



Viewing $*$ -Multiplication Operators Between Orlicz Spaces

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Abstract. In this paper, Lambert multipliers acting between Orlicz spaces are characterized based on some properties of conditional expectation operators. We provide a necessary and sufficient condition for the $*$ -multiplication operators to have closed range. Finally, a necessary condition for Fredholmness of these type of operators will be investigated.

1. Introduction and preliminaries

Operator in function spaces defined by conditional expectations appeared already in Chen and Moy paper [2] and Sidak [16], see for example Brunk [1], in the setting of L^p spaces. This class of operators was further studied by Lambert [7–9], Herron [4], Takagi and Yokouchi [18]. Later, Jabbarzadeh and Sarbaz in [5] have characterized the Lambert multipliers acting between two L^p -spaces, by using some properties of conditional expectation operator. Also, Multiplication operators have been a subject of research of many mathematicians, see for instance, [11, 14, 15]. In this paper, Lambert multipliers acting between Orlicz spaces are characterized. Also, characterizations of existence and closedness of the range of $*$ -multiplication operators are provided. The main research tool is the conditional expectation operators. In the end, Fredholmness of the corresponding $*$ -multiplication operators is investigated.

Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}^+$ be a continuous convex function satisfying the conditions: $\Phi(x) = \Phi(-x)$, $\Phi(0) = 0$ and $\lim_{x \rightarrow \infty} \Phi(x) = +\infty$, where \mathbb{R} denotes the set of real numbers. With each such function Φ , one can associate another convex function $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$ having similar properties, which is defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

The function Φ is called a *Young function*, and Ψ the *complementary function* to Φ . A Young function Φ is said to satisfy the Δ_2 -condition (globally) if $\Phi(2x) \leq k\Phi(x)$, $x \geq x_0 \geq 0$ ($x_0 = 0$) for some absolute constant $k > 0$. The simple functions are not dense in $L^\Phi(\Sigma)$ but if Φ satisfies the Δ_2 -condition, then the class of simple functions is dense in $L^\Phi(\Sigma)$. Throughout this paper we assume that Φ satisfies Δ_2 -condition.

Let (X, Σ, μ) be a complete σ -finite measure space and Φ be a Young function, then the set of Σ -measurable functions

$$L^\Phi(\Sigma) := \{f : X \rightarrow \mathbb{C}; \int_X \Phi(\varepsilon|f|)d\mu < \infty, \text{ for some } \varepsilon > 0\},$$

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is a Banach space, with respect to the norm

$$\|f\|_\Phi = \inf\{\varepsilon > 0 : \int_X \Phi(|f|/\varepsilon)d\mu \leq 1\}.$$

Such a space is known as a *Orlicz space*. For more details concerning Young function and Orlicz spaces, we refer to Labuschagne, Rao and Ren in [6, 12].

If $\Phi(x) = |x|^p$, $1 \leq p < \infty$, then $L^\Phi(\Sigma) = L^p(\Sigma)$, the usual p -integrable functions on X .

Let $\mathcal{A} \subseteq \Sigma$ be a complete σ -finite subalgebra. We view $L^p(\mathcal{A}) = L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$ as a Banach subspace of $L^p(\Sigma)$. Denote the vector space of all equivalence classes of almost everywhere finite valued Σ -measurable functions on X , by $L^0(\Sigma)$. For each nonnegative $f \in L^0(\Sigma)$ or $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique measurable function $E(f)$ with the following conditions:

- (i) $E(f)$ is \mathcal{A} -measurable;
- (ii) If A is any \mathcal{A} -measurable set for which $\int_A f d\mu$ converges, we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$

For every complete σ -finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $f \mapsto E(f)$, from $L^p(\Sigma)$ to $L^p(\mathcal{A})$, $1 \leq p \leq \infty$, is called the *conditional expectation operator with respect to \mathcal{A}* . As an operator on $L^p(\Sigma)$, $E(\cdot)$ is contractive idempotent and $E(L^p(\Sigma)) = L^p(\mathcal{A})$. We will need the following standard facts concerning $E(f)$, for more details of the properties of E , we refer the interested reader to [4, 8, 13]:

- If g is \mathcal{A} -measurable then $E(fg) = E(f)g$;
- $|E(f)|^p \leq E(|f|^p)$;
- $\|E(f)\|_p \leq \|f\|_p$;
- If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$;
- $E(1) = 1$.

