = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} m_{jk}\eta_k = \sum_{k=1}^{\infty} \eta_k$. Hence $\widehat{\xi} \prec \eta$. This completes the proof. □

Example 2.9. *Let $\eta = \{\eta_j\}$, where $\eta_j = \frac{(-1)^j}{j^2}$ for $j \in \mathbb{N}$ and $\xi = \{\xi_j\}$, where $\xi_1 = \frac{\eta_1+\eta_2}{2}$, $\xi_n = \frac{\eta_{n-1}+\eta_{n+1}}{2}$ for $n \geq 2$. Then $\sum_{j=1}^{\infty} \xi_j = \sum_{j=1}^{\infty} \eta_j$. Also $\sum_{j=1}^{n} \xi_j^+ = \sum_{j=1}^{n} \frac{1}{2}\left[\frac{1}{(2j)^2} + \frac{1}{(2j+2)^2}\right] \leq \sum_{j=1}^{n} \frac{1}{(2j)^2} = \sum_{j=1}^{n} \eta_j^+$. As n is arbitrary, $\xi^+ \ll \eta^+$. In a similar fashion, one can derive that $\xi^- \ll \eta^-$. Thus $\xi \prec \eta$ and $\xi = D\eta$, where*

$$D = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

3. CONVEX FUNCTIONS AND MAJORIZATION

In $\mathbb{R}^n$, the theory of majorization has a close relation with convex functions (for more information, the reader can go through [10]). In this section, we prove an inequality involving majorization in l^1 and convex functions. We also give a characterization of majorization in l^1 using convex functions.

Theorem 3.1. *Let $\xi = \{\xi_j\}, \eta = \{\eta_j\} \in l^1$ and η is pure. Assume that $\xi \prec \eta$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous convex function. Then the following hold.*

1. *If $\{g(\eta_j)\} \in l_1$, then $\sum_{j=1}^{\infty} g(\xi_j) \leq \sum_{j=1}^{\infty} g(\eta_j)$.*
2. *If almost all $g(\eta_j)$'s have the same sign except finitely many terms, then $\sum_{j=1}^{\infty} g(\widehat{\xi_j}) \leq \sum_{j=1}^{\infty} g(\eta_j)$, where $\widehat{\xi} = \{\widehat{\xi_j}\}$ defined above.*

Proof. By Theorem 2.8, $\widehat{\xi} = M\eta$ where $M = (m_{ij})$ such that $m_{ij} \geq 0$, $\sum_{j=1}^{\infty} m_{ij} = 1 = \sum_{i=1}^{\infty} m_{ij}$. By convexity of g, we have $g(\widehat{\xi_j}) \leq \sum_{k=1}^{\infty} m_{jk} g(\eta_k)$ for all $j \in \mathbb{N}$. If $\{g(\eta_j)\} \in l_1$, then $g(0) = 0$. Hence

$$\begin{aligned}
\sum_{j=1}^{\infty} g(\widehat{\xi_j}) &= \sum_{j=1}^{\infty} g(\xi_j) \\
&\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m_{jk} g(\eta_k) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} m_{jk} g(\eta_k), \quad \text{as } \{g(\eta_j)\} \in l_1 \\
&= \sum_{k=1}^{\infty} g(\eta_k).
\end{aligned}$$

This completes the proof of (1). Now for (2), as all $g(\beta_j)$'s but finitely many are of the same sign, we have

$$\begin{aligned}
\sum_{j=1}^{\infty} g(\widehat{\xi_j}) &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m_{jk} g(\eta_k) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} m_{jk} g(\eta_k) \\
&= \sum_{k=1}^{\infty} g(\eta_k).
\end{aligned}$$

□

Remark 3.2. *It is to be noted that, in the above theorem one cannot expect* $\sum_{j=1}^{\infty} g(\xi_j) \leq \sum_{j=1}^{\infty} g(\eta_j)$, *always. If one of the series* $\sum_{j=1}^{\infty} g(\xi_j)$ *and* $\sum_{j=1}^{\infty} g(\eta_j)$ *is conditionally convergent say* $\sum_{j=1}^{\infty} g(\xi_j)$, *then by the Riemann rearrangement theorem, there exist two rearrangements* $\sigma_1(n)$ *and* $\sigma_2(n)$ *with* $\sum_{j=1}^{\infty} g(\xi_{\sigma_1(j)}) < \infty$ *and* $\sum_{j=1}^{\infty} g(\xi_{\sigma_2(j)}) = \infty$.