Let Φ be a Young function and $f \in L^\Phi(\Sigma)$, since Φ is convex, by Jensen's inequality,

- $\Phi(|E(f)|) \leq E(\Phi(|f|))$;

and consequently,

$$\int_X \Phi\left(\frac{E(f)}{\|f\|_\Phi}\right) d\mu = \int_X \Phi\left(E\left(\frac{f}{\|f\|_\Phi}\right)\right) d\mu \leq \int_X E\left(\Phi\left(\frac{f}{\|f\|_\Phi}\right)\right) d\mu = \int_X \Phi\left(\frac{f}{\|f\|_\Phi}\right) d\mu,$$

this implies that,

- $\|E(f)\|_\Phi \leq \|f\|_\Phi$,

that is, E is contraction on Orlicz spaces. Now let

$$D(E) := \{f \in L^0(\Sigma) : E(|f|) \in L^0(\mathcal{A})\},$$

then f is said to be *conditionable with respect to E* if $f \in D(E)$. For f and g in $D(E)$, we define

$$f * g = fE(g) + gE(f) - E(f)E(g).$$

Let Φ and Ψ be Young functions. A measurable function u in $D(E)$ for which $u * f \in L^\Psi(\Sigma)$ for each $f \in L^\Phi(\Sigma)$ is called a *Lambert multiplier*. In other words, $u \in D(E)$ is a Lambert multiplier if and only if the corresponding $*$ -multiplication operator $T_u : L^\Phi(\Sigma) \rightarrow L^\Psi(\Sigma)$ defined as $T_u f = u * f$ is bounded. Our exposition

regarding Lambert multipliers follows [4, 9]. As a more results of Lambert multipliers, we mention the upcoming paper [3], in which the present authors showed that the set of all Lambert multipliers acting between L^p -spaces are commutative Banach algebra.

In the following sections, Lambert multipliers on Orlicz spaces are considered. We give necessary and sufficient condition on the $*$ -multiplication operators to be closed range. Also, Fredholmness of the corresponding $*$ -multiplication operators is investigated.

2. Characterization of Lambert multipliers

Let Φ and Ψ be Young functions. Define $K_{\Phi, \Psi}^*$, the set of all Lambert multipliers from $L^\Phi(\Sigma)$ into $L^\Psi(\Sigma)$, as follows:

$$K_{\Phi, \Psi}^* := \{u \in D(E) : u * L^\Phi(\Sigma) \subseteq L^\Psi(\Sigma)\} .$$

$K_{\Phi, \Psi}^*$ is a vector subspace of $D(E)$. Put $K_{\Phi, \Phi}^* = K_\Phi^*$. In the following theorem, similar to theorem 2.1 of [5], we characterize the members of K_Φ^* .

Theorem 2.1. *Let Φ be a Young function and $u \in D(E)$. Then $u \in K_\Phi^*$ if and only if $E(\Phi(|u|)) \in L^\infty(\mathcal{A})$.*

Proof. Let $E(\Phi(|u|)) \in L^\infty(\mathcal{A})$ and $f \in L^\Phi(\Sigma)$. Since $\Phi(|E(u)|) \leq E(\Phi(|u|)) \leq \|E(\Phi(|u|))\|_\infty$ a.e., that is, $|E(u)| \leq \Phi^{-1}(\|E(\Phi(|u|))\|_\infty)$, it follows (Prop. 3 page 60 in [12]) that,

$$\int_X \Phi\left(\frac{|E(u)f|}{\Phi^{-1}(\|E(\Phi(|u|))\|_\infty)\|f\|_\Phi}\right) d\mu \leq \int_X \Phi\left(\frac{|f|}{\|f\|_\Phi}\right) d\mu \leq 1 .$$

Hence $\|E(u)f\|_\Phi \leq \Phi^{-1}(\|E(\Phi(|u|))\|_\infty)\|f\|_\Phi$. A similar argument, using the fact that $E(fE(g)) = E(f)E(g)$, we also have

$$\int_X \Phi\left(\frac{|uE(f)|}{\Phi^{-1}(\|E(\Phi(|u|))\|_\infty)\|f\|_\Phi}\right) d\mu \leq \int_X \Phi\left(\frac{|f|}{\|f\|_\Phi}\right) d\mu \leq 1 .$$

Thus $\|E(u)E(f)\|_\Phi \leq \|uE(f)\|_\Phi \leq \Phi^{-1}(\|E(\Phi(|u|))\|_\infty)\|f\|_\Phi$. Now, we get that

$$\|u * f\|_\Phi \leq \|E(u)f\|_\Phi + \|uE(f)\|_\Phi + \|E(u)E(f)\|_\Phi \leq 3\Phi^{-1}(\|E(\Phi(|u|))\|_\infty)\|f\|_\Phi .$$

It follows that $u * f \in L^\Phi(\Sigma)$, hence $u \in K_\Phi^*$.

Now, let $u \in K_\Phi^*$. A straightforward application of the closed graph theorem shows that the operator $T_u : L^\Phi(\Sigma) \rightarrow L^\Phi(\Sigma)$ given by $T_u f = u * f$ is bounded. Define a linear functional Λ on $L^1(\mathcal{A})$ by

$$\Lambda(f) = \int_X E(\Phi(|u|))f d\mu, \quad f \in L^1(\mathcal{A}) .$$

We show that Λ is bounded. Since Φ satisfies Δ_2 -condition, we have (by the Corollary 5 page 26 in [12]):

$$\begin{aligned} |\Lambda(f)| &\leq \int_X E(\Phi(|u|))|f| d\mu = \int_X E(\Phi(|u|)|f|) d\mu \\ &\leq \int_X E(C|u|^\alpha|f|) d\mu = \int_X E\left(C(|u||f|^{\frac{1}{\alpha}})^\alpha\right) d\mu \\ &= \int_X C(|u||f|^{\frac{1}{\alpha}})^\alpha d\mu = C\|T_u|f|^{\frac{1}{\alpha}}\|_\alpha^\alpha \\ &\leq C\|T_u\|^\alpha\| |f|^{\frac{1}{\alpha}} \|_\alpha^\alpha = C\|T_u\|^\alpha\|f\|_1 , \end{aligned}$$

for some $\alpha > 1, C > 0$. Consequently, Λ is a bounded linear functional on $L^1(\mathcal{A})$ and $\|\Lambda\| \leq C\|T_u\|^\alpha$. By the Riesz representation theorem, there exists a unique function $g \in L^\infty(\mathcal{A})$ such that

$$\Lambda(f) = \int_X gfd\mu, \quad f \in L^1(\mathcal{A}).$$

Therefore, $g = E(\Phi(|u|))$, a.e. on X and hence $E(\Phi(|u|)) \in L^\infty(\mathcal{A})$. \square

Let m be the collection $\{T_u : u \in K_\Phi^*\}$. An easy consequence of the closed graph theorem shows that m consist of continuous linear transformations. Since $T_u T_v = T_{u*v}$ (it is obvious), m is commutative algebra. Similar argument as in the proof of Theorem 4.1 in [10], m is maximal abelian and hence it is norm closed.

For $u \in K_\Phi^*$, we define its norm by $\|u\|_{K_\Phi^*} := \Phi^{-1}(\|E(\Phi(|u|))\|_\infty)$ such that $(K_\Phi^*, \|u\|_{K_\Phi^*})$ is respected as a normed space. The next result reads as follows:

Theorem 2.2. *Let $u \in K_\Phi^*$, then the following holds:*

- (i) $\|u\|_{K_\Phi^*} \leq \|T_u\| \leq 3\|u\|_{K_\Phi^*}$,
- (ii) $(K_\Phi^*, \|u\|_{K_\Phi^*})$ is a Banach space.