Finally, we provide a characterization of majorization in l^1 through convex functions and it is an analogue of Theorem II.1.3 in [3], for an infinite dimensional settings.

Theorem 3.3. *Let* $\alpha = \{\alpha_j\}, \beta = \{\beta_j\} \in l^1$. *Then the following are equivalent*

1. $\alpha \prec \beta$
2. $\sum_{j=1}^{\infty} (\alpha_j - t)^+ \leq \sum_{j=1}^{\infty} (\beta_j - t)^+$, $\sum_{j=1}^{\infty} (t - \alpha_j)^+ \leq \sum_{j=1}^{\infty} (t - \beta_j)^+$ *for all* $t \in \mathbb{R}$ *and* $\sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} \beta_j$.

Proof. Assume that $\alpha \prec \beta$. Then $\sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} \beta_j$. For $t > 0$, both series $\sum_{j=1}^{\infty} (t - \alpha_j)^+$ and $\sum_{j=1}^{\infty} (t - \beta_j)^+$ diverge to ∞. So, $\sum_{j=1}^{\infty} (t - \alpha_j)^+ \leq \sum_{j=1}^{\infty} (t - \beta_j)^+$. If $t > \alpha_1^+$, then $0 = \sum_{j=1}^{\infty} (\alpha_j - t)^+ \leq \sum_{j=1}^{\infty} (\beta_j - t)^+$. Otherwise, there exists a

$k \in \mathbb{N}$ such that $\alpha^+_{k+1} \le t \le \alpha^+_k$. Now

$$\begin{aligned}\sum_{j=1}^{\infty}(\alpha_j - t)^+ &= \sum_{j=1}^{k}\alpha_j^+ - kt\\ &\le \sum_{j=1}^{k}\beta_j^+ - kt\\ &= \sum_{j=1}^{k}(\beta_j^+ - t)\\ &\le \sum_{j=1}^{\infty}(\beta_j - t)^+.\end{aligned}$$

In a similar manner, one can prove (2), when $t < 0$.

Conversely, suppose $\sum_{j=1}^{\infty}(\alpha_j - t)^+ \le \sum_{j=1}^{\infty}(\beta_j - t)^+$, $\sum_{j=1}^{\infty}(t-\alpha_j)^+ \le \sum_{j=1}^{\infty}(t-\beta_j)^+$ for all $t \in \mathbb{R}$ and $\sum_{j=1}^{\infty}\alpha_j = \sum_{j=1}^{\infty}\beta_j$. Fix some $k \in \mathbb{N}$. Let us consider $t = \beta^+_{k+1}$. Then $\sum_{j=1}^{\infty}(\beta_j - t)^+ = \sum_{j=1}^{k}\beta_j^+ - kt$ and

$$\begin{aligned}\sum_{j=1}^{k}\alpha_j^+ - kt &= \sum_{j=1}^{k}(\alpha_j^+ - t)\\ &\le \sum_{j=1}^{\infty}(\alpha_j - t)^+\\ &\le \sum_{j=1}^{\infty}(\alpha_j - t)^+\\ &= \sum_{j=1}^{k}\beta_j^+ - kt.\end{aligned}$$

Thus $\sum_{j=1}^{k}\alpha_j^+ \le \sum_{j=1}^{k}\beta_j^+$ and hence $\alpha^+ \ll \beta^+$. In a similar way, one can show that $\alpha^- \ll \beta^-$. Hence $\alpha \prec \beta$. □

Corollary 3.4. *Let $\alpha, \beta \in l^1$ with $\sum_{j=1}^{\infty}\alpha_j = \sum_{j=1}^{\infty}\beta_j$. If $\sum_{j=1}^{\infty}g(\alpha_j) \le \sum_{j=1}^{\infty}g(\beta_j)$ for any convex function g on $\mathbb{R}$, then $\alpha \prec \beta$.*

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