Proof. In order to prove (i), assume that $u \in K_\Phi^*$ and $f \in L^1(\mathcal{A})$. Then, $E(\Phi(|u|)) \in L^\infty(\mathcal{A})$, and

$$\|E(\Phi(|u|))\|_\infty = \sup_{\|f\| \leq 1} \int_X E(\Phi(|u|)) |f| d\mu \leq C\|T_u\|^\alpha.$$

That is, $\Phi^{-1}(\|E(\Phi(|u|))\|_\infty) \leq \|T_u\|$. It follows that $\|u\|_{K_\Phi^*} \leq \|T_u\|$. On the other hand, by the properties of conditional expectation operators, it is easy to see that for each $f \in L^\Phi(\Sigma)$ with $\|f\|_\Phi \leq 1$,

$$\max\{\|E(u)f\|_\Phi, \|uE(f)\|_\Phi, \|E(u)E(f)\|_\Phi\} \leq \Phi^{-1}(\|E(\Phi(|u|))\|_\infty),$$

and so $\|T_u\| \leq 3\|u\|_{K_\Phi^*}$.

For the proof of (ii), assume that $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{K_\Phi^*}$. Let $f \in L^\Phi(\Sigma)$ and $g \in L^\Psi(\Sigma)$ be arbitrary elements, then

$$\left| \int_X T_{u_n - u_m}(f) \bar{g} d\mu \right| \leq 3\|u_n - u_m\|_{K_\Phi^*} \|f\|_\Phi \|g\|_\Psi,$$

that is, $\{T_{u_n}\}_{n=1}^\infty$ is a Cauchy sequence in the weak operator topology. The subalgebra m is maximal abelian and so it is weakly closed. Therefore, $\{T_{u_n - u_0}\}_{n=1}^\infty$ is weakly convergent to zero, for some $u_0 \in K_\Phi^*$. By the dominated convergence theorem we have

$$\int_X \lim_{n \rightarrow \infty} (u_n - u_0) f g d\mu = \lim_{n \rightarrow \infty} \int_X T_{u_n - u_0}(f) g d\mu = 0.$$

Thus, $\lim_{n \rightarrow \infty} (u_n - u_0) = 0$, a.e. on X and since E is a contraction map, then, $\lim_{n \rightarrow \infty} E(\Phi(|u_n - u_0|)) = 0$, a.e. on X . Finally, $\|u_n - u_0\|_{K_\Phi^*} \rightarrow \infty$ as $n \rightarrow \infty$. \square

3. Fredholm *-multiplication operators

In the following theorem, we establish a condition for a *-multiplication operator T_u to have closed range. We use the symbols $\mathcal{N}(T_u)$ and $\mathcal{R}(T_u)$ to denote the kernel and the range of T_u , respectively.

Theorem 3.1. *Let $u \in K_\Phi^*$. Then T_u is closed range if and only if there exists $\delta > 0$ such that $E(\Phi(|u|)) > \delta$, almost everywhere on the support of $E(u)$.*

Proof. Let $S := \{x \in X : E(u)(x) \neq 0\}$ be the support of $E(u)$. If T_u has closed range, then it is bounded below on $L^\Phi(S)$, i.e., there exists a constant $k > 0$ such that

$$\|T_u f\|_\Phi \geq k \|f\|_\Phi, \quad f \in L^\Phi(S).$$

Let $\delta = k/2$, and put $U := \{x \in S : E(\Phi(|u|))(x) < \delta\}$. Suppose on contrary $\mu(U) > 0$. Since $(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is a σ -finite measure space, we can find a set $B \in \mathcal{A}$ such that $Q := B \cap S \subseteq U$ with $0 < \mu(Q) < \infty$. Then the \mathcal{A} -measurable characteristic function χ_Q lies in $L^\Phi(S)$. It is known that, $\|\chi_Q\|_\Phi = \frac{1}{\Phi^{-1}(1/\mu(Q))}$ and

$$\begin{aligned} \|T_u \chi_Q\| &= \inf\{\varepsilon : \int_S \Phi(|u \chi_Q|/\varepsilon) d\mu \leq 1\} \\ &< \inf\{\varepsilon : \int_S \Phi(|\Phi^{-1}(\delta) \chi_Q|/\varepsilon) d\mu \leq 1\} \\ &= \|\Phi^{-1}(\delta) \chi_Q\|_\Phi = \Phi^{-1}(\delta) \|\chi_Q\|_\Phi, \end{aligned}$$

which is a contradiction. Therefore, $\mu(U) = 0$, i.e., $E(\Phi(|u|)) > \delta$ a.e. on S . Conversely, suppose $E(\Phi(|u|)) > \delta$ a.e. on S and $\{T_u f_n\}_{n=0}^\infty$ be an arbitrary sequence in $\mathcal{R}(T_u)$, such that $\|T_u f_n - g\|_\Phi \rightarrow 0$ as $n \rightarrow \infty$, for some $g \in L^\Phi(\Sigma)$. Hence

$$E(T_u f_n) = E(u)E(f_n) \xrightarrow{L^\Phi(\Sigma)} E(g), \text{ as } n \rightarrow \infty.$$

Since, $E(1/\Phi(|u|))\chi_S = (1/E(\Phi(|u|)))\chi_S$, then we have $\Phi(E(1/|u|))\chi_S \leq 1/\delta$, and so we get that $E(1/|u|) \leq \Phi^{-1}(1/\delta)$ a.e. on S . Therefore, we have

$$\begin{aligned} \left\| \frac{E(g)}{E(u)} \chi_S \right\|_\Phi &= \inf\{\varepsilon > 0 : \int_S \Phi(|\frac{E(g)}{E(u)} \chi_S|/\varepsilon) d\mu \leq 1\} \\ &= \inf\{\varepsilon > 0 : \int_S \Phi(|E(g)E(\frac{1}{u}) \chi_S|/\varepsilon) d\mu \leq 1\} \\ &\leq \inf\{\varepsilon > 0 : \int_S \Phi(|E(g)E(\frac{1}{|u|}) \chi_S|/\varepsilon) d\mu \leq 1\} \\ &\leq \inf\{\varepsilon > 0 : \int_S \Phi(|\frac{|E(g)|}{\Phi^{-1}(\delta)} \chi_S|/\varepsilon) d\mu \leq 1\} \\ &\leq \frac{1}{\Phi^{-1}(\delta)} \|g\|_\Phi. \end{aligned}$$

This follows that $\frac{E(g)}{E(u)} \chi_S \in L^\Phi(S)$. Consequently,

$$E(f_n) \xrightarrow{L^\Phi(\Sigma)} \frac{E(g)}{E(u)} \chi_S, \text{ as } n \rightarrow \infty,$$

and so there exist $f \in L^\Phi(\Sigma)$ such that,

$$f_n \xrightarrow{L^\Phi(\Sigma)} f, \text{ as } n \rightarrow \infty.$$

Thus $T_u f_n \xrightarrow{L^\Phi(\Sigma)} T_u f$, as $n \rightarrow \infty$, and hence $g = T_u f$, which implies that T_u is closed range. \square

Recall (See [19]) that T_u is said to be a *Fredholm operator* if $\mathcal{R}(T_u)$ is closed, $\dim \mathcal{N}(T_u) < \infty$, and $\text{codim} \mathcal{R}(T_u) < \infty$. Also, recall that an \mathcal{A} -atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atoms is called *non-atomic*. It is a well-known fact that every σ -finite measure space $(X, \mathcal{A}, \mu|_{\mathcal{A}})$ can be partitioned uniquely as

$$X = \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cup B,$$

where $\{A_n\}_{n \in \mathbb{N}}$ is a countable collection of pairwise disjoint \mathcal{A} -atoms and B , being disjoint from each A_n , is non-atomic .

The following Theorem is a generalization of Prop. 3 due to Takagi in [17]:

Theorem 3.2. *Let $u \in K_\Phi^*$ and \mathcal{A} is a non-atomic measure space. If the operator T_u is Fredholm on $L^\Phi(\Sigma)$, then we have $|E(u)| \geq \delta$, almost everywhere on X , for some $\delta > 0$.*

Proof. Suppose that T_u is a Fredholm operator. We first claim that T_u is onto. Suppose the contrary, pick $g \in L^\Phi(\Sigma) \setminus \mathcal{R}(T_u)$. Since $\mathcal{R}(T_u)$ is closed, Φ and Ψ are complementary Young functions and Φ satisfies Δ_2 -condition, it follows from [12] (Corollary 5, page 77 and Theorem 7 page 110) that we can find a function $g^* \in L^\Psi = (L^\Phi)^*$, such that

$$(I) \quad \int_X g g^* d\mu = 1 ,$$

and

$$(II) \quad \int_X g^* T_u f d\mu = 0, f \in L^\Phi(\Sigma) .$$

Now (I) yields that the set $B_r = \{x \in X : |E(gg^*)(x)| \geq r\}$ has positive measure for some $r > 0$. As \mathcal{A} is non-atomic, we can choose a sequence $\{A_n\}$ of subsets of B_r with $0 < \mu(A_n) < \infty$ and $A_m \cap A_n = \emptyset$ for $m \neq n$. Put $g_n^* = \chi_{A_n} g^*$. Clearly, $g_n^* \in L^\Psi(\Sigma)$ and is nonzero, because

$$\int_X |g g_n^*| d\mu \geq \int_{A_n} |g g_n^*| d\mu = \int_{A_n} E(|g g^*|) d\mu \geq \int_{A_n} |E(g g^*)| d\mu \geq r \mu(A_n) > 0 ,$$

for each n . Also, for each $f \in L^\Phi(\Sigma)$, $\chi_{A_n} f \in L^\Phi(\Sigma)$ and so (II) implies that

$$\int_X T_u^* g_n^* f d\mu = \int_X g_n^* T_u f d\mu = \int_{A_n} g^* T_u f d\mu = \int_X g^* T_u (\chi_{A_n} f) d\mu = 0 ,$$

which implies that $T_u^* g_n^* = 0$ and so $g_n^* \in \mathcal{N}(T_u^*)$. Since all the sets in $\{A_n\}_n$ are disjoint, the sequence $\{g_n\}_n$ forms a linearly independent subset of $\mathcal{N}(T_u^*)$. This contradicts the fact that $\dim \mathcal{N}(T_u^*) = \text{codim} \mathcal{R}(T_u) < \infty$. Hence T_u is onto. Let $Z(E(u)) := \{x \in X : E(u)(x) = 0\}$. Then $\mu(Z(E(u))) = 0$. Since, if $\mu(Z(E(u))) > 0$, then there is an $F \subseteq Z(E(u))$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(T_u)$, then there exists $f \in L^\Phi(\Sigma)$ such that $T_u f = \chi_F$. Then

$$\mu(F) = \int_X \chi_F d\mu = \int_F T_u f d\mu = \int_F E(T_u f) d\mu = \int_F E(u) E(f) d\mu = 0 ,$$

and this is a contradiction. So $\chi_F \in L^\Phi(\Sigma) \setminus \mathcal{R}(T_u)$, which contradicts the fact that T_u is onto. For each $n = 1, 2, \dots$, let

$$H_n = \{x \in X : \frac{\|E(\Phi(|u|))\|_\infty}{(n+1)^2} < \Phi(|E(u)|)(x) \leq \frac{\|E(\Phi(|u|))\|_\infty}{n^2}\} ,$$

and $H = \{n \in \mathbb{N} : \mu(H_n) > 0\}$. Then the H_n 's are pairwise disjoint, $X = \bigcup_{n=1}^\infty H_n$ and $\mu(H_n) < \infty$ for each $n \geq 1$. Take

$$f(x) = \begin{cases} |E(u)| \Phi^{-1}(1/\mu(H_n)) & x \in H_n, n \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \int_X \Phi\left(\frac{|f(x)|}{\|E(\Phi(|u|))\|_\infty}\right) d\mu &= \sum_{n=1}^\infty \left(\int_{H_n} \Phi\left(\frac{\Phi(|E(u)|)(x)}{\|E(\Phi(|u|))\|_\infty} \Phi^{-1}\left(\frac{1}{\mu(H_n)}\right)\right) d\mu \right) \\ &\leq \sum_{n=1}^\infty \left(\int_{H_n} \Phi\left(\frac{1}{n^2} \Phi^{-1}\left(\frac{1}{\mu(H_n)}\right)\right) d\mu \right) \\ &\leq \sum_{n=1}^\infty \frac{1}{n^2} \frac{1}{\mu(H_n)} \int_{H_n} d\mu = \sum_{n=1}^\infty \frac{1}{n^2} < \infty . \end{aligned}$$

Therefore, $f \in L^\Phi(\mathcal{A})$ and so there exist $g \in L^\Phi(\Sigma)$ such that $T_u g = f$. Hence $E(u)E(g) = E(T_u g) = f$. Since $E(g) = f/E(u)$ except of $Z(E(u))$ and $\mu(Z(E(u))) = 0$, it follows that

$$\begin{aligned} \int_X \Phi(|g|) d\mu &= \int_X E(\Phi(|g|)) d\mu \geq \int_X \Phi(|E(g)|) d\mu \\ &= \int_X \Phi\left(\frac{f}{|E(u)|}\right) d\mu = \sum_{n \in H} \frac{1}{\mu(H_n)} = \sum_{n \in H} 1 . \end{aligned}$$

This implies that H must be a finite set. So there is an n_0 such that $n \geq n_0$ implies $\mu(H_n) = 0$. Together with $\mu(Z(E(u))) = 0$, we obtain

$$\mu\left(\left\{x \in X : \Phi(|E(u)|)(x) \leq \frac{\|E(\Phi(|u|))\|_\infty}{n_0^2}\right\}\right) = \mu\left(\bigcup_{n=n_0}^\infty H_n \cup Z(E(u))\right) = 0 ,$$

that is, $|E(u)| \geq \Phi^{-1}\left(\|E(\Phi(|u|))\|_\infty/n_0^2\right) := \delta$, a.e. on X . \square

References

- [1] H. D. Brunk, On an extension of the concept conditional expectation, Proc. Amer. Math. Soc., 14 (1963), 298-304.
- [2] S. T. C. Moy, Characterizations of conditional expectation as a transformation on function spaces, Pacific J. Math., 4 (1954), 47-63.
- [3] J. Cheshmavar, S.K. Hosseini, and S.K. Sarbaz, A note on Lambert multipliers between L^p spaces, under review.
- [4] J. D. Herron, Weighted Conditional Expectation Operators of L^p Space, ProQuest LLC, Ann Arbor, MI, 2004.
- [5] M. Jabbarzadeh and S.K. Sarbaz, Lambert multipliers between L^p -spaces, Czech. Math. J., 60 (2010), 31-43.
- [6] L. E. Labuschagne, Multipliers on noncommutative Orlicz spaces, Quaest. Math., 37 (2014), 531-546.
- [7] A. Lambert, Hyponormal composition operators, Bull. London Math. Soc., 18 (1986), 395-400.
- [8] A. Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Roy Soc. Edinburgh Ser. A, 118 (1991), 111-118.
- [9] A. Lambert, L^p multipliers and nested sigma-algebras, Oper. Theory Adv. Appl., 104 (1998), 147-153.
- [10] A. Lambert and T.G. Lucas, Nagata's principle of idealization in relation to module homomorphisms and conditional expectations, Kyungpook Math. J., 40 (2000), 327-337.
- [11] K. Raj, S. K. Sharma and A. Kumar, Multiplication operators on Musielak-Orlicz spaces of Bochner type, J. Adv. Stud. Topol., 4 (2012), 1-7.
- [12] M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, Marcel Dekker, Inc. New York, 1991.
- [13] M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.
- [14] H. A. Seid, Cyclic multiplication operators on L^p -spaces, Pac. J. Math., 51 (1974), 542-562.
- [15] S.K. Sharma and B.S. Komal, Multiplication operators on weighted Hardy spaces, Lobachevskii J. Math., 2 (2012), 165-169.
- [16] S. Sidak, On relations between strick sense and wide sense conditional expectation, Theory of Prob. Appl., 2 (1957), 267-271.
- [17] H. Takagi, Fredholm weighted composition operators, Integr. equ. oper. theory, 16 (1993), 267-276.
- [18] H. Takagi and K. Yokouchi, Multiplication and composition operators between two L^p -spaces, Contemp. Math., 232 (1999), 321-338.
- [19] A. C. Zaanen, Integration, 2nd ed. North-Holland, Amsterdam, 1967.